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ELASTIC SCATTERING IN SHORT-RANGE POTENTIALS

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Abstract. Elastic scattering in short-range potentials is formulated in terms of the theory of the potential, and the scattering amplitude is obtained by constructing an approximate representation of the solution of the equation of the potential (Born scattering). The basic features of the scattering amplitude are identified on this representation of the solution. The solution is exact for δ-potentials. The effective-range theory of scattering is obtained (Fermi scattering length), and the resonance scattering is derived, both for high bound states and low virtual levels (Wigner formula). The resonance scatering on "quasi-discrete" levels is derived (Wigner-Breit formula). The quasi-classical solution for high-energy scattering is presented.

Key words: elastic scattering, scattering length, resonance, quasi-classical scattering.

Scattering problem. The typical scattering experiment consists in free particles moving along the z-direction, as described by a plane wave e^{ikz}, where the wavevector k is related through $k=\sqrt{2mE/\hbar^2}$ to the energy E and mass m of the particles; and a potential $V(\mathbf{r})$ which modifies the motion of the particles, in particular scatters them ellastically, as described by the outgoing wave e^{ikr}/r at large distances. The scattering is conveniently treated in the center-of-mass frame, so m is the reduced mass of the colliding particles interacting through the potential $V(\mathbf{r})$. The elastic scattering means that the internal states of the particles do not change during collision. In addition, scattering means also $E > 0$.

Therefore, the wavefunction describing the scattering must have the asymptotic form

$$\psi(\mathbf{r})=e^{ikz}+f\cdot\frac{e^{ikr}}{r}, \tag{1}$$

where f is the scattering amplitude. The number of scattered particles across the surface of area $dS = r^2 d\Omega$ per unit time is $v|f|^2 d\Omega$,where v is the particle velocity and $d\Omega$ is the solid angle. The number of incoming particles per unit area and unit time, *i.e.*, the flow of particles, is v, assuming the velocity does not change, as for elastic scattering. The ratio of these two quantities is the differential cross-section $d\sigma = |f|^2 \, d\Omega$, and σ is the total cross-section. It is an area. The scattering problem is to find out the asymptotic form of the wavefunction ψ for the motion in potential $V(\mathbf{r})$ with energy $E > 0$, in order to get the scattering amplitude f and the cross-section. It is assumed, in this way that we get information about the potential $V(\mathbf{r})$.

Phase shifts. It is generally assumed, following the classical theory of waves, that the incoming wave changes its phase during the scattering process. This phase shift may depend on energy and on the impact parameter of the collision, *i.e.*, the angular momentum, for central potentials, but otherwise it is a constant. The scattering amplitude can be expressed by means of these phase shifts according to well-known formulae. The phase shifts can in general be estimated by solving in fact the Schrödinger equation, which is not always possible. Quasi-classical formulae are often used to this end. Obviously, this picture assumes that there is more or less a well-defined, internal region where the potential acts and a corresponding external region where the potential is practically zero, so that the continuity conditions for the wavefunction at the border of these two regions lead indeed to a constant phase shift in the free wave, one for each partial wave, *i.e.*, depending on the angular momentum and depending also on energy. The scattering amplitude depends then on the scattering angle θ only. The phase shift theory follows the example of the Schrödinger motion in a potential well, *i.e.*, it obviously refers properly to short-range potentials. It is favourable for the phase shift representation to have a potential that vanishes exactly beyond a certain finite range. If not, second-order corrections may appear, which amount to limiting the validity of the well-known representation $\exp(2i\delta)$ to $1+2i\delta$, where δ is a generic phase shift. The consequence of such a limitation is the loss of the imaginary part of the scattering amplitude, which affects the "optical theorem".

Not surprisingly, this phase- shift theory must be employed with caution for two most important potentials we know in classical physics, namely the zero-range δ-type potential and the long-range Coulomb potential. In the former case, associated very closely with nuclear forces for instance, special care must be taken in order to compute the phase shifts, with reference to particular procedures of taking the limit. Particularly interesting it is in this context the Born approximation, which is not valid. For the Coulomb potential, the theory of the phase shifts gives infinite shifts, and their regularization must be done. Indeed, for a δ-type potential there is no inside region of interest, and motion is free in the entire space. In the latter case it is easy to see that for slowly decreasing potentials, like the long-range Coulomb potential, the shifts may change with distance, albeit slowly, reflecting the non-vanishing scattering at small angles even for large impact parameters. We approach the scattering theory herein by employing the theory of the potential.

The theory of the potential. The Schrödinger equation reads

$$\left(\Delta + k^2\right)\psi = 4\pi U\left(\mathbf{r}\right)\psi \tag{2}$$

where $k^2 = 2mE/\hbar^2$ and $U(\mathbf{r}) = mV(\mathbf{r})/2\pi\hbar^2$. According to (1), we write the solution of equation (2) in the form $\psi\left(\mathbf{r}\right) = e^{ikz} + \varphi\left(\mathbf{r}\right)$, where $\varphi\left(\mathbf{r}\right)$ behaves as an outgoing wave at infinity. As is well-known from the theory of the potential, this solution reads

$$\varphi\left(\mathbf{r}\right) = -\int d\mathbf{r}'\cdot U\left(\mathbf{r}'\right)\left[e^{ikz'} + \varphi\left(\mathbf{r}'\right)\right]\frac{e^{ik\left|\mathbf{r}-\mathbf{r}'\right|}}{\left|\mathbf{r}-\mathbf{r}'\right|}. \tag{3}$$

It is the integral Schrödinger equation, and its *rhs* contains the Green function of the Schrödinger operator. If the potential $U(\mathbf{r})$ decreases conveniently fast at infinity we may disentangle the integration with respect to $\mathbf{r}'$ in (3) from the $\mathbf{r}$-dependence, by writing $\left|\mathbf{r}-\mathbf{r}'\right| \simeq r - \mathbf{r}\mathbf{r}'/r$ for $r \gg r'$, and get the asymptotic form of solution

$$\varphi\left(\mathbf{r} \to \infty\right) = -\left[U\left(\mathbf{q}\right) + \int d\mathbf{r}'\cdot U\left(\mathbf{r}'\right)\varphi\left(\mathbf{r}'\right)e^{-i\mathbf{k}'\mathbf{r}'}\right]\frac{e^{ikr}}{r}, \tag{4}$$

where

$$U\left(\mathbf{q}\right) = \int d\mathbf{r}\cdot U\left(\mathbf{r}\right)e^{-i\mathbf{q}\mathbf{r}} \tag{5}$$

is the Fourier transform of the potential, $\mathbf{k}' = k\mathbf{r}/r$ is the wavevector of the scattered wave ($\mathbf{k}$ being the wavector of the incoming wave) and $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the wavevector transfer, $q = |\mathbf{k}' - \mathbf{k}| = 2k\sin\theta/2$. Equation (4) represents the Born scattering theory, and $-U(\mathbf{q})$ is the well-known Born's scattering amplitude.

In order to solve completely equation (3) for $\varphi(\mathbf{r})$ we need also the wavefunction $\varphi(\mathbf{r})$ for small values of $\mathbf{r}$, *i.e.* for r comparable with the range of the potential. Let us denote it by $\varphi(\mathrm{r} \to 0)$ and the range of the potential by b. We can write from equation (3)

$$\varphi(\mathbf{r} \to 0) \simeq -\left[U(-\mathbf{k}) + \int d\mathbf{r}' \cdot U(\mathbf{r}')\varphi(\mathbf{r}')\right]\frac{e^{ikb}}{b}, \tag{6}$$

or

$$b'\varphi(\mathbf{r} \to 0) \simeq -U(-\mathbf{k}) - \int d\mathbf{r}' \cdot U(\mathbf{r}')\varphi(\mathbf{r}'), \tag{7}$$

where $b' = be^{-ikb} \simeq b(1-ikb)$; b'^{-1} is an order-of-magnitude estimation for the factor $e^{ik|\mathbf{r}-\mathbf{r}'|}/|\mathbf{r}-\mathbf{r}'|$.

According to equation (7) function $\varphi(\mathbf{r} \to 0)$ does not depend on **r**. It follows that the approximation used is valid for rather weak potentials. Equation (3) is an eigenvalue integral equation. For positive potentials $V > 0$ its eigenfunctions behave like exponentials near origin, with characteristic exponents $\kappa \sim \sqrt{V}$; for negative potentials $V < 0$ its eigenfunctions are periodic with characteristic period $\kappa \sim \sqrt{-V}$. It follows, in the latter case, that φ in equation (3), and the scattering amplitude in equation (4), may become infinite, periodically, for sufficiently deep potential wells. This corresponds to resonance scattering on upper bound states, or on low virtual levels. The approximation employed in equation (7) is valid for potentials which are sufficiently weak, as, for instance, potential wells that may have one bound state at most.

We multiply equation (7) by $U(\mathbf{r})$ and integrate to get

$$b'\int d\mathbf{r} \cdot U(\mathbf{r})\varphi(\mathbf{r}) \simeq -U(\mathbf{k}=0)U(-\mathbf{k}) - U(\mathbf{k}=0)\int d\mathbf{r}' \cdot U(\mathbf{r}')\varphi(\mathbf{r}'), \tag{8}$$

whence

$$\int d\mathbf{r} \cdot U(\mathbf{r})\varphi(\mathbf{r}) = -\frac{U(\mathbf{k}=0)U(-\mathbf{k})}{b' + U(\mathbf{k}=0)}, \tag{9}$$

and

$$\varphi(\mathbf{r} \to 0) \simeq -\frac{U(-\mathbf{k})}{b' + U(\mathbf{k}=0)}. \tag{10}$$

The asymptotic solution (4) becomes now

$$\psi(\mathbf{r} \to \infty) = -\left[U(\mathbf{q}) - \frac{U(-\mathbf{k})U(\mathbf{k}')}{b' + U(\mathbf{k}=0)}\right]\frac{e^{ikr}}{r}. \tag{11}$$

It follows that the scattering amplitude of the Born theory can be represented as

$$f = -U(\mathbf{q}) + \frac{U(-\mathbf{k})U(\mathbf{k}')}{b' + U(\mathbf{k}=0)}. \tag{12}$$

The general scheme of estimating the scattering amplitude given by (12) is to notice that $U(\mathbf{k})$ can be represented as $U(\mathbf{k}) \sim b(V/\varepsilon)$, where V is the average interaction and $\varepsilon \sim \hbar^2/mb^2$ is the localization energy over the potential range; then, the scattering amplitude reads $f \sim -b(V/\varepsilon)/(1+V/\varepsilon)$. It is worth noting its non-perturbational character.

We note that the second term in the scattering amplitude given by (12) differs formally from the well-known higher-order terms of the perturbation theory. This tells that the usual expansion of the wavefunction as a perturbation series is not warranted in general, as the closed, finite result obtained above indicates.[1] However, these two results do coincide in fact, when the conditions are met for the perturbation theory to be applicable.[2] In addition, we note that the "reciprocity theorem" $f(\mathbf{k},\mathbf{k}') = f(-\mathbf{k}',-\mathbf{k})$ (time-reveral symmetry) holds for the scattering amplitude given by equation (12). For central potentials the angular dependence in equation (12) comes through $k^2\cos\theta$ in Born's amplitude, so that in an expansion $\sum(2l+1)f_l P_l$ in Legendre polynomials, the partial-wave amplitudes f_l go like $k^{2l} \sim E^{-l}$, where l is the angular momentum.

Equation (12) implies a fundamental result for the scattering theory. We arrive at it by making use of $b' = be^{-ikb} \simeq b - ikb^2$. Then, we get straightforwardly that the scattering amplitude f given by equation (12) can be written as

$$f \simeq \frac{1}{g - ik}, \tag{13}$$

where g is a real function of **k** and **k**'. According to (12) it is expressible in terms of the interaction. This is a basic result, that can be inferred from equation (4). It

[1] The general scheme of such a statement is that from a generic expression like $x = 1 + ax$, *i.e.* $x = 1/(1 - a)$, is not always permissible of course to write $x = 1 + a + a^2 = ...$ The situation described herein is similar to the non-perturbational solutions of Dyson equations in quantum electrodynamics ("Landau's pole").

[2] These conditions are the "low-energy" regime $kb \ll 1$ where the inequality $|U| \ll 1/b^2$ must be satisfied (*i.e.* $V/\varepsilon \ll 1$), and the "high-energy" regime $kb \gg 1$ where the inequality $|U| \ll (kb)k^2$ must be satisfied. In both cases the zeroth order approximation of the perturbation theory is Born's amplitude.

shows that f is finite and never singular.[3] Finally, it can be checked on equation (12) that the "optical theorem" $Im\, f(\mathbf{k},\mathbf{k}) = Im\, f(0) = k\sigma/4\pi$ is conveniently satisfied; the partial-wave amplitudes are then such that $f_l = (g_l - ik)^{-1}$. In the long wavelengths limit $kb \ll 1$ the Fourier transform of the potential in equation (12) has a weak k-dependence, so we may neglect it; the "optical theorem" is then satisfied. (A short-range potential has a weak k-dependence for $kb \ll 1$ and a decreasing oscillatory tail for $kb \gg 1$. The s-wave amplitude given by equation (12) is very close to the exact result for a spherical potential well). In the short wavelengths limit equation (12) gives the same results as the quasi-classical theory of scattering, the "optical theorem" included, for the second-order of the perturbation theory, as is shown at the end of this paper. In the intermediate range $kb \approx 1$ the main contribution to (12) comes from Born's amplitude. Its partial waves must then be corrected by $f_l \to f_l/(1 - ikf_l)$ in order to satisfy the "optical theorem". This is a small correction, of the order of the second term left aside in equation (12).

This completes the basic result of the scattering theory for reasonable, short-range potentials.

***δ*-potential**. Let us assume a potential $V(\mathbf{r}) = C\delta(\mathbf{r})$, *i.e.* $U(\mathbf{r}) = mC\delta(\mathbf{r})/2\pi\hbar^2 = U_0\delta(\mathbf{r})$. Equation (3) has then the exact solution

$$\varphi(r) = -U\left[1 + \varphi(0)\right]\frac{e^{ikr}}{r}. \tag{14}$$

This equation has no acceptable solution; for instance, $\varphi(r) = 0$ for any finite r, and $\varphi(r) = -1$. We write $C = Vb^3$ where $b \to 0$ and $V \to \infty$ such that C is finite. Then, we get $\varphi(0) = -U_0/(b' + U_0) \simeq -1 + b'/U_0 + ...$ and

$$\varphi(r) = -\frac{b'U_0}{b' + U_0}\cdot\frac{e^{ikr}}{r} \simeq -b'\frac{e^{ikr}}{r}, \tag{15}$$

and the corresponding scattering amplitude $f = -b'U_0/(b' + U_0) \simeq -b'$, which coincides with (12) for $U(\mathbf{k}) = U_0$ and $b \to 0$. The representation (13) reads

[3] *i.e.*, the scattering amplitude is analytic in energy, for $E > 0$; it may be extended to $E < 0$, where it has discrete poles, corresponding to bound states (which can also be approached with the integral equation of the potential); the origin $E = 0$ is a branch point, and since the imaginary part for $ReE > 0$ cannot be changed, it follows that a cut must be practised on the right half of the energy axis.

$$f=\frac{1}{-U_0^{-1}-b^{-1}-ik}\cong\frac{1}{-b^{-1}-ik}. \quad (16)$$

This is an exact solution.

Effective-range theory and resonance scattering. For central potentials the Fourier transforms in equation (12) can be expanded as

$$U(\mathbf{k})=\int d\mathbf{r}\cdot U(\mathbf{r})e^{-i\mathbf{kr}}=U_0-\left(k^2/2\right)\int d\mathbf{r}\cdot U(\mathbf{r})r^2\cos^2\theta+...=U_0-Ak^2+..., \quad (17)$$

where $U_0=U(\mathbf{k}=0)$ and $A>0$. Such an expansion holds for $kb\ll 1$. The scattering amplitude given by (12) can then be written as $f=1/(g-ik)$, where

$$g=-\frac{b+U_0}{bU_0}-2A\frac{b+U_0}{bU_0^2}k^2+2A\left(\frac{b+U_0}{bU_0}\right)^2k^2\cos\theta. \quad (18)$$

Particularly interesting is the case $U_0\cong -b$ in equation (18), when g vanishes and the scattering amplitude is very large. This is a resonance scattering (at low energy, $kb\ll 1$). We note that the contribution of the *p*-wave is of second order in comparison with that of the *s*-wave, so we can limit ourselves to the *s*-wave amplitude in equation (18). It can be written as

$$f_0=\frac{1}{-a^{-1}+r_0k^2/2-ik}, \quad (19)$$

where

$$a=\frac{bU_0}{b+U_0} \quad (20)$$

is the scattering length and $r_0=-4A(b+U_0)/bU_0^2$ is the parameter of the "effective range". The *s*-wave amplitude is the effective range.

For U_0 slightly below $-b\left(U_0<-b\right)$ the scattering length is positive $(a>0)$ and r_0 is positive $(r_0>0)$. The dependence of the total cross-section on energy can be followed experimentally, and the parameters a and r_0 can be determined, characterizing thus the potential. It is likely to exist, in this case, a bound state of small energy $\varepsilon=-\hbar^2\kappa^2/2m<0$ and radius $R=1/\kappa$. The real part of equation (19) reads then $f_0^{-1}=-a^{-1}-r_0/2R^2=-1/R$. This may provide another check for the two parameters a and r_0. Historically, this was the procedure employed for determining the scattering length $a\cong 4\,fm$ for the triplet neutron-proton scattering, corresponding to a bound state $-\varepsilon \approx 2\mathrm{MeV}$ and a potential depth $V \approx -10\mathrm{MeV}$ for a range $b \approx 1-2\,fm$ of the nuclear forces ($1\,fm = 10^{-15}\,m$).

Leaving aside the k^2-term in equation (19), and writing $-a^{-1} = -1/R = -\kappa$, the scattering amplitude reads

$$f_0 = -\frac{1}{\kappa + ik} = -\left(2m/\hbar^2\right)^{1/2}\frac{1}{\sqrt{|\varepsilon|} + i\sqrt{E}}. \tag{21}$$

This is Wigner's formula for resonance scattering on an upper discrete level. For U_0 slightly above $-b\left(-b < U_0 < 0\right)$, the scattering length is negative ($a < 0$) and r_0 is negative also ($r_0 < 0$). There is not likely to exist now a bound state, so the k^2-term in equation (19) can be left aside. Equation (21) reads now

$$f_0 = -\frac{1}{\kappa + ik} = -\left(2m/\hbar^2\right)^{1/2}\frac{1}{-\sqrt{\varepsilon} + i\sqrt{E}}, \tag{22}$$

where $\kappa = a^{-1} < 0$; $\varepsilon > 0$ is said to be a virtual energy level. Knowing the triplet scattering length and measuring the coherent scattering of neutrons from hydrogen molecule, the scattering length $a \leq -2\,fm$ was obtained for the singlet neutron-proton scattering.[4] This suggested the spin dependence of the nuclear forces, and the fact that the singlet neutron-proton nearly misses a bound state. It has a virtual energy level.

For a partial-wave $f_l = 1/\left(g_l - ik\right)$, the function g_l starts with $k^{-2l} = E^{-l}$, and continues with $k^{-2l+2} = E^{-l+1}$. Keeping the first two terms in such an expansion, the partial scattering amplitude reads

$$f_l = \frac{1}{cE^{-l}\left(-\varepsilon + E\right) - ik}, \tag{23}$$

where c and ε are two parameters. For $\varepsilon < 0$ we have a pole $E = -|\varepsilon|$, *i.e.*, a discrete level. The scattering amplitude is however low, because of the small factor E^l. The resonance scattering of higher-order partial waves on bound states is low. For $\varepsilon > 0$, there is no bound state, but the resonance is high on the virtual level $\varepsilon > 0$. The scattering amplitude looks practically like $f = 1/\left(-ik\right)$ which, for low energies, means a very long lifetime. By (23), the width of this resonance is given by $\Delta E/\varepsilon \approx E^l k/k^2 = k^{2l-1}$. For low energy, this is a very sharp resonance. The

[4] Similar negative scattering lengths are known for (singlet) neutron-neutron and proton-proton.

centrifugal barrier leaves little chance for the motion to escape too soon. This is different from the resonance in *s*-wave.

Wigner-Breit formula. During collision, particles can be absorbed for a while on the target, and additional particles can be released by the target after a while, by disintegration. Additional contribution is then present in the scattering amplitude.

Equation (22) can be written generically as $f=\left(\hbar^2/2m\right)^{1/2}/\sqrt{E}$ (negative scattering length). Its variation is therefore $\delta f=-\left(\delta E/2k\right)/E$. Near a resonance level $E_0\left(E_0>0\right)$ the denominator *E* means $E-E_0$. On the other hand, the level E_0 is only a "quasi-discrete" level, *i.e.*, it is represented as $E_0-i\Gamma/2$, such that its uncertainty (the width) is $\delta E=\Gamma\left(\Gamma>0\right)$. We get

$$\delta f=-\frac{\Gamma/2k}{E-E_0+i\Gamma/2}. \tag{24}$$

This is the Wigner-Breit formula for resonance scattering on a "quasi-discrete" level. Obviously, the width Γ must be much smaller than the levels separation. On the other hand, *E* is close to E_0.

The wavefunction of the "quasi-discrete" (and quasi-stationary) level goes like $\psi\sim\exp\left(-iE_0t/\hbar-\Gamma t/2\hbar\right)$, *i.e.*, $\left|\psi\right|^2\sim\exp\left(-\Gamma t/\hbar\right)$; the level decays with the probability $w=\hbar/\Gamma$ per unit time, and has a lifetime $\tau=\Gamma/\hbar=1/w$. The decaying levels are positive, and their spectrum is quasi-continuous in fact. The second-order perturbation theory gives the energy correction

$$\delta E=\int\frac{\left|V_{n\nu}\right|^2 d\nu}{E_\nu-E_\nu+i0} \tag{25}$$

with usual notations, and, since $1/\left(x+i0\right)=P\left(1/x\right)-i\pi\delta\left(x\right)$, we get

$$\Gamma=2\pi\int\left|V_{n\nu}\right|^2\delta\left(E_n-E_\nu\right)d\nu. \tag{26}$$

The scattering amplitude reads then $f=f_0+\delta f$, where f_0 is the potential part of the scattering amplitude derived before; δf is the resonance part.

For resonance level E_0 close to zero, its width must vanish, since $E\to 0$. It follows that in

$$\delta f=\left(\hbar^2/2m\right)^{1/2}\delta\left(1/\sqrt{E}\right)=-\frac{\hbar}{\sqrt{2m}}\frac{\delta E/2\sqrt{E}}{E} \tag{27}$$

the width $\Gamma=\delta E$ must have the form $\Gamma=2\gamma\sqrt{E}$, where γ is finite. The Wigner-Breit formula becomes

$$\delta f=-\frac{\hbar}{\sqrt{2m}}\frac{\gamma}{E-E_0+i\gamma\sqrt{E}}. \tag{28}$$

For $E\rightarrow 0$ this δf is now finite, in contrast with equation (24).

If $E \ll \gamma^2$ it is easy to see that in (28) we may neglect the *E*-term. Equation (28) recovers then the resonance sacttering on a vanishing level, either positive or negative. Indeed, the scattering amplitude can then be writen as

$$\delta f=-\frac{1}{ik-\sqrt{2m}E_0/\hbar\gamma}. \tag{29}$$

We define $-\kappa=\sqrt{2m}E_0/\hbar\gamma$, and get $\delta f=-1/\left(\kappa+ik\right)$. This corresponds to a resonance scattering on a virtual level for $\kappa<0\left(a<0\right.$, according to (22)), or a true discrete level for $\kappa>0$ (accccording to (21).

The limit of high energy. In the high-energy limit $k\rightarrow\infty$ the second term in equation (12) goes to zero $\left(U\left(\mathbf{k}\right)\rightarrow 0\right)$. The first term in equation (12) (Born's amplitude $-U\left(\mathbf{q}\right)$) vanishes also for any finite θ in $q=2k\sin\theta/2$ in the limit $k\rightarrow\infty$. It follows that the scattering proceeds mainly forwards, $\theta\rightarrow\infty$, such that $q \simeq k\theta$ stays finite and small for $k\rightarrow\infty$. The small momentum transfer $\mathbf{q}$ is perpendicular to $\mathbf{k}$. Born's amplitude reads then

$$U\left(\mathbf{q}\right)=\int d\mathbf{r}\cdot U\left(\mathbf{r}\right)e^{-i\mathbf{qr}}=\int d\rho\cdot\left[\int_{-\infty}^{+\infty}dz\cdot U\left(z,\rho\right)\right]e^{-i\mathbf{q}\rho}, \tag{30}$$

where ρ is the transverse variable with respect to the wavevector $\mathbf{k}$. For central potentials, the integral in (30) can be estimated by noting that the rapidly oscillating exponential is not vanishing over a radius ρ only, given by $\left\langle\left(\mathbf{q}\rho\right)^2\right\rangle=q^2\rho^2/2=2,$ *i.e.* $\rho=2/q=2/k\Delta\theta$.[5] The corresponding area is therefore $\pi\rho^2=4\pi/k^2\left(\Delta\theta\right)^2$. We get

[5] $\left\langle\exp\left(i\varphi\right)\right\rangle=1-\left\langle\varphi^2\right\rangle/2=0,$ *i.e.* $\left\langle\varphi^2\right\rangle=2$, which means $\left\langle\left(\mathbf{q}\rho\right)^2\right\rangle=q^2\rho^2/2=2$, and $\rho=2/q$; the area factor is therefore $\pi\rho^2=4\pi/q^2=4\pi/k^2\left(\Delta\theta\right)^2$.

$$U\left(\mathbf{q}\right)=\frac{4\pi}{k^{2}\left(\Delta\theta\right)^{2}}\int_{-\infty}^{+\infty}dz\cdot U\left(z,0\right). \tag{31}$$

Similarly, the total cross-section is given by

$$\sigma=\int\left|U\left(\mathbf{q}\right)\right|^{2}d\Omega=\int dzdz'd\rho d\rho'\cdot U\left(z,\rho\right)U\left(z'\rho'\right)e^{-i\mathbf{q}\left(\rho-\rho'\right)}dz, \tag{32}$$

or

$$\sigma=\frac{4\pi^{2}}{k^{2}}\int d\rho\cdot\left(\int dzU\right)^{2}, \tag{33}$$

and the optical theorem gives $Im\, f\left(0\right)=k\sigma/4\pi$. One can see that $Im\, f\left(0\right)$ is a second-order correction to Born's amplitude in equation (12). Equation (32) is the classical expression of the cross-section integrated with respect to the impact parameter ρ.

Obviously, the forward scattering is the eikonal regime, and Schrodinger's equation (2) can be written as

$$\left[\Delta+k^{2}-4\pi U\left(r\right)\right]\psi=\left(\Delta+\kappa^{2}\right)\psi, \tag{34}$$

where $\kappa^{2}=k^{2}-4\pi U.$ For large k the solution reads

$$\psi=\exp\left(i\int^{z}dz\cdot\kappa\right), \tag{35}$$

providing the potential has a smooth variation (quasi-classical approximation). Indeed, the terms neglected in the laplacean of (35) are of the order of $\partial\kappa/\partial z$ which must be compared to κ^{2} (and similar derivatives with respect to the other coordinates). That means $\partial\lambda/\partial z\ll 1$, which is the typical condition of validiy for the quasi-classical approximation. It can also be written as $\partial U/\partial z\ll\kappa^{3}$, or $\left|U\right|\ll\left(\kappa b\right)\kappa^{2}$, a condition which may also be read as $\left|U\right|\ll\left(kb\right)k^{2}$, since k is much greater than κ. However, it is worth noting that this condition is much weaker that $\left|U\right|/k^{2}\ll 1$, in virtue of $kb\gg 1$. Therefore, we may use (35) with this weaker condition, without resorting to approximating κ by $k-2\pi U/k$. It is worth noting that such a first-order approximation in (35) renders in fact $\psi=\exp\left(i\int^{z}dz\cdot\kappa\right)=\exp\left(ikz\right)\left[1-\left(2\pi i/k\right)\int^{z}dz\cdot U\right]$, and it would not give an imaginary part of the scattering amplitude.

Equation (35) can also be written as

$$\psi=e^{ikz}S\left(z,\rho\right), \tag{36}$$

where

$$S(z,\rho)=\exp\left[i\int^{z}dz\cdot(\kappa-k)\right]=\exp\left[-\frac{4\pi i}{k}\int^{z}dz\cdot\frac{U}{\sqrt{1-4\pi U/k^{2}}+1}\right]. \tag{37}$$

One can see from equation (37) that the phase variation is of the order of $|U|b/k$. It holds for $|U|\ll k^{2}$. The quasi-classical approximation is still valid $\left(|U|\ll k^{2}(kb)\right)$, even for $kb\gg 1$, providing this stronger condition is satisfied. We get then $|U|b/k\ll 1$, or, since $U=mV/2\pi\hbar^{2}$, $|V|\ll(\hbar/mb)kb^{2}=\hbar v/b$, where v is the particle velocity.[6] This condition tells that U can be treated as a perturbation, and, again, ψ can be approximated by $\exp(ikz)\left[1-(2\pi i/k)\int^{z}dzU\right]$.

The scattering amplitude can be obtained from equation (3), written as

$$\varphi(\mathbf{r})=-\int d\mathbf{r}'\cdot U(\mathbf{r}')\psi(\mathbf{r}')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{38}$$

which becomes asymptotically

$$\varphi(\mathbf{r})=-\int d\mathbf{r}'\cdot U(\mathbf{r}')e^{-i\mathbf{q}\mathbf{r}'}S(z',\rho')\cdot\frac{e^{ikr}}{r}. \tag{39}$$

Hence, the scattering amplitude

$$f=-\int d\rho dz\cdot U(z,\rho)S(z,\rho)e^{-i\mathbf{q}\rho}, \tag{40}$$

where $S(z,\rho)$ is given by (37) with the lower limit of integration $-\infty$. One can see that the first term in (40) is Born's amplitude. We may define

$$\int dz\cdot U(z,\rho)S(z,\rho)=(ik/2\pi)\left[S(\rho)-1\right] \tag{41}$$

and write the scattering amplitude as

$$f=-(ik/2\pi)\int d\rho\cdot\left[S(\rho)-1\right]e^{-i\mathbf{q}\rho}. \tag{42}$$

Then we get $Im\,f(0)=-(k/2\pi)\int d\rho\cdot Re\left[S(\rho)-1\right]$ and the total cross section

[6] For a Coulomb potential e^{2}/r it means $e^{2}\ll\hbar v$, *i.e.* the "Bohr radius" $\hbar^{2}/me^{2}$ is much larger than the wavelength λ. The Coulomb potential may then be treated as a perturbation to the scattering in a short range potential.

$$\sigma=-2\int d\rho\cdot Re\left[S\left(\rho\right)-1\right]. \tag{43}$$

As regards an advanced computation of $S\left(\rho\right)$ in (41) we note that it is $S\left(+\infty,\rho\right)$ providing $S\left(z,\rho\right)$ in (37) is approximated by

$$S\left(z,\rho\right)=\exp\left[-\frac{2\pi i}{k}\int_{-\infty}^{z}dz\cdot U\right]; \tag{44}$$

indeed, we have then, formally, $US=\left(ik/2\pi\right)dS/dz$ and the integration in (41) gives $\left(ik/2\pi\right)\left[S\left(+\infty,\rho\right)-1\right]$. However, as it was said above, retaining only U in (37) amounts to write

$$S\left(\rho\right)=S\left(+\infty,\rho\right)=1-\frac{2\pi i}{k}\int_{-\infty}^{+\infty}dz\cdot U, \tag{45}$$

which leads to $Re\left(S-1\right)=0$, *i.e.* no imaginary part in the scattering amplitude, and no scattering according to the "optical theorem".

The actual computation in equation (40) gives in fact

$$f=\frac{1}{-k^{2}\left(\Delta\theta\right)^{2}/4\pi A-ik\left(\Delta\theta\right)^{2}/4}, \tag{46}$$

where $A=\int dz\cdot U\left(z,0\right)$ and $S\left(z,0\right)=1-\left(2\pi i/k\right)\int^{z}dz\cdot U\left(z,0\right)$ is used. The partial-wave amplitudes f_l in the expansion $f=\sum\left(2l+1\right)f_lP_l\left(\cos\theta\right)$, where P_l are the Legendre polynomials, are given by

$$f_l=\frac{1}{-k^{2}/\pi A-ik}, \tag{47}$$

in accordance with the "optical theorem".

Similarly, making use of (40), we get

$$f\left(0\right)=-\int d\rho\cdot A\left(1-\frac{i\pi}{k}A\right), \tag{48}$$

where $A=\int dz\cdot U\left(z,\rho\right)$ and $Imf\left(0\right)=\left(\pi/k\right)\int d\rho\cdot A^{2}$. By (40) also, the total cross-section reads

$$\sigma=\pi\left(\Delta\theta\right)^{2}\int d\rho d\rho' dzdz'\cdot USU'S'e^{-i\mathbf{q}\left(\rho-\rho'\right)}=\frac{4\pi^{2}}{k^{2}}\int d\rho\left(\int dz\cdot US\right)^{2}, \tag{49}$$

which is $(4\pi/k) Im\, f(0)$ in agreement with the "optical theorem" and Born's scattering amplitude.

It is worth estimating the contribution of the second term in equation (12) for high-energy scattering. We employ the following successive approximations

$$\begin{aligned} U(\mathbf{k}) &= \int d\mathbf{r}\cdot U(\mathbf{r})e^{-i\mathbf{kr}} = \int d\rho dz\cdot U(z,\rho)e^{-ikz} = \\ &= \pi b^2\int dz\cdot U(z,0)e^{-ikz} = (\pi b^2/bk)\int dz\cdot U(z,0) \end{aligned} \tag{50}$$

in this case, and

$$U(\mathbf{k}') = \int d\rho dz\cdot U(z,\rho)e^{-ikz}e^{-i\mathbf{q}\rho} = \left[4\pi/k^2(\Delta\theta)^2\right](1/bk)\int dz\cdot U(z,0). \tag{51}$$

Then, $U(\mathbf{k}=0)$ can be neglected in the denominator of (12) and $b'^{-1} = b^{-1} + ik$. The contribution of b^{-1} can also be neglected within this approximation, so we get finally

$$f = -U(\mathbf{q}) + ik\frac{4\pi^2}{k^4(\Delta\theta)^2}\left[\int dz\cdot U(z,0)\right]^2, \tag{52}$$

which coincides with (46). We may say that the representation given by equation (12) for the scattering amplitude can be viewed as a fairly adequate one, within the given conditions.