Abstract. A non-linear dynamical system with a finite number of degrees of freedom is generated starting from the main equations for the nonabelian gauge field. The dynamics of this system presents a class of periodical solutions and other various aspects of integrable systems. Such an aspect is the similarity that, in a special case, could be established with the Henon-Heiles system.

Key words: mechanical Yang-Mills model, Henon-Heiles system, integrability.

PACS: 02.30.Ik; 03.50.Kk; 05.45-a

1. INTRODUCTION

The Yang-Mills theory offers one of the most fruitful examples of gauge theories of an intrinsic importance, but also is able to act as a test model for various investigation procedures. It is very familiar for the studies on the universal interactions, as a gauge type theory. The Dirac-Bergmann algorithm or the BRST technique found their first applications on the Yang-Mills models [1].

A very interesting development of the Yang-Mills theory was approached on the line of the constrained dynamical systems, by transforming the field system with an infinite number of degrees of freedom in a "mechanical model", with a finite number of them [2], [3]. In recent years a lot of such models have been intensively investigated [4], [5].

The present paper proposes a study of special kinds of solutions specific to a 2-degree mechanical model. As a gauge theory, the Yang-Mills fields usually follow a chaotic dynamics. Under these circumstances, the pointing out of periodical solutions, or even of quasi-periodical ones, would be remarkable. Moreover, a very interesting similarity between our mechanical model and the Henon-Heiles system, a fruitful model for the integrability studies, has been established. These are the main results of our work.
The paper is structured in 5 sections: after this introductory part, the second section will start from the general Yang-Mills theory and will present the mechanical model to be investigated. In the third section the periodical solutions will be presented and the behavior of the system around them will be studied. In the fourth section the similitude of our mechanical model with the Henon-Heiles system will be investigated. Despite the difference between the phase portraits of the two models, a “swirl” type transformation allows a pretty exact identification. Some comments and concluding remarks will end the paper.

2. THE MECHANICAL MODEL

Let us consider the general Yang-Mills theory for fields without external sources. The equations of motion have the form:

$$\partial_\mu F^\mu_{\nu} + ge^{abc} A^b_\mu F^c_{\mu\nu} = 0, \quad \mu, \nu = 0, 1, 2, 3; \quad a, b, c = 1, 2, 3$$

(1)

Where $A^a_\mu$ is the quadri-potential and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ge^{abc} A^b_\mu A^c_\nu$$

(2)

If we imposed the requirement that the potential should depend only on time, that is to say $A^a_0 \equiv 0; \qquad A^a_i = A^a_i(t)$ and $A^a_{ij} = 0$, the equation of motion would bear the form [2]:

$$\ddot{A}^a_i - g^2 A^a_j A^b_j A^b_i + g^2 A^a_i A^b_i A^b_j = 0; \quad i, j = 1, 3$$

(3)

We look for a solution of equation (3) in a $3 \times 3$ matrix form [3]:

$$A^a_i = \frac{1}{g} f^{(a)}(t) O^a_i; \quad i = 1, 3; \quad a = 1, 2, 3$$

(4)

If we should ask for the matrix $O^a_i$ to be an orthogonal one, $O^a_i O^b_i = \delta^{ab}$, we would get a system of equation for the color factors:

$$\ddot{f}^{(a)} + f^{(a)}(f^2 - f^{(a)2}) = 0$$

(5)

Various particular cases of this equation have been studied [2,3] and the important property of owning periodical solutions has been pointed out. The mentioned works treat of the one-dimensional system, where only one color parameter is significant. We will investigate a “non-symmetric” case, with two nonvanishing parameters, namely

$$f^{(3)} \equiv 0; \quad f^{(1)} \equiv x, \quad f^{(2)} \equiv y$$

(6)
System (5) will become a system of two equations:

\[
\begin{align*}
\dot{x} + xy^2 &= 0 \\
\dot{y} + x^2y &= 0
\end{align*}
\]  

(7)

In fact, we could consider a more general system of the form:

\[
\begin{align*}
\dot{x} &= -x(y^2 + c) \\
\dot{y} &= -y(x^2 + c)
\end{align*}
\]  

(8)

where \(c\) is a true constant parameter. An important remark is that system (8) is a Hamiltonian system, with the Hamiltonian:

\[
H = \frac{1}{2}(x^2 + y^2 + cx^2 + cy^2 + x^2y^2)
\]  

(9)

In [3] the case \(c = 0\) was studied. Next we will consider the case \(c = 1\), that is to say:

\[
\begin{align*}
\dot{x} &= -x(y^2 + 1) \\
\dot{y} &= -y(x^2 + 1)
\end{align*}
\]  

(10)

It is apparently a non-significant change, but, in fact, it moves the system into another type of limiting motion (see Fig. 2).

It is very interesting to remark that (10) admits periodical solutions, to be studied in the next section, as well as that, by a “swirl”-type transformation, it could be transformed to look like the Henon-Heiles system. So, starting from a gauge field, with a typical chaotic behavior, we arrive at a dynamical system admitting integrable cases.

3. PERIODICAL SOLUTIONS

Due to its pronounced non-linearity, it is not possible to obtain analytical solutions for system (10). Solving this system will imply the use of numerical methods. We will be interested in a particular class of solutions, that is to say the periodical ones (or, more accurately, quasi-periodical). The use of numerical methods and of the approximating computations might induce certain errors, so that it would be hard to decide if a solution is indeed periodical.

We will define a new notion, the one of “\((\varepsilon, T)\)-periodical solution”. Let us state:

**Definition:** A nonlinear dynamical system admits a \((\varepsilon, T)\)-periodical solution if the evolution of the system follows a curve that after a time \(T\), comes back to the same point, with an approximation \(\varepsilon\).

**Remark:** In a good sense, the \((\varepsilon, T)\)-periodical solutions could be considered as quasi-periodical, or even as periodical. This will be the case in all our next
considerations: we will neglect the computational errors and we will admit that the 
\((\varepsilon, T)\)-periodical solutions are periodical ones.

![Diagram](image)

Fig. 3 – Some periodical solutions for the initial conditions: \(\alpha = 0\) and a) \(y_0 = 0.08\); b) \(y_0 = 0.42\); c) \(y_0 = 1.005\); d) \(y_0 = 1.183881555\); e) \(y_0 = 1.2624\); f) \(y_0 = 1.514923\).
An essential question for the study of the solutions of system (10) is related to their obvious dependence upon the initial conditions. This kind of dependence is specific to the dynamical chaotic systems, among which the abelian gauge field stands. For this reason, the expectation for this case is that the system will follow a nonregular evolution only. Although, it was strange for us to observe the existence of periodical solutions for system (10). Their form depends on the initial conditions and on the value of the energy of the system. The presentation of this class of solutions represents the essential result of this paragraph. In [6] such solutions for a quite different system, generated by the Yang-Mills field too, and at large energies were studied.

Our effective studies were done by considering the following set of initial conditions: 

\[ x_0 = 0; \quad y_0; \quad \dot{x}_0 = \sqrt{2E \cos \alpha}; \quad \dot{y}_0 = \sqrt{2E \sin \alpha}, \]

where \( y_0 \) and \( \alpha \) are considered as parameters and \( E \) represents the energy of the system. In Fig. 3 we considered \( y_0 = 0 \) and various values for \( \alpha \). Fig. 3 corresponds to the case \( \alpha = 0 \) and \( y_0 \) variable. In both cases we choose the energy \( E = 2 \). The dashed lines represent the equipotential curves.

4. TOWARDS THE HENON HEILES SYSTEM

The Henon-Heiles system is a well known model admitting many cases of integrability. The most general Hamiltonian describing this dynamical system is:

\[
H = \frac{1}{2} \dot{x}_1^2 + \frac{1}{2} \dot{y}_1^2 + ax_1^2 + by_1^2 + c x_1^2 y_1 + dy_1^3 + f y_1 + g \frac{1}{x_1^2} \tag{11}
\]

where \( a, b, c, d, f, g \) are constant quantities. There are many forms of Henon-Heiles Hamiltonians considered in the literature. For example, in [7] three cases of integrable Henon-Heiles system in two dimensions are presented and the concrete forms of the two constants of motions are written down:

\[
\text{case 1: } a = b = g = f = 0; \quad c = \frac{1}{2}; \quad d = 1
\]
\[
\Rightarrow F = -2x_1 \dot{x}_1 \dot{y}_1 + 2y_1 \dot{x}_1^2 - \frac{1}{4} x_1^4 - x_1^2 y_1^2
\]

\[
\text{case 2: } a = b = f = g = 0; \quad c = \frac{1}{2}; \quad d = 1
\]
\[
\Rightarrow F = -2x_1 \dot{x}_1 \dot{y}_1 + 2y_1 \dot{x}_1^2 - \frac{1}{4} x_1^4 - x_1^2 y_1^2 - f x^2 \tag{12}
\]

\[
\text{case 3: } a = b = f = 0; \quad c = \frac{1}{2}; \quad d = 1
\]
\[
\Rightarrow F = -2x_1 \dot{x}_1 \dot{y}_1 + 2y_1 \dot{x}_1^2 - \frac{1}{4} x_1^4 - x_1^2 y_1^2 + 4g \frac{y}{x^2}
\]
We will consider the nonintegrable case when:

\[ a = b = \frac{1}{2}; \quad c = 1; \quad d = -\frac{1}{3}; \quad f = g = 0 \]  

(13)

In this case, the equations of motion look like:

\[
\begin{align*}
\dot{x}_1 &= -x_1 - 2x_1y_1 \\
\dot{y}_1 &= -y_1 - x_1^2 + y_1^2
\end{align*}
\]  

(14)

This is a Hamiltonian system, with the corresponding Hamiltonian function of the form:

\[
H_1 = \frac{1}{2} \left( x_1^2 + y_1^2 + x_1^2 + y_1^2 + 2x_1^2y_1 - \frac{2}{3}y_1^3 \right)
\]  

(15)

A compatibility condition with the “mechanical model” (10) will be imposed, by identifying the Hamiltonians (9) and (15). We obtain:

\[ y_1 = y = 0 \quad \text{or} \quad x^2 - \frac{1}{2}x^2y - \frac{1}{3}y^2 = x_1^2 - \frac{1}{2}x_1^2y_1 - \frac{1}{3}y_1^2 = 0 \]  

(16)

Let us consider, for the two models, the surfaces of sections corresponding to \( y_1 = y = 0 \). We are interested in the aspect of the phase-space of the two systems in the considered section. A numerical analysis shows some very interesting pictures, attesting a straight connection between the two systems.

In Fig. 4 the projections of the orbits of the Hamiltonian (9), in the surface space \( y = 0 \) are shown for various values of the energy.

The main feature of the system consists in the simultaneous presence of regular trajectories and of regions of stochasticity. For the former the trajectory strikes the surface in some fixed points, called also \textit{periodic points} and determines a closed \textit{invariant curve}. The latter can be identified by the fact that the successive intersections of the trajectory with the considered surface of section do not come twice to the same point and densely cover the surface during long periods of time.

It is interesting to notice that, by increasing the energy, regular orbits disappear and

![Fig. 4 – Evolution of the dynamical system to chaotic behavior, depending on the energy level: a) \( E = 0.1 \); b) \( E = 2 \); c) \( E = 10 \).](image)
chaotic areas are extended. For the case $E = 2$ one can see how the periodic points could generate islands of stability. They are destroyed when the energy increases.

A similar study made for (14) is represented in Fig. 5. The considered energy was $E = 1/12$, closed with $E = 0.1$ from Fig. 4a.

At first sight, the phase portraits from Fig. 4a and Fig. 5 corresponding to pretty similar energies are quite different. Nevertheless, an interesting similarity can be noticed when we apply in this surface a change of coordinates of the form:

\[
\begin{align*}
    x_1 &= \sqrt{x^2 + \dot{x}^2} \cos \left( \arctan(x/x) + \theta \frac{x^2 + \dot{x}^2}{2E} - \pi/6 \right) \\
    \dot{x}_1 &= \sqrt{x^2 + \dot{x}^2} \sin \left( \arctan(x/x) + \theta \frac{x^2 + \dot{x}^2}{2E} - \pi/6 \right)
\end{align*}
\]

where $E$ is the energy of the system, $\theta = -0.492\pi$ is a rotation angle and all the coordinates are scaled as non-dimensional quantities. This transformation is a

Fig. 5 – The Hénon-Heiles surface of section $y = 0$ for the energy $E = 1/12 = 0.084$.

Fig. 6 – “Swirl” transformation (17) applied on the Henon-Heiles system at $E = 1/12$. 
“swirl”-type, like the ones encountered in the turbulent fluids [8]. It is surprising to see that, by applying the transformations (17) for the Henon-Heiles system (14), the phase portrait changes as Fig. 4 shows. A very similar portrait with those of the mechanical Yang-Mills model (10), at the same energy, \( E = 1/12 = 0.1 \) is obtained. This fact suggests the possibility of establishing some similarities between the two considered models. As Henon-Heiles admits the integrable solutions (12), we could hope to recover integrability traces for the initial mechanical Yang-Mills model.

5. CONCLUSIONS

The present work was intended to investigate the connections between the chaotic behavior and the integrability of some constrained dynamical systems. The starting point was the model of the Yang-Mills field transposed into a mechanical version. Two main results arise from our study. First of all, we pointed out the existence of some periodical classes of solutions for the considered mechanical model. A second important remark was the surprising similarities between the mechanical model (10) and the Henon-Heiles system. It seems that, by applying a special type of swirl transformation, the phase portrait of the mechanical model in a section of surface where the Hamiltonians can be identified, falls on the similar portrait of Henon-Heiles. As the Henon-Heiles system admits many integrable solutions, we could expect that our mechanical model should also admit. This might be considered as integrability traces of the gauge fields, in suitable gauge fixing conditions.

REFERENCES

Fig. 1 – The behavior in the motion space for: the $c = 0$ Yang-Mills system (a) and our modified $c = 1$ Yang-Mills system, (b), for the energy value $E = 2$.

Fig. 2 – Some periodical solutions for the initial conditions: $y_0 = 0$ and
a) $\alpha = -\pi/4$; b) $\alpha = -0.702508621$; c) $\alpha = -0.3005$; d) $\alpha = -0.5112$;
e) $\alpha = -1.003$; f) $\alpha = -1.275$. 