HIGHER ORDER DISPERSION EFFECTS ON THE MODULATIONAL INSTABILITY OF NLS TYPE EQUATIONS

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Abstract. The modulational instability in a nonlinear Schrödinger equation with a fourth order dispersion term (extended NLS eq.) is studied, from a statistical point of view. An integral stability equation is found, which is solved for different choises of the initial spectrum. The results are compared with the similar ones obtained for the usual NLS equation. In the extended NLS case an instability region, even for short wave length, is found, in contrast with the NLS equation when the instability is possible only in the long wave domain.
Key words: NLS, modulational instability, dispersion effects

INTRODUCTION

It is well known that the development of a self-organized structure in a nonlinear system is preceded by an instability. When the propagation of a quasi-monochromatic wave in a weakly nonlinear medium is considered, this is the so-called "*modulational instability*" (MI) or Benjamin-Feir instability [1]. It is a common phenomenon observed in various fields of physics: hydrodynamics, plasma physics, nonlinear optics, quasi one-dimensional nonlinear molecular chains, etc. A plane wave solution with constant amplitude (a Stokes wave) is unstable at small modulations of the amplitude. Two distinct ways to study this problem are possible. The first is a deterministic approach. A linear stability analysis around a Stokes wave is performed and a linear homogenous system of coupled equations is obtained. Usually the system is unstable in the long wave limit, and the instability depends also on the systems parameters. This approach is well known and appears in any textbook on nonlinear wave propagation (see [2] – [4] and references therein).

A less used approach is a statistical one. The statistical properties of the medium where the instability develops are taken into account through a two-point correlation function [5] – [11] and a kinetic equation is written down for it. A linear stability analysis around a translational invariant initial condition is found. The main result can be formulated as: the MI is possible only if the correlation properties in the initial state are not of too short range.

In the present paper the problem of MI will be discussed for two equations. The first is the well-known nonlinear Schrödinger equation (NLS)

$$i\frac{\partial A}{\partial t}+\lambda\frac{\partial^2 A}{\partial x^2}+\nu\left|A\right|^2 A=0 \tag{1}$$

which is a generic equation describing the propagation of quasi-monochromatic waves in dispersive and weakly nonlinear media. It is a completely integrable system, having solitonic solutions. The second is an extension of NLS eq., containing a higher order dispersive term

$$i\frac{\partial A}{\partial t}+\lambda\frac{\partial^2 A}{\partial x^2}+\alpha\frac{\partial^4 A}{\partial x^4}+\nu|A|^2 A=0 \qquad (2)$$

where α although a small quantity cannot be treated as a perturbation. We shall see that new features of MI will result for (2). These will be compared with a similar analysis for the discrete self-trapping equation.

$$i\frac{\partial a_n}{\partial t}+\lambda\left(a_{n+1}-2a_n+a_{n-1}\right)+\nu|a_n|^2 a_n=0 \qquad (3)$$

whose long wave limit is (2) ($a_n \to A(x)$, $a_{n\pm1} \to A(x\pm l)$, l is the lattice constant and $A(x\pm l)$ is expanded in a Taylor series)

In the next section the deterministic approach of MI for (2) is briefly reviewed. In the section three the statistical approach is used and several forms for the initial spectrum are investigated (δ-spectrum, restricted white spectrum, Lorentzian spectrum). Some concluding remarks are given in the last section.

2. MI of extended NLS eq. Deterministic approach

The equation (2) addmits a plane wave solution $A(x,t)=ae^{i(kx-\omega t)}$ with constant amplitude, but with an amplitude dependent dispersion relation $\omega(k)=\lambda k^2-\alpha k^4-\nu|a|^2$ (a Stokes solution). To study the MI we write:

$$A(x,t)=a\left[1+\varepsilon A_1(x,t)\right]e^{i(kx-\omega t)} \qquad (4)$$

and in first order of ε, $A_1(x,t)$ satisfies the linear equation

$$i\frac{\partial A_1}{\partial t}+\lambda\left(2ik\frac{\partial A_1}{\partial x}+\frac{\partial^2 A_1}{\partial x^2}\right)+\nu|a|^2\left(A_1+A_1^*\right)+$$
$$\alpha\left(4ik\frac{\partial^3 A_1}{\partial x^3}-6k^2\frac{\partial^2 A_1}{\partial x^2}-4ik^3\frac{\partial A_1}{\partial x}+\frac{\partial^4 A_1}{\partial x^4}\right)=0$$

Looking for plane wave solutions

$$A_1(x,t)=C_1e^{i(Qx-\Omega t)}+C_2e^{-i(Qx-\Omega^* t)}$$

the compatibility condition gives

$$\Omega=2\lambda kQ-4\alpha kQ\left(k^2+Q^2\right)+$$
$$iQ\sqrt{[\lambda-\alpha(6k^2+Q^2)]\{2\nu|a|^2-Q^2[\lambda-\alpha(6k^2+Q^2)]\}} \qquad (5)$$

If α=0 this reduces to the NLS result:

$$\Omega^{(NLS)}=2\lambda kQ+i|\lambda|Q\sqrt{2\frac{\nu}{\lambda}|a|^2-Q^2} \qquad (6)$$

An instability is associated with $\mathrm{Im}(\Omega)>0$. For the NLS case this is realized if λ, ν have the same sign (the focusing case) and if $Q<\sqrt{2\frac{\nu}{\lambda}|a|^2}$. As $|a|$ is a small quantity one sees that MI appears in the long wave regim of the perturbation.

For eq. (2) if we denote $\alpha Q_0^2 = \lambda - 6\alpha k^2$ (α and λ must have the same sign) and

$$Q_{1,2} = \frac{Q_0}{\sqrt{2}}\left(1 \mp \sqrt{1 - 8\frac{\nu}{\alpha}\frac{|a|^2}{Q_0^4}}\right)^{1/2}$$

two regions of instabilities exist, namely $Q \in (0, Q_1)$ and $Q \in (Q_2, Q_0)$. While the first is in the long wave region, the second is inn the short range region, and this is a new qualitative aspect of the problem.

3. Statistical approach

In the statistical approach A(x,t) is considered as a random variable. The analysis contains several steps. First a kinetic equation is written down for the two-point correlation function

$$\rho(x_1, x_2, t) = \left\langle A(x_1, t) A^*(x_2, t) \right\rangle \tag{7}$$

where by $\langle \ldots \rangle$ we mean an ensemble average. This is obtained in the following way: write (2) in the point $x = x_1$ and multiply by $A^*(x_2, t)$; add to it the complex conjugated of (2) for $x = x_2$ and multiply by $A(x_1, t)$ and finally take an ensemble average. Besides the two-point correlation function $\rho(x_1, x_2)$, four-point correlation functions $\left\langle A(x_1) A^*(x_1) A(x_1) A^*(x_2) \right\rangle$ and $\left\langle A(x_2) A^*(x_2) A^*(x_2) A(x_1) \right\rangle$ will appear. If the random process is Gaussian, an exact factorization of the 4-point correlation functions in products of 2-point correlation functions exists, namely

$$\left\langle A(x_1) A^*(x_1) A(x_1) A^*(x_2) \right\rangle = 2\overline{a^2}(x_1) \rho(x_1, x_2) \tag{8}$$

where

$$\overline{a^2}(x_1) = \left\langle A(x_1) A^*(x_1) \right\rangle \tag{9}$$

is the ensemble average mean square amplitude.

We shall use (8) even for other random processes and this represents the only approximation of the present analysis. In this way, we get

$$i\frac{\partial \rho}{\partial t} + \lambda\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\rho + \alpha\left(\frac{\partial^4}{\partial x_1^4} - \frac{\partial^4}{\partial x_2^4}\right)\rho + 2\nu\left[\overline{a^2}(x_1) - \overline{a^2}(x_2)\right]\rho = 0 \tag{10}$$

The next step is to use a Wigner-Moyal transform [12]. The center of mass coordinate $X = \frac{x_1 + x_2}{2}$ and the relative coordinate $x = x_1 - x_2$ are introduced and a Fourier transform over the relative coordinate is performed.

$$F(k, x, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx} \rho(x, X, t)\, dx \tag{11}$$

The initial correlation function is assumed to be translationally invariant, depending only on the relative coordinate x. During the time evolution a dependence on the center of mass coordinate appears, growing exponentially in time.

Using the new variables X and x, and expanding the difference $\overline{a^2}\left(X+\frac{x}{2}\right)-\overline{a^2}\left(X-\frac{x}{2}\right)$ in a Taylor series, after a Fourier transformation of (10) one obtains

$$\begin{aligned}&\frac{\partial F}{\partial t}+2\lambda k\frac{\partial F}{\partial X}+\alpha k\left(\frac{\partial^2}{\partial X^2}-4k^2\right)\frac{\partial F}{\partial X}+\\&2\nu\sum_{j=0}^{\infty}2\frac{(-1)^j}{2^{2j+1}(2j+1)!}\frac{\partial^{2j+1}\overline{a^2}(X)}{\partial X^{2j+1}}\frac{\partial^{2j+1}F(k,X)}{\partial k^{2j+1}}=0\end{aligned}\tag{12}$$

Further on we shall consider

$$F(k,X,t)=F_0(k)+\varepsilon F_1(k,X,t)\tag{13}$$

where $F_0(k)$ is independent of X and t. It corresponds to an initial 2-point correlation function dependent only on the relative coordinate x (translationally invariant). From the definition (9) of $\overline{a^2}(X)$ we have

$$\overline{a^2}(X,t)=\int_{-\infty}^{+\infty}F(k,X,t)dk\tag{14}$$

so we can write

$$\overline{a^2}(X,t)=\overline{a_0^2}+\varepsilon\overline{a_1^2}(X,t)\tag{15}$$

where

$$\overline{a_0^2}=\int_{-\infty}^{+\infty}F_0(k)dk\,,\qquad \overline{a_1^2}(X,t)=\int_{-\infty}^{+\infty}F_1(k,X,t)dk\tag{16}$$

In first order of ε we get

$$\begin{aligned}&\frac{\partial F_1}{\partial t}+\left[2\lambda k+\alpha k\left(\frac{\partial^2}{\partial X^2}-4k^2\right)\right]\frac{\partial F_1}{\partial X}+\\&2\nu\sum_{j=0}^{\infty}2\frac{(-1)^j}{2^{2j+1}(2j+1)!}\frac{\partial^{2j+1}F_0(k)}{\partial k^{2j+1}}\frac{\partial^{2j+1}\overline{a_1^2}(X)}{\partial X^{2j+1}}=0\end{aligned}\tag{17}$$

We look for plane wave solutions of (17)

$$F_1(k,X,t)=f_1(k)e^{i(KX-\Omega t)}\qquad(18)$$

Introducing (18) in (17) after straight forward calculations the following stability integral equation is found

$$1=2\nu\int_{-\infty}^{+\infty}\frac{F_1\left(k+\frac{K}{2}\right)-F_0\left(k+\frac{K}{2}\right)}{\Omega-2\lambda Kk[1-\alpha(\frac{K^2}{2}+2k^2)]}dk\tag{19}$$

If α=0 (19) transforms into the result known for the NLS eq.

$$1 = 2\nu \int_{-\infty}^{+\infty} \frac{F_1\left(k+\frac{K}{2}\right) - F_0\left(k+\frac{K}{2}\right)}{\Omega - 2\lambda k K} dk \tag{20}$$

Both equations (19) and (20) are very similar with the integral stability equation obtained in the study of Landau damping in plasma physics [13]

Several forms for the initial condition $F_0(k)$ will be considered. We start with a δ-function spectrum

$$F_0(k) = \overline{a_0^2}\delta(k) \tag{21}$$

this corresponds to a constant initial condition $\rho_0(x) = \overline{a_0^2}$, which is quite an unrealistic situation, but it can be considered as a limit case to which all the other initial conditions can be compared. From the point of view of MI it represents the most favourable case as all the particles are correlated with the same probability. The integration in (19) is straight-forward and considering Ω purely imaginary, $\Omega = i\Omega_i$, we get

$$\Omega_i = |\lambda| K \sqrt{(1-\alpha K^2)\left[4\frac{\nu}{\lambda}\overline{a_0^2} - K^2(1-\alpha K^2)\right]} \tag{22}$$

Introducing $K_0^2 = \frac{1}{\alpha}, K_{1,2}^2 = \frac{1}{2\alpha}\left[1 \mp \sqrt{1-16\alpha\frac{\nu}{\lambda}\overline{a_0^2}}\right]$ (we consider λ, ν, α positive quantities) the instability regions corespond to

$K \in (0, K_1)$ and $K \in (K_2, K_0)$

and besides the long wave region of instability, like in the deterministic approach, a short wave region exists. If α=0 we find

$$\Omega_i = 2\lambda K \sqrt{\frac{\nu}{\lambda}\overline{a_0^2} - \frac{K^2}{4}} \tag{23}$$

and MI is possible only in the long wave region.

The next example we shall consider is a "restricted white spectrum"

$$F_0(k) = \begin{cases} \frac{1}{2\Lambda}\overline{a_0^2}, if |K| < \Lambda \\ 0, otherwise \end{cases} \tag{24}$$

For the NLS case we get

$$1 = \frac{p_0^2}{2}\frac{1}{K\Lambda}\ln\frac{\left(\Lambda+\frac{K}{2}\right)^2+\omega^2}{\left(\Lambda-\frac{K}{2}\right)^2+\omega^2}$$

where $p_0^2 = \frac{\nu}{\lambda}\overline{a_0^2}$ and $\omega = \frac{\Omega}{2\lambda K}$. This is easily solved with respect to ω^2 giving

$$\omega^2 = K\Lambda cth\frac{K\Lambda}{p_0^2} - \left(\Lambda^2 + \frac{K^2}{4}\right) \tag{25}$$

For $K \to 0$, $\omega^2 \to p_0^2 - \Lambda^2$ and consequently the width Λ of the spectrum must be smaller than p_0. If this condition is satisfied we have a restriction on K, $K<K_{max}$,

where K_{max} is the intersection point of $K\Lambda cth\frac{K\Lambda}{p}$ with $\Lambda^2 + \frac{K^2}{4}$. On this simple example we can see for the first time the influence of the statistical properties of the medium on the development of MI. This can be formulated in the following way: if the width of the initial spectrum is too large the MI cannot develop, and any initial perturbation will be damped.

The last example is a Lorentzian form for $F_0(k)$

$$F_0(k) = \frac{\overline{a_0^2}}{\pi}\frac{p}{k^2 + p^2} \tag{26}$$

which is a more realistic assumption, corresponding to an exponentially decaying initial condition

$$\rho_0(x) = \overline{a_0^2}e^{-px}$$

For the NLS case the integration in (20) is easily done in the k-complex plane, and finally one obtains

$$\Omega_i = 2\lambda K\left(\sqrt{\frac{\nu}{\lambda}\overline{a_0^2} - \frac{K^2}{4}} - p\right) \tag{27}$$

Here again we see that a real, positive Ω_i is obtained if $p < p_0$, and if this condition is satisfied the wave vector K has to be smaller than $2\sqrt{p_0^2 - p^2}$.

With the notation ($\omega = \frac{\Omega_i}{2\lambda K}$)

$$P(k) = i\omega - \left(1 - 2\alpha\frac{K^2}{4}\right)k + 2\alpha k^3 \tag{28}$$

the integral stability equation (19) writes

$$\Lambda = \frac{\nu}{\lambda K}\int_{-\infty}^{+\infty} F_0(k)\left[\frac{1}{P\left(k - \frac{K}{2}\right)} - \frac{1}{P\left(k + \frac{K}{2}\right)}\right]dk \tag{29}$$

Performing the integration in the k – complex plane with $F_0(k)$ given by the Lorentzian spectrum (26) and closing the contour in the lower half-plane, we shall take into consideration only the pole $k = -ip$ of $F_0(k)$ (for α small the contributions of the complex zeroes of $P(k)$ in the lower half-plane are assumed small and can be neglected). The calculations are straight forward and give:

$$\frac{\Omega_i}{2\lambda K} = \sqrt{\frac{\nu}{\lambda}\overline{a_0^2} - \frac{K^2}{4} + \alpha\left(6p^2 - K^2\right)\left[\frac{\nu}{\lambda}\overline{a_0^2} - \frac{K^2}{2} + \alpha\left(6p^2 - K^2\right)\right] - p\left[1 - 2\alpha\left(p^2 - K^2\right)\right]} \qquad (30)$$

For α=0, (30) goes into (27), the result for the NLS case. The dependence on p and K is now more complicated. Under the square root we have a second order algebraic equation. Let us denote by K_1^2, K_2^2 its roots $\left(0 < K_1^2 < K_2^2\right)$. Then a real Ω_i is obtained if $K^2 < K_1^2$ (the long wave regime) and for $K^2 > K_2^2$. But this short wave region is limited by the condition to have a positive Ω_i. A new restriction $K^2 < K_3^2$ results, and the instability occurs if $K_2^2 < K^2 < K_3^2$. The calculations are lengthy so we restrict ourselves to these qualitative considerations, which are correct at least for small α.

3. CONCLUSIONS

In conclusion, the MI has been studied for a NLS type equation, containing an additional fourth order dispersion term. The results were compared with the similar ones, known for the usual NLS equation. Besides an instability region in the long wave domain, a new region for short waves was obtained, and this is entirely due to the presence of the higher order dispersion term. But this fact has to be considered with caution, because a similar MI analysis for the discrete self-trapping equation (3) gives an instability only in the long wave region [10], and the extended NLS equation (2) is obtained from (3) in a continuum approximation expanding $\left(a_{n+1} - 2a_n + a_{n-1}\right)$ up to fourth order terms. It would be interesting to analyze situations where more nonlinear terms are introduced in the equation, to compensate the stronger dispersive properties of eq. (2), and to see if this new instability region survives.

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