Romanian Reports in Physics, Vol. 57, No. 4, P. 553–559, 2005

MATHEMATICAL AND GENERAL PHYSICS

ON A SPECIAL CLASS OF PERIODIC SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION

D. GRECU, ANCA VISINESCU

Department of Theoretical Physics
"Horia Hulubei" National Institute of Physics and Nuclear Engineering
Bucharest
email: dgrecu@theory.nipne.ro

(Received June 15, 2005)

Abstract. A special class of periodic solutions, expressed through Jacobi elliptic functions, is obtained using the method of Akhmediev and Korneev.

Key words: non-linear Schrödinger equation, periodic solutions, Akhmediev and Korneev method.

1. INTRODUCTION

The nonlinear Schrödinger equation (NLS) is one member of the family of completely integrable nonlinear evolution equations. It is a generic equation which appears any time when a quasi-monochromatic wave is propagating in a dispersive and weakly nonlinear medium, and was used to describe a large variety of phenomena in hydrodynamics, nonlinear optics, especially in nonlinear optical fibers, quasi-one-dimensional nonlinear molecular systems, Bose-Einstein condensate, to mention only some of the fields where this equation appears [1–2]. With well initial conditions (decaying fast enough at infinity) the solution is found using the inverse scattering method (IST). A special class of solutions, the solitons, is easily obtained solving the corresponding linear integral equation of the inverse problem (Gelfand-Levitan-Marchenko equation). The problem is more difficult if we are looking for periodic solutions [1, 3, 4], and more modern methods are required (a regular Riemann problem). But periodic solutions can be found using several direct methods, and usually they are in the class of elliptic functions (see [1, 5–8] where more references can be found).

Of special interest for the present work is the very simple method of Akhmediev and Korneev [5], where a special class of periodic solutions of NLS equations is obtained using elementary methods. As there are some obscure points in their reasoning we reconsider the method and find the whole class of these special periodic solutions, larger than the solutions determined in [5].

2. BASIC RELATIONS

Let us consider the NLS equation

$$iu_t + \frac{1}{2}u_{xx} - u + |u|^2 u = 0 \tag{1}$$

and write the equations for the real and imaginary part of u $(u = v + iw)$

$$\begin{aligned} v_t - w + \frac{1}{2}w_{xx} + (v^2 + w^2)w = 0 \\ -w_t - w + \frac{1}{2}v_{xx} + (v^2 + w^2)v = 0 \end{aligned} \tag{2}$$

Here by subscripts t and x we mean partial derivative with respect to t and x respectively. The basic assumption used in [5] is a very simple relation (a linear one) between the real part v of u and its imaginary part w, namely

$$v(x,t) = \alpha(t) + \beta(t)w(x,t) \tag{3}$$

where the coefficients of this linear relation are depending only on the time variable. Introducing (1) in (2) and eliminating the second order derivative w_{xx} we get

$$\left(1+\beta^2\right)w_t + \beta\left(\beta_t - 2\alpha^2\right)w - \alpha\left(1+\beta^2\right)w^2 + \left[\beta\alpha_t + \left(1-\alpha^2\right)\alpha\right] = 0 \tag{4}$$

As α and β are for the moment completely arbitrary functions of t we can impose several constraints to reduce (4) to a simpler form. These are

$$\begin{aligned} \beta\alpha_t &= -\left(1-\alpha^2\right)\alpha \\ \beta_t &= -\left(1+\beta^2 - 2\alpha^2\right) \end{aligned} \tag{5}$$

The first ensures us that the free term in (4) vanishes, while the second makes the coefficient of w in (4) to be proportional to $(1+\beta^2)$. With these constraints eq. (4) becomes

$$w_t - \beta w - \alpha w^2 = 0 \tag{6}$$

Now the solving scheme is the following. First we have to find the general solution of the system (5). Next using this solution in (6), the general dependence of w on the time variable is determined. It contains an integration constant depending on x. Using one of the equations (2) a differential equation for it is found. All these differential equations are very simple and their solutions are easily obtained by elementary integrations.

3. INTEGRATION SCHEME

Eliminating β from the two equations (5) we remain with the following differential equation for α.

$$\alpha_{tt} + \frac{\alpha}{1-\alpha^2}\alpha_t^2 + \alpha\left(1-\alpha^2\right) = 0 \tag{7}$$

As this equation doesn't contain the time variable explicitly, it can be reduced to a first order differential equation. Indeed, writing $\alpha_t = p(\alpha)$ we have $\alpha_{tt} = pp'$ where we denoted $p' = \mathrm{d}p/\mathrm{d}\alpha$. Introducing the new variable $z(\alpha) = p^2$ this satisfies the first order differential equation

$$z' + \frac{2\alpha}{1-\alpha^2}z + 2\alpha\left(1-\alpha^2\right) = 0 \tag{8}$$

It is of the form

$$z' - \frac{f'}{f}z = f\,f' \tag{9}$$

where $f(\alpha) = 1-\alpha^2$. The general solution of (9) is

$$z = e^{-F}\left\{K + \int f\,f' e^F\,\mathrm{d}\alpha\right\}$$

where K is an integration constant, and F

$$F(\alpha) = -\int \frac{f'}{f}\mathrm{d}\alpha = -\ln(1-\alpha^2)$$

Then it is easily shown that the time dependence of α is given by the equation

$$\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2 = \left(1-\alpha^2\right)\left(K+1-\alpha^2\right) \tag{10}$$

Several classes of solutions are obtained, depending on the value of the constant K.

Case 1: $K > 0$

Denoting $K+1 = \frac{1}{k^2}$, $k<1$ and introducing the new time variable $\tau = t/k$, we get

$$\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right)^2 = \left(1-\alpha^2\right)\left(1-k^2\alpha^2\right) \tag{11}$$

which is the equation satisfied by the Jacobi elliptic function $\pm\mathrm{sn}(\tau, k)$. Then

$$\begin{aligned} \alpha &= -\mathrm{sn}(\tau, k) \\ \beta &= -k\frac{\mathrm{sn}\tau\,\mathrm{cn}\,\tau}{\mathrm{dn}\,\tau}, \quad \tau = \frac{1}{k}t \end{aligned} \tag{12}$$

In the limiting case $k \to 1$ we get $\alpha = \beta = -\mathrm{th}\,t$.

Case 2: $K < -1$

Then $K + 1 < 0$ and it is convenient to write $K + 1 = -a^2$. Thus eq. (10) becomes

$$\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2 = \left(\alpha^2 - 1\right)\left(a^2 + \alpha^2\right), \quad \alpha > 1 \tag{13}$$

Straightforward transformations lead us to

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 = \left(1 - z^2\right)\left(1 - k^2 z^2\right)$$

where

$$\begin{aligned} z &= \sqrt{1 + a^2}\,\frac{1}{\sqrt{a^2 + \alpha^2}} = \mathrm{sn}(\tau, k) \\ k &= \frac{a}{\sqrt{1 + a^2}}, \quad \tau = \sqrt{1 + a^2}\,t \end{aligned} \tag{14}$$

Finally we get $\left(k' = \sqrt{1 - k^2}\right)$

$$\alpha = \frac{1}{k'}\frac{\mathrm{dn}(\tau, k)}{\mathrm{sn}(\tau, k)} = \frac{1}{k'}\mathrm{ds}(\tau, k) \tag{15}$$

It is singular in the point $\tau = 0$. The inequality $\alpha \geq 1$ is satisfied.

A second solution is also possible, namely

$$\alpha = \frac{1}{\mathrm{cn}(\tau, k)} \tag{16}$$

with the same definition (14) of k and τ, which is singular for $\tau = K(k)$ ($K(k)$ being the complete elliptical function of first kind).

Case 3: $-1 < K < 0$

Then $0 < K + 1 < 1$. With the notation $K + 1 = k^2$ and introducing the new function $z = \frac{1}{k}\alpha$, this will satisfy the differential equation for Jacobi elliptic function $\mathrm{sn}(t, k)$, so

$$\alpha = k\,\mathrm{sn}(t,k) \tag{17}$$

Further on we shall restrict ourselves only to the case 1. With the change of variable $y = \frac{1}{w}$ and $t = k\tau$, eq. (6) becomes

$$y_\tau + k\beta u + k\alpha = 0 \tag{18}$$

Using the expression (12) for $\alpha(\tau)$ and $\beta(\tau)$ the solution of (18) is

$$w(t,x) = \frac{C(x)\,\mathrm{dn}\,\tau}{1 - k\,\mathrm{cn}\,\tau\, C(x)} \tag{19}$$

where $C(x)$ is an integration constant independent of t but depending on x. The function $C(x)$ satisfies a differential equation obtained introducing (19) (and $v = \alpha + \beta w$) in any of the equations (2). Straightforward calculations lead us to the following equation

$$C_{xx} + \frac{2k\,\mathrm{cn}\,\tau}{1 - k\,\mathrm{cn}\,\tau\, C} C_x^2 + \frac{2}{1 - k\,\mathrm{cn}\,\tau\, C}\left[-\frac{1}{k}\mathrm{cn}\,\tau + C + k\,\mathrm{cn}\,\tau\, C^2 + k'^2 C^3\right] = 0$$

which has to be satisfied for any value of τ. It is easily seen that this condition is equivalent with the pair of equations

$$\begin{aligned} &C_{xx} + 2C + 2k'^2 C^3 = 0 \\ &C_x^2 + k'^2 C^4 + 2C^2 - \frac{1}{k^2} = 0 \end{aligned} \tag{20}$$

In the limit $k \to 0$ this reduces to the result of Akhmediev and Korneev [5]. The two equations (20) are compatible, the second representing a prime integral of the previous (the integration constant equal with $-1/k^2$). We can write the second eq. (20) in the form

$$\left(\frac{\mathrm{d}C}{\mathrm{d}x}\right)^2 = k'^2\left(C_1^2 + C^2\right)\left(C_2^2 - C^2\right) \tag{21}$$

where $-C_1^2$ and C_2^2 are the two roots of the equation

$$k'^2 C^4 + 2C^2 - \frac{1}{k^2} = 0$$

namely

$$\begin{aligned} C_1^2 &= \frac{1}{k'^2}\left(1 + \frac{1}{k}\right) = \frac{1}{k(1-k)} \\ C_2^2 &= \frac{1}{k'^2}\left(\frac{1}{k} - 1\right) = \frac{1}{k(1+k)} \end{aligned} \tag{22}$$

The solution of (21) can be also expressed in term of Jacobi elliptic functions. Writing

$$\begin{aligned} \frac{1+k}{2} &= \kappa'^2 \qquad && \frac{1}{2} \le \kappa'^2 \le 1 \\ \frac{1-k}{2} &= \kappa^2 \qquad && 0 \le \kappa^2 \le \frac{1}{2} \end{aligned} \tag{23}$$

and $C = C_2 \overline{C}$ the equation satisfied by $\overline{C}$ is $(x = \sqrt{k/2}\,\xi)$

$$\left(\frac{\mathrm{d}\overline{C}}{\mathrm{d}\xi}\right)^2 = \left(1-\overline{C}^2\right)\left(\kappa'^2 + \kappa^2\overline{C}^2\right) \tag{24}$$

which is the differential equation satisfied by $\mathrm{cn}(\xi, \kappa)$. Then

$$C(x) = \frac{1}{\sqrt{k(1+k)}}\,\mathrm{cn}(\xi, \kappa) \tag{25}$$

Let us summarize the obtained results. The solution of eq. (1) is given by

$$u(x,t) = \alpha(t) + \left(i + \beta(t)\right) w(x,t)$$

where

$$\begin{cases} \alpha(t) = -\mathrm{sn}(t/k, k) \\ \beta(t) = -k\dfrac{\mathrm{sn}(t/k,k)\,\mathrm{cn}(t/k,k)}{\mathrm{dn}(t/k,k)}, \qquad 0 < k < 1 \end{cases}$$

and

$$w(x,t) = \frac{C(x)\,\mathrm{dn}(t/k,k)}{1 - kC(x)\,\mathrm{cn}(t/k,k)}$$

$$C(x) = \frac{1}{\sqrt{k(1+k)}}\,\mathrm{cn}\left(\sqrt{\frac{2}{k}}\,x, \sqrt{\frac{1-k}{2}}\right)$$

This is the general solution of (1) (in the case 1) and is improving the result obtained in [5]. In a similar way we find the solutions in the other cases 2 and 3.

Acknowledgement. Helpful discussions with Dr. A. S. Cârstea are kindly acknowledged.

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