Romanian Reports in Physics, Vol. 57, No. 4, P. 561–572, 2005

BEL-ROBINSON TENSOR FOR THE BIANCHI TYPE I UNIVERSE

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(Received October 26, 2005)

Abstract. Within the framework of Bianchi type-I (BI) space-time we study the Bel-Robinson tensor and its impact on the evolution of the Universe. We use different definitions of the Bel-Robinson tensor existing in the literature and compare the results.

Key words: Bianchi type I model, super-energy tensors.

PACS: 03.65.Pm and 04.20.Ha

1. INTRODUCTION

The quantum field theory in curved space-time has been a matter of great interest in recent years because of its applications to cosmology and astrophysics. The evidence of the existence of strong gravitational fields in our Universe led to the study of the quantum effects of material fields in external classical gravitational field. Since the appearance of Parker's paper on scalar fields [1] and spin-$\frac{1}{2}$ fields [2], several authors have studied this subject. The present cosmology is based largely on Friedmann's solutions of the Einstein equations, which describe the completely uniform and isotropic universe ("closed" and "open" models, *i.e.*, bounded or unbounded universe). The main feature of these solutions is their nonstationarity. The idea of an expanding Universe, following from this property, is confirmed by the astronomical observations and it is now safe to assume that the isotropic model provides, in its general features, an adequate description of the present state of the Universe. Although the Universe seems homogeneous and isotropic at present, it does not necessarily mean that it is also suitable for a description of the early stages of the development of the Universe and there are no observational data guaranteeing the isotropy in the era prior to the recombination. In fact, there are theoretical arguments that support the existence of an anisotropic phase that approaches an isotropic one [3]. Interest in studying Klein-Gordon and Dirac equations in anisotropic models has increased since Hu and Parker [4] have shown that the creation of scalar particles in anisotropic backgrounds can dissipate the anisotropy as the Universe expands.

2. THE GRAVITATIONAL FIELD

A Bianchi type-I (BI) universe, being the straightforward generalization of the flat Robertson-Walker (RW) universe, is one of the simplest models of an anisotropic universe that describes a homogeneous and spatially flat universe. Unlike the RW universe, which has the same scale factor for each of the three spatial directions, a BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. It moreover has the agreeable property that near the singularity it behaves like a Kasner universe, even in the presence of matter, and consequently falls within the general analysis of the singularity given by Belinskii *et al.* [5]. Also in a universe filled with matter for $p=\zeta\varepsilon$, $\zeta < 1$, it has been shown that any initial anisotropy in a BI universe quickly dies away and a BI universe eventually evolves into a Friedmann-RW (FRW) universe [6]. Since the present-day universe is surprisingly isotropic, this feature of the BI universe makes it a prime candidate for studying the possible effects of an anisotropy in the early universe on present-day observations. In light of the importance mentioned above, several authors have studied BI universe from different aspects.

A diagonal Bianchi type-I space-time (hereafter BI) is a spatially homogeneous space-time, which admits an Abelian group G_3, acting on spacelike hypersurfaces, generated by the spacelike Killing vectors $\boldsymbol{x}_1 = \partial_1$, $\boldsymbol{x}_2 = \partial_2$, and $\boldsymbol{x}_3 = \partial_3$. In synchronous coordinates, the metric is [7, 8]:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \sum_{i=1}^{3} a_i^2(t)\mathrm{d}x_i^2. \tag{2.1}$$

If the three scale factors are equal (*i.e.*, $a_1 = a_2 = a_3$), Eq. (1) describes an isotropic and spatially flat Friedmann-Robertson-Walker (FRW) universe. The BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. Thus, a Bianchi type-I (BI) universe, being the straightforward generalization of the flat FRW universe, is one of the simplest models of an anisotropic universe that describes a homogeneous and spatially flat universe. When two of the metric functions are equal (*e.g.*, $a_2 = a_3$) the BI space-time is reduced to the important class of plane symmetric space-time (a special class of the locally rotational symmetric space-times [9, 10]), which admits a G_4 group of isometries acting multiply transitively on the spacelike hypersurfaces of homogeneity generated by the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$, and $\boldsymbol{x}_4 = x^2\partial_3 - x^3\partial_2$. The BI has the agreeable property that near the singularity it behaves like a Kasner universe, given by

$$a_1(t) = a_1^0 t^{p_1}, \quad a_2(t) = a_2^0 t^{p_2}, \quad a_3(t) = a_3^0 t^{p_3}, \tag{2.2}$$

with p_j being the parameters of the BI space-time which measure the relative anisotropy between any two asymmetry axes and satisfy the constraints

$$p_1 + p_2 + p_3 = 1, \tag{2.3}$$

$$p_1^2 + p_2^2 + p_3^2 = 1. \tag{2.4}$$

Thus out of three parameters, only one is arbitrary. One particular choice of parametrization is

$$p_1 = \frac{-p}{p^2 + p + 1}, \tag{2.5}$$

$$p_2 = \frac{p(p+1)}{p^2 + p + 1}, \tag{2.6}$$

$$p_3 = \frac{p+1}{p^2 + p + 1}. \tag{2.7}$$

The condition $0 \le p \le 1$ on p then yields the condition $-\frac{1}{3} \le p_1 \le 0$, $0 \le p_2 \le \frac{2}{3}$, $\frac{2}{3} \le p_3 \le 1$. Another particular parametrization can be given using an angle on the unit circle, since Eqs. (2) describe the intersection of a sphere with a plane in the parameter space (p_1, p_2, p_3):

$$p_1 = \frac{1}{3}(1 + \cos\vartheta + \sqrt{3}\sin\vartheta), \tag{2.8}$$

$$p_2 = \frac{1}{3}(1 + \cos\vartheta - \sqrt{3}\sin\vartheta), \tag{2.9}$$

$$p_3 = \frac{1}{3}(1 - 2\cos\vartheta). \tag{2.10}$$

Although ϑ ranges over the unit circle, the labeling of each p_j is quite arbitrary. Thus the unit circle can be divided into six equal parts, each of which span 60°, and the choice of p_j is unique within each section separately. For $\vartheta = 0$, $p_1 = p_2 = \frac{2}{3}$ and $p_3 = -\frac{1}{3}$ while for $\vartheta = \pi/3$, $p_1 = 1$ and $p_2 = p_3 = 0$.

Further we write the BI metric in the form:

$$ds^2 = dt^2 - a^2 dx^2 - b^2 dy^2 - c^2 dz^2, \tag{2.11}$$

with a, b, c being the functions of time t only. Here the speed of light is taken to be unity.

The metric (11) has the following non-trivial Christoffel symbols

$$\begin{aligned} &\Gamma_{10}^{1} = \frac{\dot{a}}{a}, \quad \Gamma_{20}^{2} = \frac{\dot{b}}{b}, \quad \Gamma_{30}^{3} = \frac{\dot{c}}{c} \\ &\Gamma_{11}^{0} = a\dot{a}, \quad \Gamma_{22}^{0} = b\dot{b}, \quad \Gamma_{33}^{0} = c\dot{c}. \end{aligned} \tag{2.12}$$

The nontrivial components of the Ricci tensors are

$$R_0^0 = -\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right), \tag{2.13a}$$

$$R_1^1 = -\left[\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right)\right], \tag{2.13b}$$

$$R_2^2 = -\left[\frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\left(\frac{\dot{c}}{c} + \frac{\dot{a}}{a}\right)\right], \tag{2.13c}$$

$$R_3^3 = -\left[\frac{\ddot{c}}{c} + \frac{\dot{c}}{c}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)\right]. \tag{2.13d}$$

From (2.13) one finds the following Ricci scalar for the BI universe

$$R = -2\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{\dot{c}}{c}\frac{\dot{a}}{a}\right). \tag{2.14}$$

The non-trivial components of Riemann tensors in this case read

$$\begin{aligned} &R^{01}{}_{01} = -\frac{\ddot{a}}{a}, \quad R^{02}{}_{02} = -\frac{\ddot{b}}{b}, \quad R^{03}{}_{03} = -\frac{\ddot{c}}{c}, \\ &R^{12}{}_{12} = -\frac{\dot{a}}{a}\frac{\dot{b}}{b}, \quad R^{23}{}_{23} = -\frac{\dot{b}}{b}\frac{\dot{c}}{c}, \quad R^{31}{}_{31} = -\frac{\dot{c}}{c}\frac{\dot{a}}{a}. \end{aligned} \tag{2.15}$$

Now having all the non-trivial components of Ricci and Riemann tensors, one can easily write the invariants of gravitational field which we need to study the space-time singularity. We return to this study at the end of this section.

3. EINSTEIN EQUATIONS AND THEIR SOLUTIONS

In this section we study the Einstein equation. In doing so let us first write the Einstein equation for the BI metric governing the evolution of the Universe. In presence of a cosmological constant Λ the Einstein equation has the form

$$\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}=\kappa T_1^1+\Lambda, \tag{3.1a}$$

$$\frac{\ddot{c}}{c}+\frac{\ddot{a}}{a}+\frac{\dot{c}}{c}\frac{\dot{a}}{a}=\kappa T_2^2+\Lambda, \tag{3.1b}$$

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}=\kappa T_3^3+\Lambda, \tag{3.1c}$$

$$\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}+\frac{\dot{c}}{c}\frac{\dot{a}}{a}=\kappa T_0^0+\Lambda. \tag{3.1d}$$

Here over-dot means differentiation with respect to t and T_μ^ν is the energy-momentum tensor of the matter field which we choose in the form:

$$T_\mu^\nu=(\varepsilon+p)u_\mu u^\nu-p\delta_\mu^\nu, \tag{3.2}$$

where u^μ is the flow vector satisfying

$$g_{\mu\nu}u^\mu u^\nu=1. \tag{3.3}$$

Here ε is the total energy density of a perfect fluid and/or dark energy density, while p is the corresponding pressure. p and ε are related by an equation of state which will be studied below in detail. In a co-moving system of coordinates from (3.2) one finds

$$T_0^0=\varepsilon, \qquad T_1^1=T_2^2=T_3^3=-p. \tag{3.4}$$

In view of (25) from (3) one immediately obtains [11]

$$a(t)=D_1\tau^{1/3}\exp\left[X_1\int\frac{\mathrm{d}t}{\tau(t)}\right], \tag{3.5a}$$

$$b(t)=D_2\tau^{1/3}\exp\left[X_2\int\frac{\mathrm{d}t}{\tau(t)}\right], \tag{3.5b}$$

$$c(t)=D_3\tau^{1/3}\exp\left[X_3\int\frac{\mathrm{d}t}{\tau(t)}\right]. \tag{3.5c}$$

Here D_i and X_i are some arbitrary constants obeying

$$D_1 D_2 D_3 = 1, \qquad X_1 + X_2 + X_3 = 0,$$

and τ is a function of t defined to be

$$\tau = abc. \tag{3.6}$$

From (3) for τ one find

$$\frac{\ddot{\tau}}{\tau} = \frac{3\kappa}{2}(\varepsilon - p) + 3\Lambda. \tag{3.7}$$

On the other hand the conservation law for the energy-momentum tensor gives

$$\dot{\varepsilon} = -\frac{\dot{\tau}}{\tau}(\varepsilon + p). \tag{3.8}$$

After a little manipulations from (3.7) and (3.8) we find

$$\dot{\tau}^2 = 3(\kappa\varepsilon + \Lambda)\tau^2 + C_1, \tag{3.9}$$

with C_1 being an arbitrary constant. Let us now, in analogy with Hubble constant, define

$$\frac{\dot{\tau}}{\tau} = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = 3H. \tag{3.10}$$

On account of (3.10) from (3.9) one derives

$$\kappa\varepsilon = 3H^2 - \Lambda - C_1/(3\tau^2). \tag{3.11}$$

It should be noted that the energy density of the Universe is a positive quantity. It is believed that at the early stage of evolution when the volume scale τ was close to zero, the energy density of the Universe was infinitely large. On the other hand with the expansion of the Universe, *i.e.*, with the increase of τ, the energy density ε decreases and an infinitely large τ corresponds to a ε close to zero. Say at some stage of evolution ε is too small to be ignored. In that case from (3.11) follows

$$3H^2 - \Lambda \to 0. \tag{3.12}$$

As is seen from (3.12) in this case Λ is essentially non-negative. We can also conclude from (3.12) that in the absence of a Λ term beginning from some value of τ the evolution of the Universe comes stand-still, *i.e.*, τ becomes constant, since H becomes trivial, whereas in the case of a positive Λ the process of evolution of the Universe never comes to a halt. Moreover it is believed that the presence of the dark energy (which can be explained with a positive Λ as well) results in the accelerated expansion of the Universe. As far as negative Λ is concerned, its presence imposes some restriction on ε, namely, ε can never be small enough to be ignored. In the case of the perfect fluid given by $p = \zeta\varepsilon$ there exists some upper limit for τ as well (note that τ is essentially nonnegative, *i.e.*, bound from below). In our previous papers we came to the same conclusion [11, 12] [with a positive Λ which in the present paper appears to be negative]. A suitable choice of parameters in this case may give rise to an oscillatory mode of expansion, whereas in the case of a Van der Waals fluid the highly nonlinear equation of state may result in an exponential expansion as well.

Inserting (3.10) and (3.11) into (3.7) one now finds

$$\dot{H} = -\frac{1}{2}\left(3H^2 - \Lambda + \frac{C_1}{3\tau^2} + \kappa p\right) = -\frac{\kappa}{2}(\varepsilon + p) - \frac{C_1}{3\tau^2}. \tag{3.13}$$

In view of (3.11) from (3.13) follows that if the perfect fluid is given by a stiff matter where $p = \varepsilon$, the corresponding solution does not depend on the constant C_1.

Let us now go back to the Eq. (3.9). It is in fact the first integral of (3.7) and can be written as

$$\dot{\tau} = \pm\sqrt{C_1 + 3(\kappa\varepsilon + \Lambda)\tau^2} \tag{3.14}$$

On the other hand, rewriting (3.8) in the form

$$\frac{\dot{\varepsilon}}{\varepsilon + p} = \frac{\dot{\tau}}{\tau}, \tag{38}$$

and taking into account that p is a function of ε, one concludes that the right-hand side of the Eq. (3.7) is a function of τ only, *i.e.*,

$$\ddot{\tau} = \frac{3\kappa}{2}(\varepsilon - p)\tau + 3\Lambda\tau = \mathscr{F}(\tau). \tag{39}$$

From a mechanical point of view Eq. (3.16) can be interpreted as an equation of motion of a single particle with unit mass under the force $\mathscr{F}(\tau)$. Then the following first integral exists [12]:

$$\dot{\tau} = \sqrt{2[\mathscr{E} - \mathscr{U}(\tau)]}. \tag{3.17}$$

Here $\mathscr{E}$ can be viewed as energy and $\mathscr{U}(\tau)$ is the potential of the force $\mathscr{F}$. Comparing the Eqs. (3.14) and (3.17) one finds $\mathscr{E} = C_1/2$ and

$$\mathscr{U}(\tau) = -\frac{3}{2}(\kappa\varepsilon + \Lambda)\tau^2. \tag{3.18}$$

Let us finally write the solution to the Eq. (3.7) in quadrature:

$$\int \frac{d\tau}{\sqrt{C_1 + 3(\kappa\varepsilon + \Lambda)\tau^2}} = t + t_0, \tag{3.19}$$

where the integration constant t_0 can be taken to be zero, since it only gives a shift in time. Eqs. (3.7) and (3.8) for perfect fluid obeying different equations of state have been thoroughly studied by us [11, 12].

4. BEL-ROBINSON TENSORS

Bel-Robinson tensor (B) first appeared in the endless search for a covariant version of gravitational energy; the analogy with the Maxwell stress tensor $T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} + *F_{\mu\alpha} * F_{\nu}^{\alpha}$. The Bel-Robinson tensor is defined in two alternative ways [13]

$$B_{\mu\nu\alpha\beta} = R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} R_{\rho\nu\sigma\beta} + *R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} * R_{\rho\nu\sigma\beta}, \tag{4.1}$$

or equivalently

$$B_{\mu\nu\alpha\beta} = R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} R_{\rho\nu\sigma\beta} + R^{\rho}{}_{\mu}{}^{\sigma}{}_{\beta} R_{\rho\nu\sigma\alpha} - \frac{1}{2} g_{\mu\nu} R_{\alpha}{}^{\rho\sigma\tau} R_{\beta\rho\sigma\tau}. \tag{4.2}$$

Here the dual curvature is $*R^{\mu\nu}{}_{\lambda\sigma} \equiv (1/2)\varepsilon^{\mu\nu}{}_{\alpha\beta} R^{\alpha\beta}{}_{\lambda\sigma}$. The Bel-Robinson tensor has the following symmetry properties:

$$B_{\mu\nu\alpha\beta} = B_{\nu\mu\alpha\beta}, \tag{4.3a}$$

$$B_{\mu\nu\alpha\beta} = B_{\mu\nu\beta\alpha}, \tag{4.3b}$$

$$B_{\mu\nu\alpha\beta} = B_{\alpha\beta\mu\nu}. \tag{4.3c}$$

The symmetry property leads to the fact that that in n-dimensional case there are $n(n+1)[n(n+1)+2]/8$ independent components of the Bel-Robinson tensor. In case of $n = 4$ out of 256 components only 55 are linearly independent. The properties (4.3a) and (4.3b) follow immediately from (4.1) thanks to the symmetry property of Riemann tensor. The property (4.3c) is strait forward from (4.1), but for (4.2) it requires

$$g_{\mu\nu} R_{\alpha}{}^{\rho\sigma\tau} R_{\beta\rho\sigma\tau} = g_{\alpha\beta} R_{\mu}{}^{\rho\sigma\tau} R_{\nu\rho\sigma\tau}. \tag{4.4}$$

Inserting (2.15) into (4.4) we obtain following additional relations:

$$\left(\frac{\ddot{b}}{b}\right)^2 + \left(\frac{\ddot{c}}{c}\right)^2 = \left(\frac{\dot{a}}{a}\right)^2 \left[\left(\frac{\dot{b}}{b}\right)^2 + \left(\frac{\dot{c}}{c}\right)^2\right], \tag{4.5a}$$

$$\left(\frac{\ddot{c}}{c}\right)^2 + \left(\frac{\ddot{a}}{a}\right)^2 = \left(\frac{\dot{b}}{b}\right)^2 \left[\left(\frac{\dot{c}}{c}\right)^2 + \left(\frac{\dot{a}}{a}\right)^2\right], \tag{4.5b}$$

$$\left(\frac{\ddot{a}}{a}\right)^2 + \left(\frac{\ddot{b}}{b}\right)^2 = \left(\frac{\dot{c}}{c}\right)^2 \left[\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{b}}{b}\right)^2\right], \tag{4.5c}$$

$$\left(\frac{\ddot{a}}{a}\right)^2 - \left(\frac{\ddot{b}}{b}\right)^2 = -\left(\frac{\dot{c}}{c}\right)^2 \left[\left(\frac{\dot{a}}{a}\right)^2 - \left(\frac{\dot{b}}{b}\right)^2\right], \tag{4.5d}$$

$$\left(\frac{\ddot{c}}{c}\right)^2 - \left(\frac{\ddot{a}}{a}\right)^2 = -\left(\frac{\dot{b}}{b}\right)^2 \left[\left(\frac{\dot{c}}{c}\right)^2 - \left(\frac{\dot{a}}{a}\right)^2\right], \tag{4.5e}$$

$$\left(\frac{\ddot{b}}{b}\right)^2 - \left(\frac{\ddot{c}}{c}\right)^2 = -\left(\frac{\dot{a}}{a}\right)^2 \left[\left(\frac{\dot{b}}{b}\right)^2 - \left(\frac{\dot{c}}{c}\right)^2\right]. \tag{4.5f}$$

As one can easily find, among the six constrains in (4.5) only three are linearly independent. After a little manipulations with them finally obtains the following relations between the metric functions:

$$\frac{\ddot{a}}{a} = \frac{\dot{b}}{b}\frac{\dot{c}}{c}, \quad \frac{\ddot{b}}{b} = \frac{\dot{c}}{c}\frac{\dot{a}}{a}, \quad \frac{\ddot{c}}{c} = \frac{\dot{a}}{a}\frac{\dot{b}}{b}. \tag{4.6}$$

As one sees, in account of (4.6) the Einstein equation (3.1) leads to $T_0^0 = T_1^1 = T_2^2 = T_3^3$, which can be realized only when the source field satisfies the following equation of state:

$$p = -\varepsilon. \tag{4.7}$$

It is well known that only vacuum satisfies the state of equation given by (4.7). Thus we see that if we are to define Bel-Robinson tensor given by (4.1) or (4.2) we should deal with the Einstein equations with the source field given by a vacuum.

It is due to the fact that in defining the Bel-Robinson tensor we used the dual term with the duality operator acting on the left pair only. To avoid this restrictions the Be-Robinson tensor can be defined by [14]

$$2B_{\mu\nu\alpha\beta} = R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha}R_{\rho\nu\sigma\beta} + {*R}^{\rho\ \ \sigma}_{\ \mu\ \ \alpha}\,{*R}_{\rho\nu\sigma\beta} + R{*}^{\rho\ \ \sigma}_{\ \mu\ \ \alpha}\,R{*}_{\rho\nu\sigma\beta} + {*R*}^{\rho\ \ \sigma}_{\ \mu\ \ \alpha}\,{*R*}_{\rho\nu\sigma\beta}, \tag{4.8}$$

where the duality operator acts on the left or on the right pair of indices according to its position. From (4.8) one easily finds

$$\begin{aligned} B_{\mu\nu\alpha\beta} = {} & R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha}R_{\rho\nu\sigma\beta} + R^{\rho\ \ \sigma}_{\ \mu\ \ \beta}R_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}R_{\alpha}^{\ \rho\sigma\tau}R_{\beta\rho\sigma\tau} - \\ & - \frac{1}{2}g_{\alpha\beta}R_{\mu}^{\ \rho\sigma\tau}R_{\nu\rho\sigma\tau} + \frac{1}{8}g_{\mu\nu}g_{\alpha\beta}R^{\rho\sigma\tau\eta}R_{\rho\sigma\tau\eta}. \end{aligned} \tag{4.9}$$

Under the new definition the symmetry properties (4.3a), (4.3b) and (4.3c) follow immediately, without any restriction to the metric functions.

Let us now write the non-trivial components of the Bel-Robinson tensor for the BI metric. In view of (2.15) we now find

$$B_{0000}=\frac{1}{2}\left\{\frac{\ddot{a}^2}{a^2}+\frac{\ddot{b}^2}{b^2}+\frac{\ddot{c}^2}{c^2}+\frac{\dot{a}^2}{a^2}\frac{\dot{b}^2}{b^2}+\frac{\dot{b}^2}{b^2}\frac{\dot{c}^2}{c^2}+\frac{\dot{c}^2}{c^2}\frac{\dot{a}^2}{a^2}\right\}, \tag{4.10a}$$

$$B_{1111}=\frac{a^2}{2}\left\{\ddot{a}^2+\dot{a}^2\left(\frac{\dot{b}^2}{b^2}+\frac{\dot{c}^2}{c^2}\right)+a^2\left(\frac{\ddot{b}^2}{b^2}+\frac{\ddot{c}^2}{c^2}+\frac{\dot{b}^2}{b^2}\frac{\dot{c}^2}{c^2}\right)\right\}, \tag{4.10b}$$

$$B_{2222}=\frac{b^2}{2}\left\{\ddot{b}^2+\dot{b}^2\left(\frac{\dot{c}^2}{c^2}+\frac{\dot{a}^2}{a^2}\right)+b^2\left(\frac{\ddot{c}^2}{c^2}+\frac{\ddot{a}^2}{a^2}+\frac{\dot{c}^2}{c^2}\frac{\dot{a}^2}{a^2}\right)\right\}, \tag{4.10c}$$

$$B_{3333}=\frac{c^2}{2}\left\{\ddot{c}^2+\dot{c}^2\left(\frac{\dot{a}^2}{a^2}+\frac{\dot{b}^2}{b^2}\right)+c^2\left(\frac{\ddot{a}^2}{a^2}+\frac{\ddot{b}^2}{b^2}+\frac{\dot{a}^2}{a^2}\frac{\dot{b}^2}{b^2}\right)\right\}, \tag{4.10d}$$

$$B_{0101}=-a\dot{a}\left\{\frac{\dot{b}}{b}\frac{\ddot{b}}{b}+\frac{\dot{c}}{c}\frac{\ddot{c}}{c}\right\}, \tag{4.10e}$$

$$B_{0202}=-b\dot{b}\left\{\frac{\dot{c}}{c}\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\ddot{a}}{a}\right\}, \tag{4.10f}$$

$$B_{0303}=-c\dot{c}\left\{\frac{\dot{a}}{a}\frac{\ddot{a}}{a}+\frac{\dot{b}}{b}\frac{\ddot{b}}{b}\right\}, \tag{4.10g}$$

$$B_{1212}=ab\left\{\ddot{a}\ddot{b}+\dot{a}\dot{b}\frac{\dot{c}^2}{c^2}\right\}, \tag{4.10h}$$

$$B_{2323}=bc\left\{\ddot{b}\ddot{c}+\dot{b}\dot{c}\frac{\dot{a}^2}{a^2}\right\}, \tag{4.10i}$$

$$B_{3131}=ca\left\{\ddot{c}\ddot{a}+\dot{c}\dot{a}\frac{\dot{b}^2}{b^2}\right\}, \tag{4.10j}$$

$$B_{0011}=\frac{1}{2}\left\{-\ddot{a}^2+\dot{a}^2\left(\frac{\dot{b}^2}{b^2}+\frac{\dot{c}^2}{c^2}\right)+a^2\left(\frac{\ddot{b}^2}{b^2}+\frac{\ddot{c}^2}{c^2}-\frac{\dot{b}^2}{b^2}\frac{\ddot{c}^2}{c^2}\right)\right\}, \tag{4.10k}$$

$$B_{0022}=\frac{1}{2}\left\{-\ddot{b}^2+\dot{b}^2\left(\frac{\dot{c}^2}{c^2}+\frac{\dot{a}^2}{a^2}\right)+b^2\left(\frac{\ddot{c}^2}{c^2}+\frac{\ddot{a}^2}{a^2}-\frac{\dot{c}^2}{c^2}\frac{\ddot{a}^2}{a^2}\right)\right\}, \tag{4.10l}$$

$$B_{0033}=\frac{1}{2}\left\{-\ddot{c}^2+\dot{c}^2\left(\frac{\dot{a}^2}{a^2}+\frac{\dot{b}^2}{b^2}\right)+c^2\left(\frac{\ddot{a}^2}{a^2}+\frac{\ddot{b}^2}{b^2}-\frac{\dot{a}^2}{a^2}\frac{\ddot{b}^2}{b^2}\right)\right\}, \tag{4.10m}$$

$$B_{1122} = \frac{1}{2}\left\{\dot{a}^2\dot{b}^2 - \ddot{a}^2 b^2 - a^2\ddot{b}^2 - \frac{\dot{c}^2}{c^2}(\dot{a}^2 b^2 + a^2\dot{b}^2) + a^2 b^2 \frac{\dot{c}^2}{c^2}\right\}, \quad (4.10n)$$

$$B_{3311} = \frac{1}{2}\left\{\dot{c}^2\dot{a}^2 - \ddot{c}^2 a^2 - c^2\ddot{a}^2 - \frac{\dot{b}^2}{b^2}(\dot{c}^2 a^2 + c^2\dot{a}^2) + c^2 a^2 \frac{\dot{b}^2}{b^2}\right\}, \quad (4.10o)$$

$$B_{2233} = \frac{1}{2}\left\{\dot{b}^2\dot{c}^2 - \ddot{b}^2 c^2 - b^2\ddot{c}^2 - \frac{\dot{a}^2}{a^2}\left(\dot{b}^2 c^2 + b^2\dot{c}^2\right) + b^2 c^2 \frac{\dot{a}^2}{a^2}\right\}, \quad (4.10p)$$

Thus we obtained the non-trivial components of the Bel-Robinson tensor for the anisotropic BI metric.

5. CONCLUSION

In view of the importance of the BI model in the study of the present day Universe we considered the most simple model with a perfect fluid as a source field. The corresponding solutions to the Einstein equations have been obtained. Two alternative definitions of Bel-Robinson tensor are considered. It is shown that one of the definitions imposes some restriction on the metric functions. In particular this definition is consistent with the Einstein equations when the source field is given by a vacuum only.

Finally we mention that it is desirable to investigate the so called “dominant super-energy property” for the Bel-Robinson tensor as a generalization of the usual dominant energy condition for energy momentum tensors. In general it is considered that the energy condition rules out exotic phenomena like closed timelike curves, superluminal signals, etc. The investigation of the energy condition for the model discussed in this paper will be the subject of forthcoming work [15].

Acknowledgments. This work was supported from the Grant No 532/15.10.2004 of the Plenipotentiary Representative of Romania to JINR, Dubna, Russia.

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