Romanian Reports in Physics, Vol. 59, No. 2, P. 515–522, 2007

Dedicated to Prof. Dorin N. Poenaru's 70th Anniversary

MULTIDIMENSIONAL SOLUTONS AND VORTICES IN NONLOCAL NONLINEAR OPTICAL MEDIA

D. MIHALACHE

"Horia Hulubei" National Institute for Physics and Nuclear Engineering (IFIN-HH)
407 Atomistilor, Magurele-Bucharest, 077125, Romania
E-mail: Dumitru.Mihalache@nipne.ro

(Received January 13, 2007)

Abstract. We give a brief overview of recent results in the area of two- and three-dimensional solitons and vortices in nonlocal nonlinear optical media.

Key words: spatiotemporal optical solitons, vortex solitons, nonlocal optical media.

Multidimensional (two- and three-dimensional) solitons, have attracted a great deal of attention in optics, see [1–4]. Of much importance are the three-dimensional spatiotemporal solitons, often refered to as "light bullets" [5]. These are multidimensional pulses, which maintain their shape in the longitudinal (temporal) and transverse (spatial) directions due to the balance between the group-velocity dispersion, diffraction, and nonlinear self-phase modulation. However, solitons in media with the cubic self-focusing nonlinearity, obeying the nonlinear Schrödinger (NLS) equation, are unstable in two and three dimensions, because of the occurrence of collapse in the same model [6]. Several possibilities to arrest the collapse were considered, such as periodic alternation of self-focusing and defocusing layers [7] and various generalizations of this setting [8], and the use of weaker instabilities, *viz.*, saturable [9] or quadratic ($\chi^{(2)}$) ones [10–13]. Tandem layered structures, composed of alternating linear and quadratic ($\chi^{(2)}$) layers, were also proposed and investigated [14]. The only successful experiment in this field was the creation of quasi-(2+1)-dimensional spatiotemporal solitons in bulk $\chi^{(2)}$ samples [4, 15]. Other theoretically developed approaches use off-resonance two-level systems [16] and self-induced-transparency media [17].

Collapse does not occur either in $\chi^{(3)}$ media whose nonlinearity is nonlocal [18], therefore they may also give rise to stable solitons, see review [19]. Two-dimensional spatial solitons stabilized by the nonlocality were observed in vapors [20] and lead glasses featuring strong thermal nonlinearity [21]; in the latter case, elliptic and vortex-ring solitons were reported. Optical one-dimensional solitons supported by a nonlocal $\chi^{(3)}$ nonlinearity were also created in liquid crystals [22]. Further, photonic lattices [23], vortices [24], spatial solitons in soft matter [25], multipole vector solitons in nonlocal nonlinear media [26], one-dimensional solitons of even and odd parities supported by competing nonlocal nonlinearities [27] were considered in the context of nonlocality. In addition, it was shown that long-range cubic nonlinearity induced by long-range interactions between atoms carrying polarized magnetic momenta in effectively two-dimensional Bose-Einstein condensates also leads to the prediction of stable two-dimensional (2D) solitons [28].

Two-dimensional vortex solitons [24] and three-dimensional (3D) fundamental (nonspinning) and spinning (with nonzero vorticity) [29] were considered in the context of nonlocality. It has been demonstrated that one-parameter families of *stable* (3+1)-dimensional spatiotemporal solitons are possible in media with nonlocal $\chi^{(3)}$ nonlinearity [29]. However, the spinning solitons are unstable in the specific nonlocal model, see below, which is valid, e.g., for the case of thermal nonlinearities.

Next we consider in detail a 3D nonlocal model based on a general system of coupled equations for the complex field amplitude q and nonlinear correction to the refractive index n (see, e.g., [30]); in a normalized form, the equations are

$$iq_\xi + (1/2)(q_{\eta\eta} + q_{\zeta\zeta} + Dq_{\tau\tau}) + qn = 0, \tag{1}$$

$$d(n_{\eta\eta} + n_{\zeta\zeta}) - n + |q|^2 = 0. \tag{2}$$

Here, η, ζ and ξ are the transverse and longitudinal coordinates, scaled, respectively, to the beam's width and its diffraction length, τ is the reduced temporal variable, and D is the ratio of the diffraction and dispersion lengths. We consider the case of *anomalous* temporal dispersion, $D > 0$ (otherwise, there is no chance for the self-trapping in the longitudinal direction), and then set $D = 1$ by means of an obvious scaling.

Lastly, $\sqrt{d}$ determines the correlation length (nonlocality scale) of the medium's nonlinear response [note that by setting $d = 0$ one turns Eqs. (2) into the ordinary 3D NLS equation with self-focusing. In fact, after setting $D = 1$, additional rescaling of Eqs. (2) makes it possible to set $d = 1$, so as to cast the system into the parameter-free form. Nevertheless, we prefer to keep d as an explicit parameter, as it directly controls the system's nonlocality degree.

The nonlocal nonlinearity in Eqs. (2) is typical for the light propagation in nematic liquid crystals, as well as for thermal nonlinearity in optical media

[19,22]. For the derivation of the model, the usual approximation of the slowly varying amplitude is adopted, along with an assumption of fast relaxation of the refractive-index perturbations in time, therefore the second equation does not contain the term $n_{\tau\tau}$. Equations (2) conserve the energy E (the norm of the multidimensional soliton) and the Hamiltonian H,

$$E = \iiint |q(\eta,\zeta,\tau)|^2 \, d\eta d\zeta d\tau, \tag{3}$$

$$H = \frac{1}{2}\iiint \left(|q_\eta|^2 + \left|q_\zeta^2\right| + D\left|q_\tau^2\right| - n|q|^2\right) d\eta d\zeta d\tau. \tag{4}$$

Note that the conservation of H is easily demonstrated by the substitution the first equation of (2) of the formal inversion of the second equation,

$$n = \left[1 - d\left(\partial_\eta^2 + \partial_\zeta^2\right)\right]^{-1}\left(|q|^2\right). \tag{5}$$

The same substitution helps to prove the conservation of the usually defined momentum in the transverse plane, and angular momentum along the longitudinal direction,

$$P_{\eta,\zeta} \equiv i \iiint q q_{\eta,\zeta}^* d\eta d\zeta d\tau, \tag{6}$$

and

$$M_\xi \equiv i \iiint q\left(\zeta q_\eta^* - \eta q_\zeta^*\right) d\eta d\zeta d\tau. \tag{7}$$

Stationary solutions to Eqs. (2) are looked for as

$$q = w(r,\tau)\exp[i(b\xi + S\theta)], \quad n = n(r,\tau), \tag{8}$$

where r and θ are the polar coordinates in the (η,ζ) plane, b is the propagation constant, the integer S is the vorticity ("spin"), and the real functions w and n obey the equations

$$\left(w_{rr} + r^{-1}w_r + w_{\tau\tau}\right) - \left(2b + r^{-2}S^2\right)w + 2wn = 0, \tag{9}$$

$$d\left(n_{rr} + r^{-1}n_r\right) - n + w^2 = 0. \tag{10}$$

Note that for this solution, the angular momentum is proportional to the energy: $M_\xi = SE$. We have numerically found families of localized solutions to these equations, dealing with the corresponding two-point boundary-value problem by dint of the standard band-matrix algorithm. Typically, grids with 241×240 and 201×360 points were used for the computations of the fundamental ($S = 0$) and spinning solitons, respectively. A noteworthy feature is that the 3D solitons exist only above a finite energy threshold. We have put into evidence the stabilizing effect of the nonlocality for the fundamental solitons ($S = 0$ solitons): except for a narrow interval of small wavenumbers, $0 < b < b_{\mathrm{cr}}$, the solitons are expected to be stable, as they satisfy the Vakhitov-Kolokolov criterion, $dE/db > 0$, which is a necessary (but, generally, not sufficient) stability condition for the soliton family [6, 31] (note that the instability of the 3D solitons in the local NLS equation precisely follows this criterion). Full stability of solitons was investigated using the equations for small perturbations linearized around the stationary solution.

Accordingly, solutions including perturbations with an infinitesimal amplitude ϵ are looked for as

$$q = e^{ib\xi+iS\theta}\left\{w(r,\tau)+\epsilon\left[f(r,\tau)e^{\delta\xi+iJ\theta}+g^*(r,\tau)e^{\delta^*\xi-iJ\theta}\right]\right\}, \qquad (11)$$

$$n = n(r,\tau)+\epsilon\left[p(r,\tau)e^{\delta\xi+iJ\theta}+p^*(r,\tau)e^{\delta^*\xi-iJ\theta}\right], \qquad (12)$$

where J is an arbitrary integer azimuthal index of the perturbation, δ is the instability growth rate, the asterisk stands for the complex conjugation, and the eigenfunctions f, g and p obey the equations

$$2i\delta f + f_{rr} + r^{-1}f_r + f_{\tau\tau} - \left[2b + (S+J)^2 r^{-2}\right] f + 2\,(nf + wp) = 0, \qquad (13)$$

$$-2i\delta g + g_{rr} + r^{-1}g_r + g_{\tau\tau} - \left[2b + (S-J)^2 r^{-2}\right] g + 2\,(ng + wp) = 0, \qquad (14)$$

$$d(p_{rr} + r^{-1}p_r) - (1 + dJ^2 r^{-2})p + w(f+g) = 0. \qquad (15)$$

The growth rate δ was found as an eigenvalue at which Eqs. (14) has a nonsingular localized solution. Stable solitons are those for which $\mathrm{Re}(\delta) = 0$ for all (integer) values of J. We have found that for the 3D spinning solitons, the perturbations with $J > 1$ destabilize a part of the families, but entire families of solitons with $S \geq 1$ are unstable against the perturbations with $J = 1$. The latter instability mode implies a trend to splitting of the vortex soliton into a set of two fundamental ones [4], which is corroborated by direct simulations below. *Stable* 3D spinning optical solitons (with $S = 1$) were only found in media with *competing* self-focusing and defocusing nonlinearities, *viz.*, $\chi^{(3)} : \chi^{(5)}$ or $\chi^{(2)} : \chi^{(3)}$ [32]. Note that for dissipative (non-Hamiltonian) systems governed by the complex cubic-quintic Ginzburg-Landau equation, stable two-dimensional and three-dimensional dissipative solitons with topological charges $S = 1$ and $S = 2$ were found, too [33, 34]. Recently, stable 2D spatial (rather than spatiotemporal) vortex rings were experimentally observed in a medium featuring the thermal nonlocal nonlinearity [21].

The predictions of the linear stability analysis were checked in direct simulations of Eqs. (2), which were run by means of a standard Crank-Nicholson scheme. The nonlinear finite-difference equations were solved using the Picard iteration method, and the resulting linear system was handled with the help of the Gauss-Seidel iterative procedure. To achieve good convergence, we needed typically, twelve Picard's and four Gauss-Seidel iterations. The initial condition for perturbed solitons were taken as the form $q(\xi = 0) = w(\eta, \zeta, \tau)(1+\epsilon\rho)$, and $n(\xi = 0) = n(\eta, \zeta, \tau)(1 + \epsilon\rho)$, where ϵ is, as above, a small perturbation amplitude, and ρ was either as a random variable uniformly distributed in the interval $[-0.5, 0.5]$, or simply as $\rho = 1$ (uniform perturbation).

First, we have checked that all the fundamental spatiotemporal solitons which were predicted above to be stable are indeed stable against random perturbations; Fig. 1 displays an example of self-healing of a stable soliton with the initial perturbation amplitude $\epsilon = 0.1$. A small uniform perturbation ($\rho = 1$) applied to a stable soliton excites its persistent oscillations, which suggests the existence of a stable intrinsic mode in the soliton. On the other hand, direct simulations show that those fundamental solitons which were predicted to be unstable decay into radiation, if slightly perturbed.

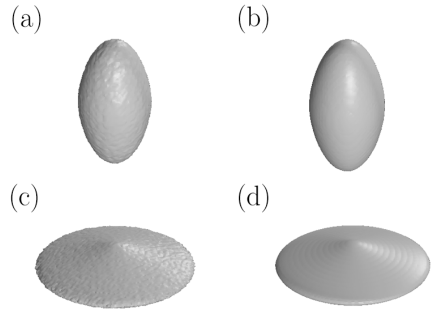

Fig. 1 – Isosurface plots illustrating the stability of a fundamental soliton corresponding to $d = 10$, $b = 1$, and $E = 178$. (a) and (c): the initially perturbed soliton, at $\xi = 0$; (b) and (d): the self-cleaned one at $\xi = 360$.

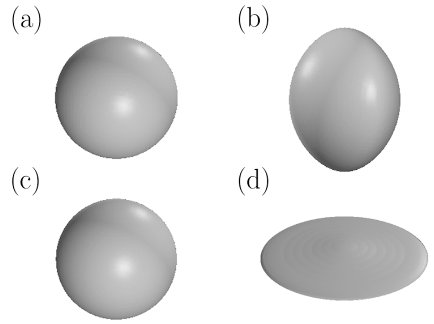

Fig. 2 – Self-trapping of a fundamental soliton, for $d = 1$: (a) and (c) – the initial Gaussian pulse with energy $E_0 = 54$; (c) and (d) – the soliton at $\xi = 60$.

For the linearly unstable nonspinning solitons we have identified two different instability scenarios: they either spread out under uniform perturbations that reduce their norm (energy), or they collapse if the soliton is perturbed by a relatively strong uniform perturbation with the strength $\epsilon = 0.05$, that increases their energy. We also simulated self-trapping of a stable fundamental soliton from an initial spatiotemporal pulse of an arbitrary form. An example is shown in Fig. 2 for an isotropic Gaussian input, $w(\xi = 0) = W_0 e^{-(\eta^2+\zeta^2+\tau^2)/\rho_1^2}$, $n(\xi = 0) = N_0 e^{-(\eta^2+\zeta^2+\tau^2)/\rho_2^2}$, which generates an anisotropic (elliptic) soliton.

We also simulated the evolution of unstable spinning solitons. Most typically, they follow the prediction of the linear stability analysis and split into two stable fundamental solitons, see an example (for $S = 1$) in Fig. 3.

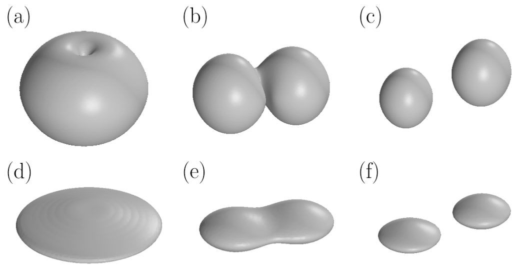

Fig. 3 – Splitting of an unstable $S = 1$ soliton with $d = 100$ and $b = 0.05$. (a) and (d) $\xi = 0$, (b) and (e) $\xi = 1400$, and (c) and (f) $\xi = 1600$.

In conclusion, we have demonstrated that nonlocal cubic nonlinearity is sufficient to stabilize 3D fundamental (nonspinning) solitons, which suggests a new approach to making of 3D spatiotemporal solitons in optics, which thus far evaded experimental observation. The stability of the fundamental solitons was demonstrated through the computation of the corresponding stability eigenvalues, and in direct simulations. Their robustness and, hence, physical relevance was demonstrated by self-trapping from arbitrary input pulses. On the other hand, all the spinning 3D solitons in the nonlocal medium with thermal nonlinearity are unstable against splitting into a set of stable fundamental solitons.

Acknowledgements. Part of the research work presented in this paper has been carried out in collaboration with Lucian-Cornel Crasovan, Yaroslav V. Kartashov, Falk Lederer, Boris A. Malomed, Dumitru Mazilu, and Lluis Torner. I am deeply indebted to all of them.

REFERENCES

1. Yu.S. Kivshar and G.P. Agrawal, *Optical solitons: From fibers to photonic crystals*, Academic Press, San Diego, 2003.

2. N.N. Akhmediev and A. Ankiewicz, *Solitons: Nonlinear Pulses and Beams*, London, Chapman and Hall, 1997.
3. G.I. Stegeman, D.N. Christodoulides, and M. Segev, IEEE J. Select. Top. Quant. Electron., **6**, 1419 (2000).
4. B.A. Malomed, D. Mihalache, F. Wise, and L. Torner, J. Opt. B: Quantum Semiclass. Opt., **7**, R53 (2005).
5. Y. Silberberg, Opt. Lett., **15**, 1282 (1990).
6. L. Bergé, Phys. Rep., **303**, 260 (1998).
7. I. Towers and B.A. Malomed, J. Opt. Soc. Am., **19**, 537 (2002).
8. B.A. Malomed, *Soliton Management in Periodic Systems*, Springer, New York, 2006.
9. D. Edmundson and R.H. Enns, Opt. Lett., **17**, 586 (1992); N. Akhmediev and J.M. Soto-Crespo, Phys. Rev., A **47**, 1358 (1993); R. McLeod, K. Wagner, and S. Blair, Phys. Rev., A **52**, 3254 (1995).
10. B.A. Malomed *et al.*, Phys. Rev., E **56**, 4725 (1997).
11. D.V. Skryabin and W.J. Firth, Opt. Commun., **148**, 79 (1998).
12. L. Torner, D. Mazilu, and D. Mihalache, Phys. Rev. Lett., **77**, 2455 (1996); D. Mihalache, F. Lederer, D. Mazilu, and L.-C. Crasovan Opt. Eng., **35**, 1616 (1996).
13. D. Mihalache *et al.*, Opt. Commun., **152**, 365 (1998); Opt. Commun. **169**, 341 (1999); Opt. Commun., **159**, 129 (1999); Phys. Rev., E **62**, 7340 (2000).
14. L. Torner *et al.*, Opt. Commun., **199**, 277 (2001).
15. X. Liu, L.J. Qian, and F.W. Wise, Phys. Rev. Lett., **82**, 4631 (1999).
16. I.V. Mel'nikov, D. Mihalache, and N.-C. Panoiu, Opt. Commun., **181**, 345 (2000).
17. M. Blaauboer, B.A. Malomed, and G. Kurizki, Phys. Rev. Lett., **84**, 1906 (2000).
18. S.K. Turitsyn, Theor. Math. Phys., **64**, 797 (1985).
19. W. Krolikowski *et al.*, J. Opt., B: Quantum Semiclass. Opt. **6**, S288 (2004).
20. D. Suter and T. Blasberg, Phys. Rev., A **48**, 4583 (1993).
21. C. Rotschild, O. Cohen, O. Manela, M. Segev, and T. Carmon, Phys. Rev. Lett., **95**, 213904 (2005).
22. X. Hutsebaut, C. Cambournac, M. Haelterman, A. Adamski, and K. Neyts, Opt. Commun., **233**, 217 (2004); C. Conti, M. Peccianti, and G. Assanto, Phys. Rev. Lett., **92**, 113902 (2004).
23. Y.V. Kartashov, V.A. Vysloukh, and L. Torner, Phys. Rev. Lett., **93**, 153903 (2004); Z. Xu, Y.V. Kartashov, and L. Torner, Phys. Rev. Lett., **95**, 113901 (2005); A.S. Desyatnikov *et al.*, Opt. Lett., **30**, 869 (2005).
24. D. Briedis *et al.*, Opt. Express **13**, 435 (2005); A.I. Yakimenko, Y.A. Zaliznyak, and Y. Kivshar, Phys. Rev., E **71**, 065603(R) (2005).
25. C. Conti, G. Ruocco, and S. Trillo, Phys. Rev. Lett., **95**, 183902 (2005).
26. Y.V. Kartashov, L. Torner, V.A. Vysloukh, and D. Mihalache, Opt. Lett., **31**, 1483 (2006).
27. D. Mihalache, D. Mazilu, F. Lederer, L.-C. Crasovan, Y.V. Kartashov, L. Torner, and B.A. Malomed, Phys. Rev., E **74**, 066614 (2006).
28. P. Pedri and L. Santos, Phys. Rev. Lett., **95**, 200404 (2005).
29. D. Mihalache, D. Mazilu, F. Lederer, B.A. Malomed, Y.V. Kartashov, L.-C. Crasovan, and L. Torner, Phys. Rev., E **73**, 025601(R) (2006).
30. W. Krolikowski and O. Bang, Phys. Rev. E **63**, 016610 (2001); M. Peccianti, C. Conti, and G. Assanto, Phys. Rev., E **68**, 025602(R) (2003).
31. M.G. Vakhitov and A.A. Kolokolov, Sov. J. Radiophys. Quantum Electr., **16**, 783 (1973).

32. D. Mihalache *et al.*, Phys. Rev. Lett. **88**, 073902 (2002); Phys. Rev., E **66**, 016613 (2002).
33. L.-C. Crasovan, B.A. Malomed, and D. Mihalache, Phys. Rev., E **63**, 016605 (2001); L.-C. Crasovan, B. A. Malomed, and D. Mihalache, Phys. Lett., A **289**, 59 (2001).
34. D. Mihalache, D. Mazilu, F. Lederer, Y.V. Kartashov, L.-C. Crasovan, L. Torner, and B.A. Malomed, Phys. Rev. Lett., **97**, 073904 (2006).