Romanian Reports in Physics, Vol. 57, No. 4, P. 649–657, 2005

STABILITY CRITERION FOR FULLY SYNCHRONIZED STATE OF PERIODIC NETWORKS

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(Received July 19, 2205)

Abstract. We derived a stability criterion for the totally synchronized state of a periodic network of coupled functional units. The functional units were nonlinear discrete dynamical systems. We assumed exponentially decaying coupling strength to reduce the number of control parameters.

Key words: synchronization, coupled oscillators, circulant matrix.

1. INTRODUCTION

From the early days of neuroanatomical investigations [6], the nervous system has been viewed as a network of interconnected nonlinear elements (neurons). The synchronization is the main mechanism for neural information processing [16, 20] and conceptual binding stimuli [8, 13]. On the other hand, the synchronization is the key mechanism involved in dynamical diseases [2, 10, 19]. The word synchronization means "to share the common time" [3]. Understanding the synchronization mechanism in large networks of coupled nonlinear functional units is of paramount importance for the physiology of brain functioning in both normal and pathological conditions. Conceptual approaches on neural synchronization use both small network models to extrapolate the results to large networks [4, 5, 9, 11, 18], and direct experiments on relevant biological networks [1, 14].

We present a general approach on the study of the completely synchronized state of a population of coupled functional units based on the circulant matrices method [7, 12]. Our method applies to periodic structures of coupled nonlinear functional units *i.e.* one-dimensional rings, two-dimensional torus, etc. We assumed that the state, or activity, of an isolated functional unit (not coupled with its neighbors) is controlled by at least one global parameter. The effect of the coupling between the current (i-th) functional unit and its neighbors is to modulate the global control parameter(s) of the current unit. If the coupling is homogeneous, meaning that the coupling coefficients are independent of the functional units position, then the stability of the synchronous solution reduces to the computation

of the eigenvalues of an associated circulant matrix [7, 12]. The method which we developed could also be applied to study partial synchronization, or clustering, in a network of coupled functional units. We used the iterated logistic map introduced by May [17] as a concrete example of nonlinear functional unit.

2. METHOD

We worked through all the mathematical details of the stability analysis for the one-dimensional chain of N coupled functional units with periodic boundary conditions (the ring network structure), and briefly outlined our approach for higher dimensional lattice spaces. Let $x_i[n]$ be the value of the state variable, or activity, associated with the i-th functional unit at the time step n. The general form of the coupled lattice map we considered in this study is

$$x_i[n+1] = g(X_i[n])f(x_i[n]), \tag{1}$$

where $g()$ is the coupling function and determines how the state of the adjacent sites influences the state of i-th functional unit through local couplings. $X_i[n]$ is the average, or mean field, activity sensed by the i-th functional unit when coupled with its neighbors. $f()$ describes the intrinsic dynamics of the isolated (no coupling, or $g() = 1$) functional unit. We defined the mean value of the activity over the neighborhood of size $2N_i$ centered at i-th site by

$$X_i[n] = \sum_{\substack{j=-N_i, \\ j\neq 0}}^{N_i} c_j x_{i+j}[n], \tag{2}$$

where N_i is called the coupling range, and $c_j \in [0, 1]$ are the coupling weights that measure the strength of the coupling between the i-th and the $(i + j)$-th functional units. Without loosing the generality of the proposed approach, we assumed that the coupling coefficients satisfy the normalization condition $\sum_{\substack{j=-N_i, \\ j\neq 0}}^{N_i} c_j = 1$.

Let us assume that a state of complete synchronization between the functional units exists, meaning that $x_i[n] = x[n] = x^*$ for all $i \in \{1, \ldots, N\}$. The completely synchronized state is the fix point of the recursion (1) namely, $x^* = g(X^*)f(x^*) = g(x^*)f(x^*)$, where $X^* = x^*$ based on (2) and the normalization condition. For small perturbations, $\delta x_i[n]$, from the synchronous solution x^*, the state of the i-th functional unit is $x_i[n] = x^* + \delta x_i[n]$. By substituting the previous solution into (1) and linearizing around the fully synchronized state x^* we get the equation of the perturbations:

$$\delta x_i[n+1] = g'(X^*)f(x^*)\delta X_i[n] + g(X^*)f'(x^*)\delta x_i[n] =$$

$$= g'(x^*)f(x^*)\delta X_i[n] + g(x^*)f'(x^*)\delta x_i[n],$$

where prime denotes the first derivative with respect to the corresponding variable. Based on (2), we get $\delta X_i[n] = \sum_{\substack{j=-N_i,\\ j\neq 0}}^{N_i} c_j \delta x_{i+j}[n]$, which reduces the above equation for the perturbations to:

$$\delta x_i[n+1] = \sum_{\substack{j=-N_i,\\ j\neq 0}}^{N_i} \alpha_j \delta x_{i+j}[n] \tag{3}$$

with $\alpha_j = g(x^*)f'(x^*)$, and $\alpha_j = c_j'(x^*)f(x^*)$ for all $j \in \{-N_i, \ldots, N_i\} - \{0\}$. If the local coupling defined by the coefficients c_j is homogeneous, meaning that c_js do not depend on the specific lattice site, then the coefficients (α) of the perturbation equation (3) form a circulant matrix defined, or generated, by the raw vector

$$C = (\alpha_0\ \alpha_1\ \alpha_2 \ldots \alpha_{Ni}\ 0 \ldots 0\ \alpha_{-Ni}\ \alpha_{-Ni+1} \ldots \alpha_{-1}).$$

The circulant matrix (α) in (3) is the square matrix whose lines were obtained by cyclically shifting the elements of the above raw vector C [7, 12]. The characteristic polynomial, $q(t)$, of any circulant matrix is determined by the elements its generating raw vector as follows

$$q(t) = \alpha_0 + \sum_{m=1}^{N_i} \alpha_m t^m + \sum_{m=-N_i}^{N_i} \alpha_m t^{N+m}. \tag{4}$$

The eigenvalues of the circulant matrix (α) are given by $q(\omega)$, where ω is the N-th root of the unity, meaning $\omega = \exp(2\pi i/N)$, where $i = \sqrt{-1}$. Therefore, the j-th eigenvalue $\lambda(j) = q(\omega^j)$ is

$$\lambda(j) = \alpha_0 + \sum_{m=1}^{N_i} \left(\alpha_m \omega^{j\cdot m} + \alpha_{-m}\omega^{-j\cdot m}\right), \tag{5}$$

where we used the fact that $\omega^N = 1$.

3. RESULTS

The totally synchronized mode x^* is linearly stable if all the eigenvalues given by (5) are inside the unit circle, meaning that $|\lambda(j)| < 1$ for all $j \in \{1, \ldots, N\}$.

If the coupling between the functional units is symmetric ($\alpha_j = \alpha_{-j}$) then all the eigenvalues (5) are real. By using the symmetric coupling and the previous expressions for the coefficients α, we get

$$\begin{aligned}\lambda(j) &= \alpha_0 + 2\sum_{m=1}^{N_i} \alpha_m \cos(2\pi jm/N) = \\ &= g(x^*)f'(x^*) + 2g'(x^*)f(x^*)\sum_{m=1}^{N_i} c_m \cos(2\pi jm/N)\end{aligned} \tag{6}$$

The stability of the totally synchronize mode with asymmetric (symmetric) coupling determined by the eigenvalues (5) depends on at least $2N_i$ (N_i) coupling parameters α_j. Besides the above mentioned control parameters, determined by the strength of the local interactions between the adjacent functional units, there is at least one more global parameter that controls the dynamics of the isolated functional unit. To extract meaningful information about the stability of the synchronized state we drastically reduced the number of the control parameters. We studied a restricted but significant class of the local couplings described by only one parameter: the exponentially scaled coupling defined by $c_{j+1}/c_j = k \in$ $\in [0, 1]$. By changing the value of k, the relative weight of the local coupling changes from first order (nearest neighbors) for $k = 0$ to long range coupling over the entire neighborhood of $2N_i + 1$ adjacent sites for $k = 1$.

Based on the normalization condition, the symmetric coupling coefficients are $c_j = c_{-j} = \frac{k^{j-1}}{2}\left(\sum_{m=1}^{N_i} k^{m-1}\right)^{-1}$, for all $j \in \{1, \ldots, N\}$. By substituting the above coupling coefficients into (6) we get

$$\lambda(j) = g(x^*)f'(x^*) + g'(x^*)f(x^*)H(j), \tag{7}$$

where the harmonic factor, $H(j)$, of the eigenvalue $\lambda(j)$ depends only on the local coupling and is given by

$$H(j) = \sum_{m=1}^{N_i} k^{m-1} \cos(2\pi jm/N)\left(\sum_{m=1}^{N_i} k^{m-1}\right)^{-1}. \tag{8}$$

The following properties of the harmonic factor $H(j)$ proved useful in establishing the upper and lower limits of the symmetric eigenvalues (7). First, based on (8) and the fact that $\cos(2\pi m(N/2 - j)/N) = \cos(2\pi m(N/2 + j)/N)$, it results that $\lambda(N/2 - j) = \lambda(N/2 + j)$. This symmetry of $H(j)$ with respect to $N/2$ means that we only need to evaluate half of the eigenvalues (for $j \in \{0, 1, \ldots, N/2\}$) in order to

determine the stability of the synchronous mode. Second, the upper bond of the eigenvalues over all indices j is determined by the maximum value of the harmonic factor $H(j)$, which is $\max\{H(j)\}_j = H(0) = 1$. Indeed, without restricting the generality of the solution, we may assume that the intrinsic dynamics of an isolated functional unit is positively defined, $f(x) \geq 0$ for all x, and that $g'(x) \geq 0$, meaning that the coupling function $g(x)$ increases by increasing the local average activity x. As a result, the coefficient $g'(x^*)f(x^*)$ of $H(j)$ in (7) is always positive, and the maximum eigenvalue is $\max\{\lambda(j)\}_j = g(x^*)f'(x^*) + g'(x^*)f(x^*) \max\{H(j)\}_j = g(x^*)f'(x^*) + g'(x^*)f(x^*) = \lambda(0)$. Third, it can be immediately checked that the harmonic factor of the eigenvalues is always bounded $H(j) \in [-1, 1]$.

Since the harmonic factor of the eigenvalues $H(j)$ plays a significant role in the stability of the synchronous mode, we also determined the lower bounds of $H(j)$. For large N, the ratio j/N could be considered a continuous variable and, therefore, the extremum values of $H(j)$ are determined by the zeroes of the first derivative of the harmonic factor $\partial H(j)/\partial(j/N) = 0$. Based on (8), the values j^* that determine the extremum value(s) of $H(j)$ are solution(s) of the equation

$$\sum_{m=1}^{N_i} mk^{m-1} \sin(2\pi jm/N) = 0. \tag{9}$$

For the first order, or nearest neighbors, coupling ($N_i = 1$) the extremum condition (9) gives $j_1^* = 0$ (the extremum is the global maximum), and $j_2^* = N/2$ (the extremum is the global minimum). For the second order coupling ($N_i = 2$), besides the above mentioned solutions of (9), we get $j_3^* = N/(2\pi)\arccos(-1/(4k))$, which exists only for $k > 1/4$ (Fig. 1A). The harmonic factor has the minimum value $H(j_3^*) = -(1 + 8k^2)(8k(1 + k))$ (Fig. 1B). For $k \leq 1/4$, the minimum value of the harmonic factor is $H(j_2^*) = -(1 - k)/(1 + k)$ (Fig. 1B). For the third order coupling ($N_i = 3$), besides $j_1^* = 0$ (global maximum) and $j_2^* = N/2$ (local minimum), the additional solution of (9) is $j_4^* = N/(2\pi)\arccos((-1 + \sqrt{(9k^2 - 2)})/(6k))$ for all $k > \sqrt{2/3}$ (Fig. 1A). In this case, for $k \leq \sqrt{2/3}$, the minimum value of the harmonic factor is $H(j_2^* = N/2) = -(k^2 - k + 1)/(k^2 + k + 1)$ (see Fig. 1B). Although in principle it is possible to get analytic expressions for the minimum values of the harmonic factor $H(j)$ for every coupling order N_i and coupling strength k, we also numerically determined the above values (Fig. 1). The index value j^* determines which eigenvalue $\lambda(j)$ is the lowest bound over $j \in \{0, 1, 2, \ldots, N/2\}$ (Fig. 1A). The maximum value of $H(j^*)$ (Fig. 1B) determines the stability domain of the synchronized solution (see next section). Numerically (Fig. 1C), we found that the maximum value of $H(j^*)$ is negative (within the limits of the numerical errors) and

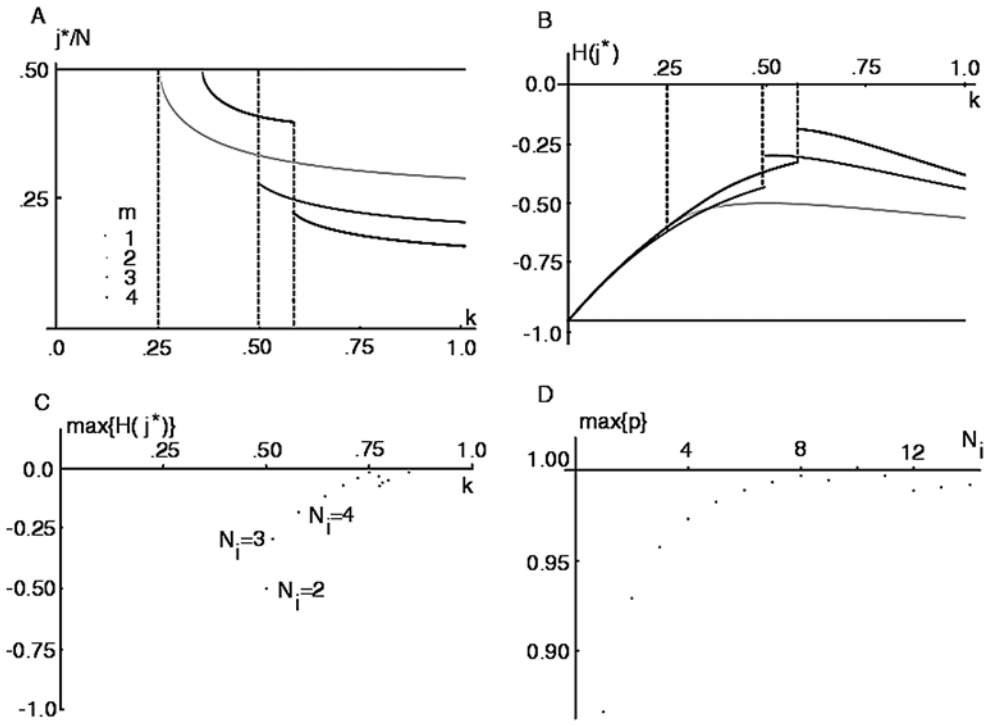

Fig. 1 – Minimum values of the harmonic factor $H(j)$ A. By increasing the coupling strength k the minimum of the harmonic factor $H(j)$ occurs at different values of j = j^*. For the second order coupling ($N_i = 2$) the bifurcation in the j^* versus k space occurs for $k = 1/4$, for $N_i = 3$ the bifurcation occurs at $k = \sqrt{2/3}$, etc. B. For coupling orders higher than two ($N_i > 2$) the function $H(j^*)$ is discontinuous at the bifurcation point(s). C. The maximum value of $H(j^*)$ for successive values of the coupling range N_i increases from –0.5 for $N_i = 2$ to zero for $N_i > 7$. D. The lowest bound for the global control parameter p is determined by the smallest eigenvalue for each coupling order N_i.

monotonically increases from –0.5 for the second order interactions to zero for long range interaction (large N_i).

Although the above derivation of the eigenvalues was done for the one-dimensional periodic structure of coupled functional units, the method of the circulant matrix could be easily extended to higher dimensional periodic structures. For example, in a two-dimensional rectangular lattice, the vertices of the periodic structure (torus) are identified by two indices *i, j*, which could be combined into one $k = i + N_x j$ where $i \in \{1, \ldots, N_x\}$ is the number of vertices along the x axis and $j \in \{1, \ldots, N_y\}$ is the number of vertices along the y axis. The stability results are still valid, with the only difference that the circulant matrix has now $N_xN_y \times N_xN_y$ elements for two-dimensional periodic networks instead of N for the one-dimensional case. The generalization to higher dimensional spaces is straightforward.

As a concrete example, we used the circulant matrix method outlined in the previous section to determine the parameter domain for the complete synchronization of the functional units described by a logistic type equation. The intrinsic dynamics of an isolated functional unit is given by $f(x) = px(1 - x)$, where $p > 0$ is the global control parameter. Following Lopez-Ruiz's results [3, 15], we assumed that the coupling modulates the global control parameter p of each individual functional unit (neuron) by a linear coupling function $g(x) = 3x + 1$. The steady solutions of (1) with the aforementioned logistic nonlinearity and coupling function are: $x_0^* = 0$ and $x_\pm^* = (1 \pm \sqrt{(4 - 3/p)})/3$. The silent state $x_{-1}^* = 0$ of the network is stable for all $0 < p < 1$, regardless of the range N_i and the strength k of the coupling between the functional units. The steady solution $x_+ > 0$ emerges at $p = 3/4$ and is stable for all $p < (15 - \sqrt{33})/8 \approx 1.157$. The steady solution x_- is unstable. According to previous studies [15], the network develops bistability for some values of the global parameter p and switches between silent and active states.

Here we use our newly proposed analytical method to derive the stability conditions for the steady synchronous state x_+^* and to study the influence of the coupling range N_i and the coupling strength k on the stability of the synchronous state. The maximum eigenvalue for the steady solution x_+^* is $\lambda(j^* = 0)$, which for the logistic dynamics and the particular coupling function becomes $\lambda(j^* = 0) = 3 - 2p(4 + \sqrt{(4 - 3/p)})/3$. The largest eigenvalue lies inside the unit circle $\left|\max\{\lambda(j)\}_j\right| = \left|\lambda(0)\right| < 1)$ for all $p \in (3/4, (15 - \sqrt{33})/8)$ and determines the maximum possible range of the control parameter p that ensures the stability of the synchronous solution x_+^*. However, the upper limit of p that ensures the stability of the synchronous state is the solution of the equation $\min\{\lambda_j\}_j = -1$, which means $2 - (2 + \sqrt{(4 - 3/p)})p + \min\{H(j)\}_j(1 + (-2 + \sqrt{(4 - 3/p)})p/3) = -1$. The upper boundary of p is a monotonically increasing function of $\min\{H(j)\}_j = H(j^*)$, which means that for each coupling range N_i and coupling strength k we could use the results regarding the maximum attained by $\min\{H(j)\}_j$ established in the previous section (Fig. 1C) to determine the upper boundaries of p (Fig. 1D). Our results are in very good agreement with those obtained in [15].

5. CONCLUSIONS

To our knowledge, this is the only attempt to give an analytic criterion for the stability of the completely synchronous mode in periodic networks of coupled functional units based on circulant matrices.

We explicitly computed the eigenvalues associated to the fully synchronized state of a generic (one-dimensional) periodic network, and outlined the extension to higher dimensional lattice spaces. We separated the contribution of the intrinsic nonlinearity of the functional units on the stability of the fully synchronized state from the contribution of the coupling, which we called harmonic factor. Useful general properties of the harmonic factor for the eigenvalues computation were also derived.

As a concrete example, we applied the circulant matrices analytical method to determine the boundary of the stability domain of the totally synchronous state developed in the periodic structure of coupled logistic functional units.

We found that by increasing the coupling range, the domain of the global control parameter that ensured the stability of the fully synchronized state widened (Fig. 1D).

The method outlined in this study could also be used for non periodic structures. For such cases, the linear stability matrix has Toeplitz form. Our preliminary results show that the bistability between active and silent state is still present in nonperiodic networks. However, the domain of stability of the completely synchronized open-ends network narrowed compared to the periodic network. Our hypothesis is that open-ends network severely truncates the long range coupling that proved to be essential for active mode stability in periodic network.

REFERENCES

1. S. M. Bawin, R. J. Gavalas-Medici, W. R. Adey, Effects of modulated very high frequency fields on specific brain rhythms in cats, Brain Research, (58): 365–384, 1973.
2. J. Belair, L. Glass, U. der Heiden, J. Milton, *Dynamical disease: mathematical analysis of human illness*, American Institute of Physics, New York, 1995.
3. S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, C. S. Zhou, The synchronization of chaotic systems, Physics Reports, (366): 1–101, 2002.
4. C. Borgers, N. Kopell, Synchronization in networks of excitatory and inhibitory neurons with sparse, random connectivity, Neural Computation., (15): 509–538, 2003.
5. S. L. Bressler, R. Coppola, R. Nakamura, Episodic multiregional cortical coherence at multiple frequencies during visual task performance, Nature, (2366): 153–156, 1993.
6. S. Caja, *New Ideas on the Structure of the Nervous System in Man and Vertebrates*, MIT Press, Cambridge, MA, 1990.
7. P. J. Davis, *Circulant Matrices*, John Wiley and Sons, 1979.
8. R. Eckhorn, in "Oscillatory event-related brain dynamics", Oscillatory and non-oscillatory synchronizations in the visual cortex of cat and monkey, page 115–134, editor Pantev C., New York, 1994.
9. R. C. Elson, A. I. Selverston, R. Huerta, N. F. Rulkov, M. I. Rabinovich, H. D. I. Abarbanel, Synchronous behavior of two coupled biological neurons, Physical Review Letters, (81): 5692–5695, 1998.
10. R. Fitzgerald, Phase Synchronization May Reveal Communication Pathways in Brian Activity, Physics Today, (56): 17–19, 1999.

11. D. Hansel, G. Mato, Asynchronous states and the emergence of synchrony in large networks of interacting excitatory and inhibitory neurons, Neural Computation, (15): 1–56, 2003.
12. D. Kalman, J. E. White, Polynomial equations and circulant matrices, American Mathematics Monthly, (108): 821–840, 2001.
13. A. K. Kreiter, W. Singer, Stimulus-dependent synchronization of neuronal responses in the visual cortex of the awake macaque monkey, Journal of Neuroscience, (16): 2381–96, 1996.
14. J. Lian, J. Shuai, D. M. Durand, Control of phase synchronization of neuronal activity in the rat hippocampus, Journal of Neural Engineering, (1): 46–54, 2004.
15. R. Lopez-Ruiz, Y. Moreno, A. F. Pacheco, S. Boccaletti, D.-U. Hwang, Awaking and Sleeping a Complex Network, arXiv: nlin. AO/ 0406053, 2004.
16. V. Makarenko, R. Llinas, Experimentally determined chaotic phase synchronization in a neural system, Proceedings of the National Academy of Science USA, (95): 15747–15752, 1998.
17. R. May, Simple mathematical models with very complicated dynamics, Nature, (261): 459–467, 1976.
18. R. C. O'Reilly, Y. Munakata, *Computational Explorations in Cognitive Neuroscience*, Cambridge, MA, MIT Press, 2000.
19. C. Schafer, M. G Rosenblum, J. Kurths, H.-H. Abel, Heartbeat synchronized with ventilation, (392): 239–240, 1998.
20. W. Singer, C. M. Gray, Visual feature integration and the temporal correlation hypothesis, Annual Review of Neuroscience, (18): 555–586, 1995.