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Dedicated to Prof. Dorin N. Poenaru's 70th Anniversary

GENERALIZED COMMUTATION RELATIONS

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Abstract. We show how to handle extended symplectic structures in classical and quantum mechanics. Some of the consequences are quite surprising: loss of gauge invariance in the presence of an electromagnetic background, or absence of a Lagrangian formulation for the equations of motion. Nevertheless, the dynamics can always be described satisfactorily using an extended Hamiltonian formalism.

Key words: noncommutative mechanics, Hamiltonian dynamics, symplectic geometry.

1. INTRODUCTION

This volume of Romanian Reports in Physics being dedicated to the aniversary of Professor Dorin Poenaru – a distinguished nuclear theorist – one may rightfully expect to find here contributions related to the real world of the atomic nucleus. The present one offers a variation by going in a somewhat different direction. Instead of aplying quantum mechanics to a given physical system and comparing the results with experiment, it speculates on how simple modifications of the basic theory can affect our perspective. In short, it explores to a certain extent the effects not of the Hamiltonian, but of the symplectic structure (Poisson brackets classically, commutation relations quantum mechanically), a structure which is given *independently* of the choice of the Hamiltonian. The ensuing discussion is not only of academic interest.

It applies to charged particles evolving in external magnetic fields (e.g. the celebrated quantum Hall effect), and has been recently of great interest in the theory of relativistic strings in various backgrounds. The possible relevance of this extended framework to the dynamics of a nucleon in the effective field of the others is an open issue. Here, we will concentrate mostly on setting up an efficient formalism, a necessary step for making reliable computations in generic models, and understanding what our expectations might be.

As is well known [1], once the degrees of freedom of a physical system are identified, the dynamics is determined by two elements:

1. The Hamiltonian, which is often quadratic in the momenta and e.g. of a finite power series form in the coordinates, symbolically

$$H \sim p^2 + V(q). \tag{1}$$

2. The symplectic structure, determined by the Poisson brackets in classical mechanics

$$\{q^i, q^j\} = 0, \quad \{q^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = 0, \tag{2}$$

and by the Heisenberg commutation relations in quantum mechanics

$$[\hat{q}^i, \hat{q}^j] = 0, \quad [\hat{q}^i, \hat{p}_j] = i\delta^i_j, \quad [\hat{p}_i, \hat{p}_j] = 0. \tag{3}$$

Standard notation is used above, q_i and p_j representing the generalized canonical coordinates and momenta of the system under consideration (and $\hbar \equiv 1$).

A basic but fundamental consequence of the above equations is the following. Due to (3) – more precisely to the fact that coordinates commute – one can of course use the coordinate representation $\hat{p}_i = -i\frac{\partial}{\partial q_i}$. Given now the (say, stationary) Schrödinger equation $\hat{H}\Psi = E\Psi$ it is clear that quadracity of $\hat{H}$ in the momenta $\hat{p}_i$ leads to the usual *second order* partial differential equation. For this to happen, we stress that we needed simultaneously a Hamiltonian quadratic in the momenta *and* $[\hat{q}^i, \hat{q}^j] = 0$. In this paper, we will mainly lift this second constraint, exploring some of the consequences of the following commutation relations

$$[\hat{q}^i, \hat{q}^j] = i\theta^{ij}(\hat{q}, \hat{p}), \quad [\hat{q}^i, \hat{p}_j] = i\delta^i_j, \quad [\hat{p}_i, \hat{p}_j] = iF_{ij}(\hat{q}, \hat{p}). \tag{4}$$

We will restrict here to the case of constant θ, the so-called noncommutative mechanics – a restricted set of references is given at the end ([2] to [19]). The review will be biased by stressing mainly aspects studied in [17, 18, 19]. For results concerning nonconstant θ one can consult for instance [20].

2. **CLASSICAL DYNAMICS**

2.1. EQUATIONS OF MOTION

We begin by studying the classical theory. In addition to the Hamiltonian, one starts from the classical analogue of (4), namely the generalized Poisson brackets

$$\{q^i, q^j\} = \theta^{ij}, \quad \{q^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = F_{ij}. \tag{5}$$

For simplicity in the notation we will work in (2+1)-dimensions, although the extension to higher dimensionalities is straightforward. We denote by x_a, $a = 1, 2, 3, 4$ the phase space coordinates, $x_{1,2,3,4} = q_1, p_1, q_2, p_2$. Eqs. (5) can then be rewritten as $\{x_i, x_j\} = \Theta_{ij}$, where

$$\Theta = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & \sigma \\ 0 & -1 & -\sigma & 0 \end{pmatrix}, \quad \text{i.e.} \quad \omega = \frac{1}{1-\theta\sigma} \begin{pmatrix} 0 & -\sigma & 1 & 0 \\ \sigma & 0 & 0 & 1 \\ -1 & 0 & 0 & -\theta \\ 0 & -1 & \theta & 0 \end{pmatrix}. \tag{6}$$

Above, $\Theta_{ij} = (\omega^{-1})_{ij}$, and ω is the symplectic form, which enters the action

$$S = \int dt \left(\frac{1}{2} \omega_{ij} x_i x_j - H(x) \right). \tag{7}$$

Independent variation of S along each x_a provides the equations of motion

$$\dot{x}_i = \{x_i, H\} = \Theta_{ij} \frac{\partial H}{\partial x_j}, \tag{8}$$

more explicitely

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + \theta \epsilon_{ij} \frac{\partial H}{\partial q_j}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \sigma \epsilon_{ij} \frac{\partial H}{\partial p_j}, \tag{9}$$

with $\epsilon_{12} = -\epsilon_{21} = 1$. If $\theta = \sigma = 0$, (9) are the usual Hamilton equations.

Important remark. By taking at the level of the equations of motion the limit $\theta\sigma = 1$, which renders Θ (and ω) singular, one reduces to one half the number of degrees of freedom having dynamics, since

$$\dot{q}_1 = -\theta \dot{p}_2 \quad \text{and} \quad \dot{q}_2 = \theta \dot{p}_1. \tag{10}$$

Let us express the equations of motion in terms of q_1 and q_2. Considering Hamiltonians of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{11}$$

one gets the coordinate equations of motion:

$$\ddot{q}_i = -(1-\theta\sigma)\frac{\partial V}{\partial q_i} + \sigma\epsilon_{ij}\dot{q}_j + \theta\epsilon_{ij}\frac{d}{dt}\frac{\partial V}{\partial q_j}, \quad i = 1, 2. \tag{12}$$

For $\theta \neq 0$, it is easy to see that equations (12) are not derivable from a Lagrangian, if the potential V is higher than quadratic in the coordinates. This is shown by finding a Lagrangean formulation (unique up to total derivatives) for one of the two equations in (12), and seeing in which cases the second equation in (12) can be derived from it [17], or by using the so-called Helmholtz conditions which are to be obeyed by a set of equations admitting a variational formulation [19].

The system does not appear to be dissipative, since a notion of conserved energy exists. For quadratic Hamiltonians, the effects of noncommutativity are trivially described; they correspond to adding either constant forces, or magnetic fields, or harmonic oscillator potentials, into the Lagrangian for the commutative case. We will not detail this here, but stress that the classical effects of noncommutativity seem to be truly relevant only in the presence of nonlinear equations of motion. The lack of a Lagrangian formulation reflects the difficulty in defining a suitable Legendre transform if $\theta \neq 0$. In fact, if one would start from the Hamiltonian (11) and perform a usual Legendre transformation, then use (9) to express $p_{1,2}$ in terms of $q_{1,2}$, one would get wrong equations of motion. The usual procedure works correctly only if $\theta = 0$. On the other hand, σ is quite harmless; it plays the role of a constant magnetic field.

The RHS of (12) contains three kinds of terms. The first, $-(1-\theta\sigma)\frac{\partial V}{\partial q_i}$, is just the usual Newtonian force, apart from the $(1-\theta\sigma)$ factor. The second term, $\sigma\epsilon_{ij}\dot{q}_j$, mimicks a magnetic field. It is the third term which prevents the Lagrangian formalism from working. However, if it is taken in isolation, it gives

$$\ddot{q}_i = \theta\epsilon_{ij}\frac{d}{dt}\frac{\partial V}{\partial q_j}, \quad \text{i.e.} \quad \dot{q}_i = \theta\epsilon_{ij}\frac{\partial V}{\partial q_j} + c_i, \quad i = 1, 2, \tag{13}$$

c_i being two arbitrary constants. Eqs. (13) allow a first-order Lagrangian formulation, since (in a reference frame moving with velocity $\vec{v} = (c_1.c_2)$) the equations $\dot{q}_i = \theta\epsilon_{ij}\frac{\partial V}{\partial q_j}$ follow from the Lagrangian $L = \frac{1}{2}(\dot{q}_1 q_2 - q_1\dot{q}_2) - V(q_i)$.

Most important, the full equations (12) admit a first order Lagrangian description, in the limit

$$\sigma \to \theta^{-1}. \tag{14}$$

In this case, the usual Newtonian force term disappears completely (this is kind of antipodal to usual Hamiltonian dynamics), and (12) becomes

$$\dot{q}_i = \epsilon_{ij}\left(\frac{q_j}{\theta} + \theta\frac{\partial V}{\partial q_j}\right) + C_i. \tag{15}$$

A first-order Lagrangian for (15) is:

$$L = \frac{1}{2}(\dot{q}_1 q_2 - \dot{q}_2 q_1) - \theta V(q_i) - \frac{1}{2\theta}(q_1^2 + q_2^2) + C_2 q_1 - C_1 q_2. \tag{16}$$

This Lagrangian contains a term which is first order in time derivatives, the usual potential V, and an additional two-dimensional harmonic oscillator potential.

The limit (14) reduces the number of degrees of freedom of the phase-space to half, from four to two, cf. (10). This is most clearly seen by noticing that (15) arises from the one-dimensional Hamiltonian

$$H = \theta V(q,p) + \frac{1}{2\theta}(q^2 + p^2) - C_2 q + C_1 p \tag{17}$$

after relabeling $q_1 = q$, $q_2 = p$. This is similar (but not identical) to the dimensional reduction involved in the Peierls substitution [21], which is based on the noncommutativity of coordinates (see [22], which also refers to earlier work). The connection is provided by the fact that in a two-dimensional first-order system, the coordinates are canonically conjugate to each other.

In conclusion, we argued that if $\theta \neq 0$ the equations of motion do not admit in general a variational formulation – a quite unexpected result. One can of course block diagonalize Θ by linear *non-canonical* transformations mixing the q's and p's. A canonical symplectic structure (2) would then result, but also a Hamiltonian not anymore quadratic in the (new) momenta. Hence one will not be able to eliminate the momenta from the equations of Hamilton, and again no explicit Lagrangian formulation will be available. Still, consistent quantization is possible, as will be shown in Section 3.

2.2. EXAMPLES

We proceed with examples which do not admit a Lagrangian formulation, and display some of their features [19].

Consider first the anisotropic harmonic oscillator potential, $V = \frac{1}{2}(a_1 q_1^2 + a_2 q_2^2)$, which gives the equations of motion

$$m\ddot{q}_1 = -(1-\theta\sigma)a_1 q_1 + (\sigma + \theta m a_2)\dot{q}_2, \tag{18}$$

$$m\ddot{q}_2 = -(1-\theta\sigma)a_2 q_2 - (\sigma + \theta m a_1)\dot{q}_1. \tag{19}$$

If we chose $\sigma + m\theta a_2 = 0$, then $\sigma + m\theta a_1 \neq 0$, provided $a_1 \neq a_2$. q_1 becomes a harmonic oscillator, whereas q_2 is a harmonic oscillator driven by a periodic force $m\theta(a_1 - a_2)\dot{q}_1$. The solution for q_1 is the usual one, $q_1(t) = q_1(0)\cos\omega_1 t + (q_1'(0)/\omega_1)\sin\omega_1 t$, whereas for q_2 it reads

$$q_2(t) = q_2(0)\cos\omega_2 t + \frac{q_2'(0)}{\omega_2}\sin\omega_2 t + \theta m \frac{q_1'(0)\cos\omega_1 t - \omega_1 q_1(0)\sin\omega_1 t}{1-\theta\sigma}. \tag{20}$$

Above, $m\omega_i^2 = (1-\theta\sigma)a_i$, $i = 1,2$. If θ is small, the last term in Eq. (20) is a perturbation which produces oscillations around the commutative trajectory. The particle goes on a wiggly path, which averages to the commutative one. If θ is big, or if $|1-\theta\sigma| \ll 1$, the "perturbation" explodes and dominates the dynamics, which becomes completely different from the commutative one. One sees a qualitative difference between a NC isotropic oscillator (which admits a Lagrangian form) and a NC anisotropic one (no Lagrangian form).

As a second example consider, commutatively speaking, a constant force along q_2, and a harmonic one along q_1, $V = \frac{1}{2}a_1 q_1^2 + bq_2$. The equations of motion are

$$m\ddot{q}_1 = -(1-\theta\sigma)a_1 q_1 + \sigma\dot{q}_2, \tag{21}$$

$$m\ddot{q}_2 = -(1-\theta\sigma)b - (\sigma + \theta m a_1)\dot{q}_1. \tag{22}$$

If $\sigma = 0$, again q_1 is a harmonic oscillator, while q_2 is driven by a constant plus periodic force. The solution is the usual harmonic oscillator for q_1, while for q_2 one has

$$q_2(t) = q_2(0) + [q_2'(0) + q_1(0)\theta a_1]t - \frac{bt^2}{2m} -$$

$$-\theta a_1 \left[\frac{q_1(0)}{\omega_1}\sin\omega_1 t - \frac{q_1'(0)}{\omega_1^2}(1-\cos\omega_1 t)\right]. \tag{23}$$

Again, the NC trajectory wiggles around the commutative one. On the other hand, if $\sigma + \theta m a_1 = 0$, q_2 feels a constant force, while the oscillator q_1 is driven by a linearly time-dependent force $\sigma\dot{q}_2$. One has the solution $q_2(t) =$ $= q_2(0) + tq_2'(0) - (1-\theta\sigma)\frac{bt^2}{2m}$, but

$$q_1(t) = q_1(0)\cos\omega_1 t + \frac{q_1'(0)}{\omega_1}\sin\omega_1 t + \frac{\sigma}{a_1}\left[\frac{q_2'(0)}{(1-\theta\sigma)} - \frac{b}{m}t\right]. \tag{24}$$

A drastic change occurs: q_1 grows linearly with time (it is not bounded anymore), and oscillates around this path as a commutative oscillator.

As a third example, consider a potential which depends only on one coordinate, say $V = V(q_1)$. If $\sigma = 0$ the equations of motion are

$$m\ddot{q}_1 = -\partial_1 V, \quad m\ddot{q}_2 = -\theta m\frac{d}{dt}\partial_1 V = -\theta m^2 \frac{d^3 q_1}{dt^3}. \tag{25}$$

If $\theta \neq 0$, q_1 transfers nontrivial dynamics to q_2. More precisely, once $q_1(t)$ is known (its implicit form is $t(q_1) = \int_0^{q_1} \frac{dq'}{\sqrt{V(0)-V(q')}}$), q_2 is fixed by the second equation in (25). To illustrate, consider the quartic potential $V(q_1) =$ $= V(0) - \frac{1}{2}m^2 q_1^2 + gq_1^4$. One can not find simple expressions for $q_1(t)$ in a nonlinear problem in general. However, the classical solution satisfying $q_1(t=-\infty)=0$ and $q_1(t=0)=\frac{m}{\sqrt{g}}=\lambda$ is simple enough

$$q_1(t)=\frac{m}{\sqrt{g}}\frac{2\mathrm{e}^{-mt}}{1+\mathrm{e}^{-2mt}}. \tag{26}$$

Calculating $q_2(t)$ via (25) one obtains

$$q_2(t)=q_2(0)+q_2'(0)t-\theta m\dot{q}_1(t), \tag{27}$$

radically different from the $\theta=0$ expression, $q_2(t)=q_2(0)+q_2'(0)t$.

Time-dependent backgrounds appearing "out-of-nowhere" (actually being induced by the dynamics of the other degrees of freedom) are thus possible in NC dynamics.

2.3. GAUGE INVARIANCE

Another simple type of Hamiltonian worth studying is

$$H=\frac{1}{2}\sum_{i=1,2}(p_i-A_i(q_j))^2, \tag{28}$$

the gauge field A_i being minimally coupled. If the symplectic structure is given by (5, 6) then

$$\dot{q}_i=(p_l-A_l)\left(\delta_{il}-\theta\epsilon_{ij}\frac{\partial A_l}{\partial q_j}\right), \tag{29}$$

$$\dot{p}_i=(p_j-A_j)\left(\frac{\partial A_j}{\partial q_i}+\sigma\epsilon_{ij}\right). \tag{30}$$

Assuming $\frac{\partial A_j}{\partial t}=0$ for simplicity, the pair (29) can be rewritten as

$$p_i=A_i+\frac{1}{\Delta}\frac{d}{dt}(q_i+\theta\epsilon_{ij}A_j), \quad i=1,2, \tag{31}$$

where $\Delta=1+\theta F_{12}+\theta^2\{A_1,A_2\}_{q_1q_2}$, with $F_{12}=\partial_1A_2-\partial_2A_1$, $\{A_1,A_2\}_{q_1q_2}=$ $=\frac{\partial A_1}{\partial q_1}\frac{\partial A_2}{\partial q_2}-\frac{\partial A_1}{\partial q_2}\frac{\partial A_2}{\partial q_1}$. Using (31) in (30), and assuming $\frac{\partial A_1}{\partial q_1}=\frac{\partial A_2}{\partial q_2}=0$, one gets

$$\ddot{q}_1=\left(1+\theta\frac{\partial A_1}{\partial q_2}\right)\left[-\dot{A}_1+\left(\frac{\partial A_2}{\partial q_1}+\sigma\right)\frac{\dot{q}_2}{1-\theta\frac{\partial A_2}{\partial q_1}}\right], \tag{32}$$

$$\ddot{q}_2=\left(1+\theta\frac{\partial A_2}{\partial q_1}\right)\left[-\dot{A}_2+\left(\frac{\partial A_1}{\partial q_2}-\sigma\right)\frac{\dot{q}_1}{1-\theta\frac{\partial A_1}{\partial q_2}}\right]. \tag{33}$$

Let us consider the case of a constant magnetic field, $B=F_{12}=\partial_1A_2-\partial_2A_1$. This can be obtained in different gauges. A striking feature of the equations

(32,33) is that they are *not* gauge invariant, unless $\theta = 0$. For instance, in the symmetric gauge, $A_1 = -q_2 B/2, A_2 = q_1 B/2$, one has

$$\ddot{q}_1 = \dot{q}_2(\sigma + B + \theta B^2/4), \quad \ddot{q}_2 = -\dot{q}_1(\sigma + B + \theta B^2/4), \tag{34}$$

whereas in the gauge $A_1 = 0,\ A_2 = q_1 B$ one gets

$$\ddot{q}_1 = \dot{q}_2 \frac{(\sigma + B)}{(1 + \theta B)}, \quad \ddot{q}_2 = -\dot{q}_1(\sigma + B)(1 + \theta B), \tag{35}$$

which is not even derivable from a Lagrangian. One sees again that σ is inoffensive – it just adds to B – whereas θ even breaks gauge invariance!

Thus, after the existence of a Lagrangian, a second cherished principle is lost due to $\theta \neq 0$ – gauge invariance. Since a non-zero σ mimicks a constant magnetic field, the remedy we propose is to account for a magnetic field B not through the Hamiltonian – which remains free, but through the symplectic form, by requiring $\{p_1, p_2\} = B$. When $\theta = 0$ this is equivalent to (28), and does not pose problems when $\theta \neq 0$.

3. **QUANTUM THEORY**

3.1. QUANTUM MECHANICS: FORMULATION

We extend the three main formalisms of quantum mechanics (operatorial, Schrödinger, path integral) to the case of noncommuting coordinates.

Operatorial quantization is trivially implemented using Eqs (4, 6):

$$\frac{d}{dt}\hat{x}_a = i[\hat{x}_a, H] = i[\hat{x}_a, \hat{x}_b]\frac{\partial H}{\partial \hat{x}_b} = \Theta_{ab}\frac{\partial H}{\partial \hat{x}_b}. \tag{36}$$

The equations of motion (36) are an extension of the usual Heisenberg ones. They are the same as (8), with the coordinates becoming operators.

A Schrödinger (wave function) formulation can be constructed as follows. First, chose a basis in the Hilbert space on which the operators $\hat{x}_a$ act, for instance $|q_1, p_2>$, i.e. the eigenstates of the operators $\hat{q}_1$ and $\hat{p}_2$. Second, for an arbitrary state $|\psi>$, define the wave function (half in coordinate space, half in momentum space)

$$\psi(q_1, p_2, t) \equiv \langle \psi(t) | q_1, p_2 \rangle. \tag{37}$$

The commutation relations (4) imply that the operators $\hat{q}_2$ and $\hat{p}_1$ have the following action on ψ:

$$\hat{q}_2\psi = i(\partial_{p_2} - \theta\partial_{q_1})\psi, \quad \hat{p}_1\psi = i(-\partial_{q_1} + \sigma\partial_{p_2})\psi. \tag{38}$$

If $H = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + V(\hat{q}_1, \hat{q}_2)$, (38) leads to the Schrödinger equation

$$i\frac{d}{dt}\psi = H\psi = \left[\frac{1}{2m}\left(p_2^2 - (\partial_{q_1} - \sigma\partial_{p_2})^2\right) + V(q_1, i\partial_{p_2} - i\theta\partial_{q_1})\right]\psi(q_1, p_2). \quad (39)$$

If $\sigma = 0$, a momentum-space wave function $\psi(p_1, p_2, t)$ also exists; it will be discussed later.

A phase space path integral for systems obeying the commutation relations (4) was constructed in [18]. Since we saw that for generic systems, if $\theta \neq 0$, equations (8) do not admit a Lagrangian formulation, one can at best hope for a phase-space path integral formulation of the quantum theory corresponding to the action (7).

This is provided [18] by the path integral

$$Z = \int \prod_{k=1}^{4} Dx_k e^{iS} = \int \prod_{k=1}^{4} Dx_k e^{i\int dt\left(\frac{1}{2}\omega_{ij}x_i\dot{x}_j - H(x)\right)}. \quad (40)$$

To put it briefly the prescription (40) is simple: if $[\hat{x}_i, \hat{x}_j] = i\Theta_{ij}$ then $Z = \int Dx e^{i\int dt(\Theta_{ij}^{-1}\frac{x_i\dot{x}_j}{2} - H)}$, and general: it applies to any Hamiltonian H. The above path integral can be derived by elementary means from the canonical formulation [18]. All one needs to know is that Z represents a transition amplitude between two states of a given Hilbert space, and that time-ordering of operators is enforced, as usual, by the path integral, $\int Dx O_1 O_2 e^{iS} = \langle T\{\hat{O}_1\hat{O}_2\}\rangle$.

3.2. QUANTIZATION: EXAMPLES

We apply the formalism to the examples considered classically in Section 2.

For a harmonic potential, it can be shown by path integrals [18], or operatorially [2], that the only change induced by NC is an anisotropy of the oscillator. However, *starting* with an anisotropic oscillator, $V = \frac{1}{2}(a_1q_1^2 + a_2q_2^2)$, $a_1 \neq a_2$, makes an important difference. The equations of motion are the same as in (18,19), with $q_{1,2}$ operators. For simplicity, assume $\sigma + m\theta a_2 = 0$; then $\sigma + m\theta a_1 \neq 0$. $\hat{q}_2$ is driven by a periodic force and, being of the form (20), transitions between the states of the quantum system will appear.

Our second example, $V = \frac{1}{2}a_1q_1^2 + bq_2$, also exhibits peculiar behaviour. If $\sigma = 0$, the operator solutions of (21,22) again involve transitions which would be absent if $\theta = 0$. If $\sigma + \theta m a_1 = 0$, changes are more dramatic. Eq. (24) shows that the particle is not bounded anymore along q_1, in contrast with the commutative case.

Third, consider the case in which the potential depends only on one coordinate, $V = V(q_1)$. If $\sigma = 0$ an interesting phenomenon takes place.

The commutation relations (4) admit a representation in the basis $|p_1, p_2>$, $\psi(p_1, p_2, t) \equiv \langle\psi(t)|p_1, p_2\rangle$:

$$\hat{q_1}\psi = (i\partial_{p_1} + \theta\alpha p_2)\psi, \quad \hat{q_2}\psi = (i\partial_{p_2} + \theta(1+\alpha)p_1)\psi(p_1, p_2), \tag{41}$$

with α a parameter, and the Schrödinger equation becomes

$$i\frac{d}{dt}\psi = \left[\frac{1}{2m}\left(p_1^2 + p_2^2\right) + V(i\partial_{p_1} + \theta\Lambda p_2, i\partial_{p_2} + \theta(1+\Lambda)p_1)\right]\psi(p_1, p_2). \tag{42}$$

This equation is (gauge) invariant under shifts of α by Λ,

$$\alpha \to \alpha - \Lambda \tag{43}$$

combined with multiplications of the momentum-space wave-function by a phase $e^{i\Lambda\theta p_1 p_2}$,

$$\psi(p_1, p_2) \to e^{i\Lambda\theta p_1 p_2}\psi(p_1, p_2). \tag{44}$$

θ plays the role of a "magnetic field" in momentum space.

In particular, when $\Lambda = \alpha$, $\hat{q}_1$ becomes θ-independent. Then, if $V = V(q_1)$ and $\sigma = 0$, the Schrödinger equation is θ-independent. It has consequently the same spectrum with the commutative problem, although classically the NC system does not even admit a Lagrangian formulation! For example, $V(q_1, q_2) = V(q_1) = V(0) - \frac{1}{2}m^2 q_1^2 + g q_1^4$, on a NC space, gives rise to a nonlinear system without classical Lagrangian formulation, but which has the same spectrum as the corresponding commutative (Lagrangian) system.

If $V = V(q_1, q_2)$ the above gauge invariance persists, but does not eliminate θ from the wave equation.

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