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UNIFIED DESCRIPTION OF THE $2\nu\beta\beta$ DECAY TO THE FIRST QUADRUPOLE PHONON STATE IN SPHERICAL AND DEFORMED NUCLEI

C.M. RADUTA[1], A.A. RADUTA[1,2]

[1] *Institute of Physics and Nuclear Engineering*
Bucharest, P.O. Box MG6, Romania, E-mail: raduta@ifin.nipne.ro
[2] *Department of Theoretical Physics and Mathematics, Bucharest University*
P.O. Box MG11, Romania

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Abstract. The Gamow-Teller transition operator is written as a polynomial in the dipole proton-neutron and quadrupole charge conserving QRPA boson operators, using the prescription of the boson expansion technique. Then, the $2\nu\beta\beta$ process ending on the first 2^+ state in the daughter nucleus is allowed through one, two and three boson states describing the odd-odd intermediate nucleus. The approach uses a single particle basis which is obtained by projecting out the good angular momentum from an orthogonal set of deformed functions. The basis for mother and daughter nuclei have different deformations. The GT transition amplitude as well as the half lives were calculated for six transitions. Results are compared with the available data as well as with some predictions obtained with other methods.

Key words: double beta decay, particle-hole interaction, particle-particle interaction, neutrino, half life, quasiparticle-proton-neutron random phase approximation.

One of the most exciting subject of nuclear physics is that of double beta decay. The interest is generated by the fact that in order to describe quantitatively the decay rate one has to treat consistently the neutrino properties as well as the nuclear structure features. The process may take place in two distinct ways: a) by a $2\nu\beta\beta$ where decay the initial nuclear system, the mother nucleus, is transformed in the final stable nuclear system, usually called the daughter nucleus, two electrons and two anti-neutrinos; b) by the $0\nu\beta\beta$ process where the final state does not involve any neutrino. The latter decay mode is especially interesting since one hopes that its discovery might provide a definite answer to the question whether the neutrino is a Majorana or a Dirac particle. The contributions over several decades have been reviewed by many authors. [1, 2, 3, 4, 5, 6].

Although none of the double beta emitters is a spherical nucleus most formalisms use a single particle spherical basis.

In the middle of 90's we treated the $2\nu\beta\beta$ process in a pnQRPA formalism using a projected spherical single particle basis which resulted in having a unified description of the process for spherical and deformed nuclei [7, 8]. Recently the single particle basis [9, 10] has been improved by accounting for the volume conservation while the mean field is deformed [11, 12]. The improved basis has been used for describing quantitatively the double beta decay rates as well as the corresponding half lives [13, 14]. The results were compared with the available data as well as with the predictions of other formalisms. The manners in which the physical observable is influenced by the nuclear deformations of mother and daughter nuclei are in detail commented. Two features of the deformed basis are essential: a) the single particle energy levels do not exhibit any gap; b) the pairing properties of the deformed system are different from those of spherical system. These two aspects of the deformed nuclei affect the overlap matrix of the pnQRPA states of mother and daughter nuclei. Moreover, considering the Gamow-Teller (GT) transition operator in the single particle-space generated by the deformed mean-field, one obtains an inherent renormalization with respect to the one acting in a spherical basis.

In Ref. [15] we studied the higher pnQRPA effects on the GT transition amplitude, by means of the boson expansion technique for a spherical single particle basis. Considering higher order boson expansion terms in the transition operator, significant corrections to the GT transition amplitude are obtained especially when the strength of the two body particle-particle (*pp*) interaction approaches its critical value where the lowest dipole energy is vanishing. As we showed in the quoted reference, there are transitions which are forbidden at the pnQRPA level but allowed once the higher pnQRPA corrections are included. An example of this type is the $2\nu\beta\beta$ decay leaving the daughter nucleus in a collective excited state 2^+. The electrons resulting in this process can be distinguished from the ones associated to the ground to ground transition by measuring, in coincidence, the gamma rays due to the transition $2^+ \to 0^+$ in the daughter nucleus [16].

The aim of this work is to study the double beta decay $0^+ \to 2^+$ where 0^+ is the ground state of the emitter while 2^+ is a single quadrupole phonon state describing the daughter nucleus. The adopted procedure is the boson expansion method as formulated in our previous paper [15] but using a projected spherical single particle basis.

In order to fix the necessary notations and to be self-contained, in the present work we describe briefly the main ideas underlying the construction of the projected single particle basis. The single particle mean field is determined by a particle-core Hamiltonian:

$$\tilde{H} = H_{sm} + H_{core} - M\omega_0^2 r^2 \sum_{\lambda=0,2} \sum_{-\lambda\leq\mu\leq\lambda} \alpha^*_{\lambda\mu} Y_{\lambda\mu}\,, \tag{1}$$

where H_{sm} denotes the spherical shell model Hamiltonian while H_{core} is a harmonic quadrupole boson (b^+_μ) Hamiltonian associated to a phenomenological core. The interaction of the two subsystems is accounted for by the third term of the above equation, written in terms of the shape coordinates $\alpha_{00}, \alpha_{2\mu}$. The quadrupole shape coordinates and the corresponding momenta are related to the quadrupole boson operators by the canonical transformation:

$$\alpha_{2\mu} = \frac{1}{k\sqrt{2}}(b^\dagger_{2\mu} + (-)^\mu b_{2,-\mu}), \quad \pi_{2\mu} = \frac{ik}{\sqrt{2}}((-)^\mu b^\dagger_{2,-\mu} - b_{2\mu}), \tag{2}$$

where k is an arbitrary C number. The monopole shape coordinate is determined from the volume conservation condition. In the quantized form, the result is:

$$\alpha_{00} = \frac{1}{2k^2\sqrt{\pi}}\left[5 + \sum_\mu (2b^\dagger_\mu b_\mu + (b^\dagger_\mu b^\dagger_{-\mu} + b_{-\mu}b_\mu)(-)^\mu)\right]. \tag{3}$$

Averaging $\tilde{H}$ on the eigenstates of H_{sm}, hereafter denoted by $|nljm\rangle$, one obtains a deformed boson Hamiltonian whose ground state is, in the harmonic limit, described by a coherent state

$$\Psi_g = \exp[d(b^+_{20} - b_{20})]|0\rangle_b, \tag{4}$$

with $|0\rangle_b$ standing for the vacuum state of the boson operators and d a real parameter which simulates the nuclear deformation. On the other hand, the average of $\tilde{H}$ on Ψ_g is similar to the Nilsson Hamiltonian [17]. Due to these properties, it is expected that the best trial functions to generate a spherical basis are:

$$\Psi^{pc}_{nlj} = |nljm\rangle\Psi_g. \tag{5}$$

The projected states are obtained by acting on these deformed states with the projection operator

$$P^I_{MK} = \frac{2I+1}{8\pi^2}\int {D^I_{MK}}^*(\Omega)\hat{R}(\Omega)d\Omega. \tag{6}$$

The subset of projected states:

$$\Phi^{IM}_{nlj}(d) = \mathcal{N}^I_{nlj} P^I_{MI}[|nljI\rangle\Psi_g] \equiv \mathcal{N}^I_{nlj}\Psi^{IM}_{nlj}(d), \tag{7}$$

are orthogonal with the normalization factor denoted by $\mathcal{N}^I_{nlj}$.

Although the projected states are associated to the particle-core system, they can be used as a single particle basis. Indeed, when a matrix element of a particle like operator is calculated, the integration on the core collective coordinates is performed first, which results in obtaining a final factorized expression: one factor carries the dependence on deformation and one is a spherical shell model matrix element.

The single particle energies are approximated by the average of the particle-core Hamiltonian $H' = \tilde{H} - H_{core}$ on the projected spherical states defined by Eq. (7):

$$\epsilon^I_{nlj} = \langle \Phi^{IM}_{nlj}(d)|H'|\Phi^{IM}_{nlj}(d)\rangle. \tag{8}$$

The off-diagonal matrix elements of H' is ignored at this level. Their contribution is however considered when the residual interaction is studied.

As shown in Ref. [9], the dependence of the new single particle energies on deformation is similar to that shown by the Nilsson model [17]. The quantum numbers in the two schemes are however different. Indeed, here we generate from each j a multiplet of $(2j+1)$ states distinguished by the quantum number I, which plays the role of the Nilsson quantum number Ω and runs from $1/2$ to j and moreover the energies corresponding to the quantum numbers K and $-K$ are equal to each other. On the other hand, for a given I there are $2I+1$ degenerate sub-states while the Nilsson states are only double degenerate. As explained in Ref. [9], the redundancy problem can be solved by changing the normalization of the model functions:

$$\langle \Phi^{IM}_{\alpha}|\Phi^{IM}_{\alpha}\rangle = 1 \Longrightarrow \sum_{M}\langle \Phi^{IM}_{\alpha}|\Phi^{IM}_{\alpha}\rangle = 2. \tag{9}$$

Due to this weighting factor the particle density function is providing the consistency result that the number of particles which can be distributed on the $(2I+1)$ sub-states is at most 2, which agrees with the Nilsson model. Here α stands for the set of shell model quantum numbers nlj. Due to this normalization, the states Φ^{IM}_{α} used to calculate the matrix elements of a given operator should be multiplied with the weighting factor $\sqrt{2/(2I+1)}$.

Finally, we recall a fundamental result, obtained in Ref. [12], concerning the product of two projected states, which comprises a product of two core components. Therein we have proved that the matrix elements of a two body interaction corresponding to the present scheme are very close to the matrix elements corresponding to spherical states projected from a deformed state consisting of two spherical single particle states times a single collective core wave function. The small discrepancies of the two types of matrix elements could be washed out by using slightly different strengths for the two body interaction in the two methods.

As we already stated, in the present work we are interested to describe the Gamow-Teller two neutrino double beta decay of an even-even deformed nucleus. In our treatment the Fermi transitions, contributing about 20% to the total rate, and the "forbidden" transitions are ignored, which is a reasonable approximation for the two neutrino double beta decay in medium and heavy nuclei. The $2\nu\beta\beta$ process is conceived as two successive single β^- virtual transitions. The first transition connects the ground state of the mother nucleus to a magnetic dipole state 1^+ of the intermediate odd-odd nucleus which subsequently decays to the first state 2^+ of the daughter nucleus. The second leg of the transition is forbidden within the pnQRPA approach but non-vanishing within a higher pnQRPA approach [15]. The states, involved in the $2\nu\beta\beta$ process are described by the following many body Hamiltonian:

$$\begin{aligned} H &= \sum \frac{2}{2I+1}(\epsilon_{\tau\alpha I} - \lambda_{\tau\alpha})c^\dagger_{\tau\alpha IM}c_{\tau\alpha IM} - \sum \frac{G_\tau}{4}P^\dagger_{\tau\alpha I}P_{\tau\alpha I'} \\ &+ 2\chi \sum \beta^-_\mu(pn)\beta^+_{-\mu}(p'n')(-)^\mu - 2\chi_1 \sum P^-_{1\mu}(pn)P^+_{1,-\mu}(p'n')(-)^\mu \\ &- \sum_{\tau,\tau'=p,n} X_{\tau,\tau'}Q_\tau Q^\dagger_{\tau'}. \end{aligned} \tag{10}$$

The operator $c^\dagger_{\tau\alpha IM}(c_{\tau\alpha IM})$ creates (annihilates) a particle of type $\tau(=p,n)$ in the state Φ^{IM}_α, when acting on the vacuum state $|0\rangle$. In order to simplify the notations, hereafter the set of quantum numbers $\alpha(=nlj)$ will be omitted. The two body interaction consists of three terms, the pairing, the dipole-dipole particle hole (ph) and the particle-particle (pp) interactions. The corresponding strengths are denoted by G_τ, χ, χ_1, respectively. All of them are separable interactions, with the factors defined by the following expressions:

$$\begin{aligned} P^\dagger_{\tau I} &= \sum_M \frac{2}{2I+1}c^\dagger_{\tau IM}c^\dagger_{\widetilde{\tau IM}}, \\ \beta^-_\mu(pn) &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}}\langle pIM|\sigma_\mu|nI'M'\rangle\frac{\sqrt{2}}{\hat{I}'}c^\dagger_{pIM}c_{nI'M'}, \\ P^-_{1\mu}(pn) &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}}\langle pIM|\sigma_\mu|nI'M'\rangle\frac{\sqrt{2}}{\hat{I}'}c^\dagger_{pIM}c^\dagger_{\widetilde{nI'M'}}, \\ Q^{(\tau)}_{2\mu} &= \sum_{i,k} q^{(\tau)}_{ik}\left(c^\dagger_i c_k\right)_{2\mu}, \quad q^{(\tau)}_{ik} = \sqrt{\frac{2}{2I_k+1}}\langle I_i\|r^2Y_2\|I_k\rangle. \end{aligned} \tag{11}$$

The remaining operators from Eq. (10) can be obtained from the above operators, by hermitian conjugation.

The one body term and the pairing interaction terms are treated first through the standard BCS formalism and consequently replaced by the quasiparticle one body term $\sum_{\tau IM} E_\tau a^\dagger_{\tau IM} a_{\tau IM}$. In terms of quasiparticle creation ($a^\dagger_{\tau IM}$) and annihilation ($a_{\tau IM}$) operators, related to the particle operators by means of the Bogoliubov-Valatin transformation, the two body interaction terms, involved in the model Hamiltonian, can be expressed just by replacing the operators (3.2) by their quasiparticle images. Thus, the Hamiltonian terms describing the quasiparticle correlations become a quadratic expression in the dipole and quadrupole two quasiparticles and quasiparticle density operators:

$$
\begin{aligned}
A^\dagger_{1\mu}(pn) &= \sum_{m_p,m_n} C^{I_p\ I_n\ 1}_{m_p\ m_n\ \mu} a^\dagger_{pI_pm_p} a^\dagger_{nI_nm_n}, \\
B^\dagger_{1\mu}(pn) &= \sum_{m_p,m_n} C^{I_p\ I_n\ 1}_{m_p\ -m_n\ \mu} a^\dagger_{pI_pm_p} a_{nI_nm_n} (-)^{I_n-m_n}, \\
A^\dagger_{2\mu}(\tau\tau') &= \sum_{m_\tau,m_{\tau'}} C^{I_\tau\ I_{\tau'}\ 1}_{m_\tau\ m_{\tau'}\ \mu} a^\dagger_{\tau I_\tau m_\tau} a^\dagger_{\tau' I_{\tau'} m_{\tau'}}, \\
B^\dagger_{2\mu}(\tau\tau') &= \sum_{m_\tau,m_{\tau'}} C^{I_\tau\ I_{\tau'}\ 2}_{m_\tau\ -m_{\tau'}\ \mu} a^\dagger_{\tau I_\tau m_\tau} a_{\tau' I_{\tau'} m_{\tau'}} (-)^{I_{\tau'}-m_{\tau'}},\ \tau,\tau' = p,n.
\end{aligned}
\tag{12}
$$

Since the $pnQRPA$ treatment of the dipole-dipole interaction in the particle-hole (ph) and pp channels run in an identical way as in our previous publications [13, 14], here we do not give any detail about building the dipole proton-neutron phonon operator:

$$
\Gamma^\dagger_{1\mu} = \sum_k [X_1(k) A^\dagger_{1\mu}(k) - Y_1(k) A_{1,-\mu}(k)(-)^{1-\mu}]. \tag{13}
$$

We just mention that the amplitude are determined by the pnQRPA equations and the normalization condition.

The charge conserving $QRPA$ bosons

$$
\Gamma^\dagger_{2\mu} = \sum_k [X_2(k) A^\dagger_{2\mu}(k) - Y_1(k) A_{2,-\mu}(k)(-)^{\mu}], \quad k = (p,p'),(n,n'), \tag{14}
$$

are determined by the $QRPA$ equations associated to the matrices:

$$
\begin{aligned}
\mathcal{A}_{\tau\tau'}(ik;i'k') &= \delta_{\tau\tau'}\delta_{ii'}\delta_{kk'}(E^\tau_i + E^\tau_k) - X_{\tau\tau'}\left(q^{(\tau)}_{ik}\xi^{(\tau)}_{ik}\right)\left(q^{(\tau)}_{i'k'}\xi^{(\tau)}_{i'k'}\right), \\
\mathcal{B}_{\tau\tau'}(ik;i'k') &= -X_{\tau\tau'}\left(q^{(\tau)}_{ik}\xi^{(\tau)}_{ik}\right)\left(q^{(\tau)}_{i'k'}\xi^{(\tau)}_{i'k'}\right), \quad i \le k,\ i' \le k',
\end{aligned}
\tag{15}
$$

where

$$
\xi^{(\tau)}_{ik} = \frac{(U^\tau_i V^\tau_k + U^\tau_k V^\tau_i)}{\sqrt{1+\delta_{i,k}}}. \tag{16}
$$

In order to distinguish between the phonon operators acting in the RPA space associated to the mother and daughter nuclei respectively, one needs an additional index. Also, an index labeling the solutions of the RPA equations is necessary. Thus, the two kinds of bosons will be denoted by:

$$\begin{aligned} &\Gamma^{\dagger}_{1\mu}(jk), \quad j=i,f;\ k=1,2,\ldots,N_s^{(1)};\\ &\Gamma^{\dagger}_{2\mu}(jk), \quad j=i,f;\ k=1,2,\ldots,N_s^{(2)}. \end{aligned} \tag{17}$$

Acting with $\Gamma^{\dagger}_{1\mu}(ik)$ and $\Gamma^{\dagger}_{1\mu}(fk)$ on the vacuum states $|0\rangle_i$ and $|0\rangle_f$ respectively, one obtains two sets of non-orthogonal states describing the intermediate odd-odd nucleus. By contrast, the states $\Gamma^{\dagger}_{2}(ik)|0\rangle_i$ and $\Gamma^{\dagger}_{2}(fk)|0\rangle_f$ describe different nuclei, namely the initial and final ones participating in the process of $2\nu\beta\beta$ decay. The mentioned indices are however omitted whenever their presence is not necessary.

Within the boson expansion formalism the transition GT operators are written as polynomial expansion in terms of the QRPA boson operators with the expansion coefficients determined such that the mutual commutation relations of the constituent operators $A^{\dagger}_{1\mu}(pn), A_{1\mu}(pn), B^{\dagger}_{1\mu}(pn), B_{1\mu}(pn)$ be preserved in each order of approximation [18]. One arrives at the expressions:

$$\begin{aligned}
A^{\dagger}_{1\mu}(j_pj_n) &= \sum_{k_1}\left\{\mathcal{A}^{(1,0)}_{k_1}(j_pj_n)\Gamma^{\dagger}_{1\mu}(k_1)+\mathcal{A}^{(0,1)}_{k_1}(j_pj_n)\Gamma_{1-\mu}(k_1)(-)^{1-\mu}\right\}\\
&+\sum_{k_1,k_2,k_3;l=0,2}\left\{\mathcal{A}^{(3,0);l}_{K_3k_2k_1}(j_pj_n)\left[\left(\Gamma^{\dagger}_2(k_3)\Gamma^{\dagger}_2(k_2)\right)_l\Gamma^{\dagger}_1(k_1)\right]_{1\mu}\right.\\
&+\left.\mathcal{A}^{(0,3);l}_{K_3k_2k_1}(j_pj_n)\left[(\Gamma_2(k_3)\Gamma_2(k_2))_l\,\Gamma_1(k_1)\right]_{1\mu}\right\}\\
&+\sum_{k_1,k_2,k_3;l=0,2}\left\{\mathcal{A}^{1;(2\bar{2})l}_{K_1k_2k_3}(j_pj_n)\left[\Gamma^{\dagger}_1(k_1)\left(\Gamma^{\dagger}_2(k_2)\Gamma_2(k_3)\right)_l\right]_{1\mu}\right.\\
&+\left.\mathcal{A}^{(2\bar{2})l;1}_{K_3k_2k_1}(j_pj_n)\left[\left(\Gamma^{\dagger}_2(k_3)\Gamma_2(k_2)\right)_l\Gamma_1(k_1)\right]_{1\mu}\right\},\\
B^{\dagger}_{1\mu}(j_pj_n) &= \sum_{k_1k_2}\left\{\mathcal{B}^{(2,0)}_{k_1k_2}(j_pj_n)\left[\Gamma^{\dagger}_1(k_1)\Gamma^{\dagger}_2(k_2)\right]_{l\mu}+\mathcal{B}^{(0,2)}_{k_1k_2}(j_pj_n)[\Gamma_1(k_1)\Gamma_2(k_2)]_{l\mu}\right.\\
&+\left.\mathcal{B}^{11;12}_{k_1k_2}(j_pj_n)\left[\Gamma^{\dagger}_1(k_1)\Gamma_2(k_2)\right]_{l\mu}+\mathcal{B}^{11;2l}_{k_1k_2}(j_pj_n)\left[\Gamma^{\dagger}_1(k_2)\Gamma_1(k_1)\right]_{l\mu}\right\},
\end{aligned} \tag{18}$$

where the expansion coefficients are those given in Ref. [15]. If the energy carried by leptons in the intermediate state is approximated by the sum of the rest energy of the emitted electron and half the Q-value of the double beta decay process

$$\Delta E = m_e c^2 + \frac{1}{2} Q^{(0\rightarrow 2)}_{\beta\beta}, \tag{19}$$

the reciprocal value of the $2\nu\beta\beta$ half life can be factorized as:

$$T^{2\nu}_{1/2}(0^+_i \rightarrow 2^+_f)^{-1} = F_2 |M^{(02)}_{GT}|^2, \tag{20}$$

where F_2 is the Fermi integral which characterizes the phase space of the process while the second factor is the GT transition amplitude which, in the second order of perturbation theory, has the expression:

$$M^{(02)}_{GT} = \sqrt{3} \sum_{k,m} \frac{{}_i\langle 0\|\beta^+\|k,m\rangle_i \; {}_i\langle k,m|k',m'\rangle_f \; {}_f\langle k',m'\|\beta^+\|2^+_1\rangle_f}{(E_{k,m}+\Delta E_2)^3}. \tag{21}$$

Here $\Delta E_2 = \Delta E + E_{1^+}$, with E_{1^+} standing for the experimental energy for the first state 1^+. The intermediate states $|k,m\rangle$ are k-boson states with $k = 1,2,3$ labeled by the index m, specifying the spin and the ordering label of the RPA roots. Inserting the boson expansions from Eq. (19) into the expression of the β^+ transition operator one can check that the following non-vanishing factors, at numerator, show up:

$$\begin{aligned}
&{}_i\langle 0\|\Gamma_1(i,k_1)\|1,1_{k_1}\rangle_{if}\langle 1,1_{k_2}\|\Gamma^\dagger_1(f,k_2)\Gamma_2(f,1)\|1,2_1\rangle_f,\\
&{}_i\langle 0\|\Gamma_1(i,k_1)\Gamma_2(i,k_2)\|2,1_{k_1}2_{k_2}\rangle_{if}\langle 2,1_{j_1}2_1\|\Gamma^\dagger_1(f,j_1)\|1,2_1\rangle_f,\\
&{}_i\langle 0\|\Gamma_1(i,k_1)\Gamma_2(i,k_2)\Gamma_2(i,k_3)\|3,1_{k_1}2_{k_2}2_{k_3}\rangle_{if}\langle 3,1_{j_1}2_{j_2}2_1\|\Gamma^\dagger_1(f,j_1)\Gamma^\dagger_2(f,j_2)\|1,2_1\rangle_f,\\
&{}_i\langle 0\|\Gamma_1(i,k_1)\Gamma_2(i,k_2)\|2,1_{k_1}2_{k_2}\rangle_{if}\langle 2,1_{j_1}2_{j_2}\|\Gamma^\dagger_1(f,j_1)\Gamma^\dagger_2(f,j_2)\Gamma_2(f,1)\|1,2_1\rangle_f.
\end{aligned} \tag{22}$$

The term $E_{k,m}$ from the denominator of Eq. (21) is the average of the energies of the mother and daughter states $|k,m\rangle$ normalized to the average energy of the first $pnQRPA$ states 1^+ in the initial and final nuclei. Calculations were performed for the following 6 double beta emitters: ^{48}Ca, ^{76}Ge, ^{96}Zr, ^{100}Mo, ^{116}Cd, ^{130}Te. Since the single particle space, the pairing interaction treatment, and the pnQRPA description of the dipole states describing the intermediate odd-odd nuclei used in the present paper are identical with those from Refs. [13, 14] for ground to ground transition, we don't present them again. The strength of the QQ interaction was fixed by requiring that the first root of the QRPA equation for the quadrupole charge conserving boson is close to the experimental energy of the first 2^+ state. To save the space the results of the fitting procedure will not be given.

Having the RPA states defined, the GT amplitude has been calculated by means of Eq. (21), while the half life with Eq. (20). The Fermi integral for the transition $0^+ \rightarrow 2^+$ was computed by using the analytical result given in Ref. [4]. The final results are collected in Table 1.

Table 1

The GT transition amplitudes and the half lives of the double beta decay $0^+ \to 2^+$ are given. Also the Q values are given in units of m_ec^2. ΔE_2 is the energy shift defined in the text. For comparison, we give also the available experimental results as well as some theoretical predictions obtained with other formalisms. The M_{GT} values for the ground to ground transitions are also listed. For ^{100}Mo we mention the result of Ref. [20] obtained with an SU(3) deformed single particle basis $^{a)}$ and with a spherica l basis $^{b)}$

Nucleus	$Q^{2^+}_{\beta\beta}$	$\lvert M^{(0\to2)}_{GT}\rvert$	$T^{(0\to2)}_{1/2}[yr]$		
	[m_ec^2]	[MeV^{-3}]	present	Exp.	Ref. [19]
^{48}Ca	6.432	0.901×10^{-3}	1.72×10^{24}		
^{76}Ge	2.894	0.558×10^{-3}	5.75×10^{28}	$>1.1\times10^{21}$	1.0×10^{26}
^{96}Zr	5.033	0.834×10^{-3}	2.27×10^{25}	$> 7.9\times10^{19}$	4.8×10^{21}
^{100}Mo	4.874	0.136×10^{-2}	1.21×10^{25}	$>1.6\times10^{21}$	3.9×10^{24}
					$^{a)}2.5\times10^{25}$
					$^{b)}1.2\times10^{26}$
^{116}Cd	2.967	0.507×10^{-2}	3.4×10^{26}	$>2.3\times10^{21}$	1.1×10^{24}
^{130}Te	3.902	0.620×10^{-3}	6.94×10^{26}	$>4.5\times10^{21}$	2.7×10^{23}

Therein one may find also the available experimental data as well as some theoretical results obtained with other approaches. One notices that the half life is influenced by both the phase space integral (through the Q-value) and the single particle properties which determine the transition amplitude. Indeed, for ^{128}Te and ^{134}Xe the small Q-value causes a very large half life, while in ^{48}Ca the opposite situation is met. By contrary the Q-value of ^{110}Pd is about the same as for ^{76}Ge but, due to the specific single particle and pairing properties of the orbits participating coherently to the process, the half life for the former case is more than three orders of magnitude less than in the later situation. The results for ^{134}Xe and ^{110}Pd mentioned above will be published elsewhere. The transition matrix elements reported in Ref. [19] are larger than those given here, despite the fact that the higher pnQRPA approaches in the two descriptions are similar [21]. The reason is that there a spherical single particle basis is used whereas here we use a deformed basis. The same effect of deformation on the GT matrix elements was pointed out by Zamick and Auerbach in Ref. [22]. Indeed, they calculated the GT transition matrix elements for the neutrino capture $\nu_\mu +^{12}\mathrm{C} \to^{12}\mathrm{N} + \mu^-$ using different structures for the ground states of ^{12}C and ^{12}N: a) spherical ground states; b) asymptotic limits of the wave functions and 3) deformed states with an intermediate deformation of $\delta = -0.3$. The results for the transition rate were $\frac{16}{3}$, 0 and 0.987, respectively. Similar results are obtained also for the spin M1 transitions in ^{12}C. The ratio between the transition rates obtained with spherical and deformed basis explains the factor of 5 overestimate in the calculations of Ref. [23], where a spherical basis is used. It is worth mentioning the good agreement between our prediction for ^{100}Mo and that of Ref. [20] obtained with a deformed SU(3) single particle basis.

To have a reference value for the matrix elements associated to the transition $0^+ \rightarrow 2^+$, one should compare them with the M_{GT} values for the ground to ground transitions given in Ref. [14]. The ratio of the transition $0^+ \rightarrow 0^+$ and $0^+ \rightarrow 2^+$ matrix elements is quite large for ^{76}Ge (398), ^{100}Mo (224) and ^{96}Zr (136) but small for ^{110}Pd (5.26) and ^{134}Xe (6.3). However, these ratios are not directly reflected in the half lives, since the phase space factors for the two transitions are very different from each other and moreover the differences depend on the atomic mass of the emitter.

It is worth mentioning that the double beta transitions to excited states have been considered by several authors in the past, but the calculations emphasized the role of the transition operator and some specific selection rules. Many of calculations regarded the neutrinoless process. Thus, in Ref. [24] it was shown that the neutrinoless transition to the excited 0^+ for medium heavy nuclei might be characterized by matrix elements which are larger than that of ground to ground transition and that happens since in the first transition, the change of the K quantum number is less. In Ref. [25] it has been stated that the $0^+ \rightarrow 2^+$ matrix element depends on the left-right current coupling and not on the neutrino mass. This could provide a way of fixing the strength of the left-right coupling if the transition matrix element is experimentally known. However, according to the calculations of Haxton *et al.* [2], the matrix element is strongly suppressed and therefore the mentioned method of fixing the coupling parameter would not be reliable. Although the transition operator might have a complex structure, many calculations have been performed with the approximate interaction $[\sigma(1) \times \sigma(2)]^{\lambda=2} t_+(1) t_+(2)$ in order to test some selection rules. Thus, this interaction was used in Ref. [26] for the transition $0^+ \rightarrow 2^+$ of ^{48}Ca, using a single j calculation. It has been proved that the matrix element for this transition is suppressed due to the signature selection rules. Actually, this result confirms the feature of suppression for the $0^+ \rightarrow 2^+$ double beta transition matrix element pointed out by Vergados [27] and Haxton *et al.* [2]. The transition to 0^+_1 was examined for $A = 76, 82, 100, 136$ nuclei by assuming light and heavy Majorana neutrino exchange mechanism and triliniar R-parity contribution. Higher RPA as well as renormalization effects for the nuclear matrix elements were included [28].

Here we show that the transition $0^+ \rightarrow 2^+$ in a $2\nu\beta\beta$ process is allowed by renormalizing the GT transition operator with some higher RPA corrections which results in making the matrix elements from Eq. (22) non-vanishing. The calculated M_{GT} values of the present work are smaller than those from Ref. [19] obtained with a spherical single particle basis, which agrees with the earlier calculations of Zamick and Auerbach for ^{12}C, showing that the nuclear deformation suppresses the GT matrix elements.

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