

Dedicated to Prof. Dorin N. Poenaru's
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FIRST ORDER PHASE TRANSITION IN A MODEL FOR GENERALISED STATISTICS

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Abstract. A first order phase transition is found in a model which was introduced originally by Murthy and Shankar [Phys. Rev. Lett., **73**, 3331 (1994)] to describe systems of generalised exclusion statistics. I characterise the phase transition in the canonical and grandcanonical ensembles for the case when the statistical exclusion parameter is 1, which corresponds to the Fermi exclusion statistics. We observe that in the grandcanonical ensemble the phase transition has no latent heat, but it has a finite jump in the particle number. In canonical conditions – when the particle number is held fix – the internal energy is discontinuous at the transition.

Key words: quantum ensembles, phase transitions, fractional exclusion statistics.

1. INTRODUCTION

The exclusion statistics that is intermediate between the Bose and the Fermi statistics, called *fractional exclusion statistics* (FES), was introduced by Haldane in Ref. [1] to describe, among other systems, quasiparticle excitations in fractional quantum Hall effect and spinon excitations in spin- $\frac{1}{2}$ quantum ferromagnets. For this he assumed that the many-particle system has a finite-dimensional Hilbert space whose dimension varies linearly with the particle number in the system.

The thermodynamic properties of systems with FES in the thermodynamic limit were clarified mainly by Isakov [2] and Wu [3].

One of the interacting models that leads to FES was introduced first by Murthy and Shankar in Ref. [4] and then used with slight variations in

different contexts by various authors (see for example [5, 6, 7, 8]). To explain the model, let us assume for the beginning that we have an ideal gas of bosons, with the single particle levels denoted by ϵ_i , $i = 0, 1, \dots$ and $\epsilon_i \leq \epsilon_{i+1}$ for any i . I also assume, for the sake of clarity, that the energy levels are equidistant and I introduce the *density of states*, $\sigma \equiv (\epsilon_{i+1} - \epsilon_i)^{-1}$. The interaction between particles changes the energy levels in such a way that the energy level i changes its energy to

$$\tilde{\epsilon}_i \equiv \tilde{\epsilon}(\epsilon_i) = \epsilon_i + \sigma^{-1}gN_{i-} + \sigma^{-1}h_0 \max(n_i - 1, 0), \quad (1)$$

where n_i is the number of particles on the level i , while $N_{i-} \equiv \sum_{j < i} n_j$ is the number of particles below the level i . By $\max(a, b)$ I denote the maximum of a and b . The constants g and h_0 are adimensional interaction parameters. In order to ensure the stability of the system, I impose from the beginning $h_0 \geq 0$ and $g \geq 0$. Obviously, $h_0 = g = 0$ corresponds to the ideal Bose gas.

Let us now analyse the (quasi)continuous limit. In this limit we can relax the condition of having an equidistant single particle spectrum and describe the system only by the *average* density of states σ . Then for any $\epsilon > 0$, if there are no macroscopically populated energy levels – i.e., in the absence of condensation –, $n(\epsilon)$ is much smaller than $N_{\epsilon-}$ and we can write

$$\tilde{\epsilon}(\epsilon) = \epsilon + g \int_0^\epsilon n(\epsilon) d\epsilon. \quad (2)$$

Even for $\epsilon = 0$ we can neglect the microscopic offset $\sigma^{-1}h_0n(\epsilon = 0)$ of the groundstate – which has no contribution to the thermodynamics of the system – and write $\tilde{\epsilon}(0) = 0$.

Now we can prove the generalised exclusion statistics character of the system. For this, I will take the arbitrary interval $[\tilde{\epsilon}_1, \tilde{\epsilon}_u]$, which contains a large enough number of particles, $n_{[\tilde{\epsilon}_1, \tilde{\epsilon}_u]}$, and calculate the number of states it contains. The energies $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_u$ are related to the energy levels in the noninteracting system, ϵ_1 and ϵ_u , by the relations

$$\tilde{\epsilon}_1 = \epsilon_1 + \sigma^{-1}[gN_{\tilde{\epsilon}_1-} + h_0 \max(n_{\tilde{\epsilon}_1}, 0)], \quad (3)$$

$$\tilde{\epsilon}_u = \epsilon_u + \sigma^{-1}[g(N_{\tilde{\epsilon}_1-} + n_{[\tilde{\epsilon}_1, \tilde{\epsilon}_u]}) + h_0 \max(n_{\tilde{\epsilon}_u}, 0)]. \quad (4)$$

For enough many energy states in the interval ($n_{[\tilde{\epsilon}_1, \tilde{\epsilon}_u]} \gg n_{\tilde{\epsilon}_1}, n_{\tilde{\epsilon}_u}$) I can write the number of states between $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_u$ as

$$d_{[\tilde{\epsilon}_1, \tilde{\epsilon}_u]} = \sigma(\epsilon_u - \epsilon_1) = \sigma(\tilde{\epsilon}_u - \tilde{\epsilon}_1) - n_{[\tilde{\epsilon}_1, \tilde{\epsilon}_u]}g. \quad (5)$$

Relation (5) is linear in the number of particles in the interval and gives an *effective exclusion statistics parameter* $\alpha = g$ [5, 6, 7, 8]. The case $\alpha = 0$ corresponds to Bosons, whereas $\alpha = 1$ corresponds to Fermions. Therefore the interacting gas is described as an “ideal” gas of particles of generalised statistics, with the *same*, constant density of states σ .

The thermodynamics of such a system is obtained by splitting the energy axis into intervals which are small enough, so that the energy variations caused by different rearrangements of particles in any of the intervals are insignificant, but which are also sufficiently large, so that each interval contains a large number of particles and energy levels. If we number these energy intervals by capital letters (e.g. $I = 0, 1, \dots$) and the interval I contains δn_I particles and d_I energy levels, then the number of microstates obtained by rearranging the particles in this interval is

$$w_i = \frac{(\delta n_I + d_I)!}{\delta n_I! d_I!}. \quad (6)$$

Summing up the number of microstates, multiplied by their statistical weights, one obtains the partition function, from which, by taking the logarithm and using the Stirling formula, one calculates the thermodynamic potential and entropy (see for example [2, 3, 9]). The thermodynamic potentials have been put into an universal form for any α by writing them in terms of polylogarithm functions [8], as previously done independently for Bosons ($\alpha = 0$) and Fermions ($\alpha = 1$) by Lee [10] and Viefers, Ravndal, and Haugset [11]. For example the grandcanonical potential, Ω , and the entropy, S , can be written in the form

$$\Omega = -U = \frac{1 - \alpha}{2} \frac{N^2}{\sigma} + (k_B T)^2 \sigma \text{Li}_2(-y_0), \quad (7a)$$

$$S = -k_B^2 T \sigma [2\text{Li}_2(-y_0) + \log(1 + y_0) \log y_0], \quad (7b)$$

if we define y_0 by the equation $(1 + y_0)^{1-\alpha}/y_0 = e^{-\mu/k_B T}$. In Eq. (7a) N is the total particle number in the system and μ is the chemical potential.

Equation (7b) shows that the entropy of any system of constant DOS is independent of the exclusion statistics parameter α [8]. From here follows that in canonical conditions the systems of any exclusion statistics, but of the same, constant density of states, are identical from the thermodynamic point of view. This property has been given the name of *thermodynamic equivalence* of systems of constant DOS [10].

1.1. THERMODYNAMIC EQUIVALENCE

Apparently h_0 plays no role in the model since does not enter any of the macroscopic equations (5), (6), or (7). For this reason it has been in general neglected [7], if not entirely discarded [4, 5, 6, 9, 12]. After all, a thermodynamic Bose system is usually thought of as an infinite Bose gas, with $\sigma \rightarrow \infty$, $N \rightarrow \infty$, and finite density $\rho \equiv N/\sigma$. Such a gas does not condense at any finite temperature, so there is no reason to consider h_0 .

Moreover, the model can be built from a fermionic perspective. In [8] I showed that the microscopic reason for the thermodynamic equivalence between a Bose and a Fermi system of the same, constant density of states is the possibility to make a one-to-one mapping between the microscopic configurations of bosons and the microscopic configurations of fermions, with the same excitation energy [13]. By this mapping, the bosons that lay on the same energy level are transformed into a group of “close packed” fermions, i.e. into a group of fermions which occupy all the states into a single-particle energy interval (more details are given in Refs. [8, 13]). In such a case, the single particle energies of the interacting Fermi gas corresponding to Eq. (1) should be written as

$$\tilde{\epsilon}_i \equiv \tilde{\epsilon}(\epsilon_i) = \epsilon_i + \sigma^{-1}g^{(F)}N_{i-} + \sigma^{-1}h_0^{(F)}\max(n_I^{(F)} - 1, 0), \quad (8)$$

with $g^{(F)} = 1 - g$ and $h_0^{(F)} = h_0 - 0.5$. In this description, $n_I^{(F)}$ is the number of fermions in the “close-packed” group that contains the particle on the level i . Since at temperature T in any small energy interval $\delta\epsilon$, centered at $\tilde{\epsilon}$, there are on average $\sigma\delta\epsilon/\{1 + \exp[\beta(\mu - \tilde{\epsilon})]\}$ ($\neq 0$) holes, we cannot expect that a macroscopical interval along the energy axis will be “close-packed,” so from this perspective also – maybe even more clearly – the parameter h_0 does not appear to have any macroscopic significance.

1.2. THE ROLE OF h_0

Infinite systems of constant density of states are indeed not condensed at any finite temperature, as stated before. Nevertheless, in finite systems, a significant fraction of particles may accumulate on the ground state at low enough temperature in Bose systems, or it can arrange in a close-packed (or degenerate) configuration on the lowest energy levels in Fermi systems [14]. Therefore, at low enough temperature we should take into account separately the population of the ground state in Bose systems or the corresponding degenerate fermionic subsystem and I shall refer to them as the Bose or the Fermi condensates [14].

By doing so, in Ref. [8] I showed that if $h_0 < 0.5g$ the system undergoes a first order phase transition for any α . In this paper I will describe in detail the phase transition for $\alpha = 1$. In Section 2 I calculate the partition functions and I show how the phase transition occurs in grandcanonical (Section 2.1) and canonical systems (Section 2.2). In Section 3 I calculate the discontinuities that appear in the thermodynamical functions at the phase transitions. I shall show that in the grandcanonical ensemble the internal energy is continuous at the transition, while the particle number has a jump. However, in the canonical ensemble, since the particle number is conserved, the internal

energy is discontinuous at the phase transition. Last section is reserved for conclusions.

2. THE PHASE TRANSITION

2.1. THE GRANDCANONICAL ENSEMBLE

From now on I shall discuss the case $\alpha = g = 1$. Although the analysis is equally easy from Bose and Fermi perspectives, for easier reference to the calculations in [8] I shall adopt here the bosonic picture.

To emphasise the condensate, I write the grandcanonical partition function as a sum of terms corresponding to different values of n_0 (the population of the condensate) [8],

$$\begin{aligned} \mathcal{Z} &= \sum_{n_0} \mathcal{Z}_{n_0} \equiv \sum_{n_0} e^{-\beta \frac{h_0 n_0^2}{\sigma} + \beta \mu n_0} \cdot \prod_{i=1}^{\infty} e^{-\beta(\tilde{\epsilon}_i - \mu)n_i} \\ &\equiv \sum_{n_0} e^{-\beta \frac{h_0 n_0^2}{\sigma} + \beta \mu n_0} \cdot \mathcal{Z}_{\text{ex}}(n_0, \beta). \end{aligned} \quad (9)$$

The ‘‘partial’’ partition function, \mathcal{Z}_{n_0} , is proportional to the probability that a configuration with n_0 particles in the condensate occurs. As it happens usually at a first order phase transition, the partition function in some parameter space develops two maxima. In each of the phases one maximum is dominant and the equilibrium state of the system corresponds to it. As we change from one phase to the other by varying some of the thermodynamic parameters of the system (e.g. the temperature or the chemical potential), the difference between the two maxima decreases and they become equal at the phase transition. Moving further into the second phase, the maximum corresponding to it becomes dominant. For our system, the parameter with which we describe the phase transition is the population of the condensate, n_0 .

Therefore, instead of calculating the whole partition function, I calculate \mathcal{Z}_{n_0} and analyse its maxima. It is easier to work with $\ln \mathcal{Z}_{n_0}$, which is

$$\ln \mathcal{Z}_{n_0} = -\beta \frac{h_0 n_0^2}{\sigma} + \beta \mu n_0 + \ln \mathcal{Z}_{\text{ex}}(n_0, \beta). \quad (10)$$

I introduce the dimensionless quantities $a \equiv \beta \mu$, $\xi \equiv \beta N / \sigma$, and $\xi_0 \equiv \beta n_0 / \sigma$. In these notations Eq. (10) becomes

$$\beta \sigma^{-1} \ln \mathcal{Z}_{n_0} = -h_0 \xi_0^2 + a \xi_0 + \beta \sigma^{-1} \ln \mathcal{Z}_{\text{ex}}(n_0, \beta). \quad (11)$$

According to the calculation procedure outlined in Section 1, $\mathcal{Z}_{\text{ex}}(n_0, \beta)$ is the partition function of a Fermi gas with the same chemical potential and with

a single particle spectrum that starts at $\tilde{\epsilon}_1$:

$$\ln \mathcal{Z}_{\text{ex}}(n_0, \beta) = \int_{\tilde{\epsilon}_1}^{\infty} \ln \left(1 + e^{a-\beta\epsilon} \right) \sigma d\epsilon. \quad (12)$$

In Ref. [8] I showed that condensation may occur only on the first energy level, so I can write $\tilde{\epsilon}_1 = \epsilon_1 + n_0\sigma^{-1} \equiv \sigma^{-1}(1 + n_0)$, which leads to

$$\begin{aligned} \beta\sigma^{-1} \ln \mathcal{Z}_{\text{ex}}(n_0, \beta) &= \int_{\sigma^{-1}(1+n_0)}^{\infty} \ln \left(1 + e^{a-\beta\epsilon} \right) \beta d\epsilon \\ &= \int_{\beta\sigma^{-1}+\xi_0}^{\infty} \ln \left(1 + e^{a-x} \right) dx. \end{aligned} \quad (13)$$

I plug this into (11) and I get

$$\begin{aligned} \beta\sigma^{-1} \ln \mathcal{Z}_{n_0} &= -h_0\xi_0^2 + a\xi_0 + \int_0^{\infty} \ln \left(1 + e^{a-\xi_0-\beta\sigma^{-1}-x} \right) dx \\ &\equiv -h_0\xi_0^2 + a\xi_0 - Li_2(-e^{a-\beta\sigma^{-1}-\xi_0}), \end{aligned} \quad (14)$$

where I used the notation

$$Li_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^{\infty} \frac{x^{n-1} dx}{e^x z - 1} = \frac{z}{1^n} + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \dots \quad (15)$$

for the polylogarithmic function of order n [15, 16].

Now we have to find and compare the maxima of $\ln \mathcal{Z}_{n_0}$. If $\ln \mathcal{Z}_{n_0}$ forms two maxima, then our system has two phases. In each of the phases one maximum is dominant and n_0 , together with all the measurable thermodynamic quantities, take the values corresponding to this maximum [17, 18, 19, 20]. Since $\ln \mathcal{Z}_{n_0}$ is a smooth function, we find its extrema at $n_0 > 0$ by equating to zero its derivatives with respect to n_0 or, more conveniently, with respect to ξ_0 :

$$0 = \frac{\partial(\beta\sigma^{-1} \ln \mathcal{Z}_{n_0})}{\partial \xi_0} = -2h_0\xi_0 + a - \ln \left(1 + e^{a-\beta\sigma^{-1}-\xi_0} \right). \quad (16)$$

We transform it a bit, to obtain

$$\beta\sigma^{-1} + \xi_0(1 - 2h_0) = \ln \left[1 + e^{-(a-\xi_0)+\beta\sigma^{-1}} \right]. \quad (17)$$

For an ideal Fermi gas, $h_0 = 1/2$ and we reobtain the equation for the Fermi condensate in ideal gases of constant σ [14]:

$$\beta\sigma^{-1} = \ln \left[1 + e^{-(a-\xi_0)+\beta\sigma^{-1}} \right]. \quad (18)$$

In general, if $h_0 > 1/2$, the derivative $\partial(\beta\sigma^{-1} \ln \mathcal{Z}_{n_0})/\partial \xi_0$ is always negative and \mathcal{Z}_{n_0} has only one maximum at $n_0 = 0$. We have therefore only one (uncondensed) phase for any α [8].

If $h_0 < 1/2$ we have a condensation [8]. In this case, since the condensate contains a macroscopic number of particles, we can neglect β/σ as compared to a and ξ_0 and write Eq. (17) in the shorter form:

$$\xi_0(1 - 2h_0) = \ln \left[1 + e^{\xi_0 - a} \right]. \quad (19)$$

Equation (19) is easy to interpret. For small enough a , the functions $\xi_0(1 - 2h_0)$ and $\ln[1 + \exp(\xi_0 - a)]$ do not cross each-other, so we have no solution for (19) and Z_{n_0} has only one maximum at $n_0 = 0$ – no condensate. This situation is plotted with continuous lines in the right upper and lower plots of Fig. 1.

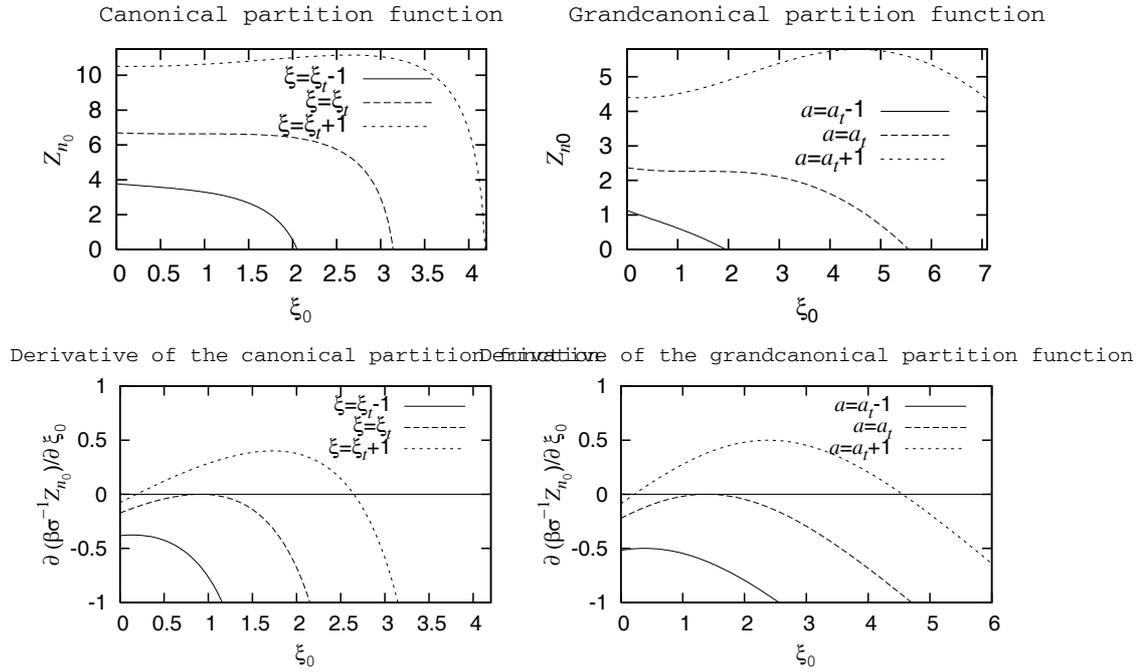


Fig. 1 – The partition functions of the canonical and grandcanonical ensembles and their derivatives.

As a increases, $\xi_0(1 - 2h_0)$ becomes tangent to $\ln[1 + \exp(\xi_0 - a)]$ when

$$\frac{d}{d\xi_0} \ln[1 + e^{\xi_0 - a}] = \frac{1}{1 + e^{a - \xi_0}} = 1 - 2h_0,$$

which happens at $\xi_{0,t} = a - \ln[2h_0/(1 - 2h_0)]$. Plugging $\xi_{0,t}$ into Eq. (19) I get

$$a_t = -\frac{2h_0 \ln(2h_0) + (1 - 2h_0) \ln(1 - 2h_0)}{1 - 2h_0}. \quad (20)$$

This situation is drawn in dashed lines in the right-hand plots of Fig. 1.

For $a > a_t$, Eq. (19) has two solutions, $\xi_{0,1}$ and $\xi_{0,2}$, with $\xi_{0,1} < \xi_{0,2}$. This means that for $\xi_0 < \xi_{0,1}$ the partition function \mathcal{Z}_{n_0} decreases with increasing n_0 , for $\xi_{0,1} < \xi_0 < \xi_{0,2}$, \mathcal{Z}_{n_0} increases with n_0 , and finally for $\xi_0 > \xi_{0,2}$, \mathcal{Z}_{n_0} decreases again. The two competing maxima of \mathcal{Z}_{n_0} , one at $n_0 = 0$ and the other one at $n_0 = k_B T \sigma \xi_{0,2}$, indicate a potential phase transition. Indeed, as a increases past a_t , the relative height of the maximum located at $\xi_0 = \xi_{0,2}$ increases with respect to the maximum at $\xi_0 = 0$, and at a certain critical value, a_{cr} , we obtain $\mathcal{Z}_{n_0=0} = \mathcal{Z}_{n_0=k_B T \sigma \xi_{0,2}}$. Above a_{cr} the second maximum is higher (see the dotted lines in the plots of Fig. 1), $\mathcal{Z}_{n_0=0} = \mathcal{Z}_{n_0=k_B T \sigma \xi_{0,2}}$, and we say that a phase-transition occurred. As we shall see in Section 3.1, the (average) internal energy of the gas is conserved at the transition, but the particle number has a jump.

This is a first order phase transition.

2.2. THE CANONICAL ENSEMBLE

Let us now analyse the system in canonical conditions. Neglecting again the microscopic energy step σ^{-1} in the system with interaction, the total particle number is

$$N \equiv n_0 + N_{\text{ex}} = n_0 + \int_{\bar{\epsilon}_1}^{\infty} \frac{\sigma d\epsilon}{e^{\beta\epsilon-a} - 1} \approx n_0 + k_B T \sigma \ln \left(1 + e^{a-\xi_0} \right). \quad (21)$$

If I denote $\xi_{\text{ex}} \equiv \beta\sigma^{-1}N_{\text{ex}}$, then

$$\xi_{\text{ex}} = \ln \left(1 + e^{a-\xi_0} \right) \quad (22)$$

and Eq. (21) becomes

$$\xi = \xi_0 + \xi_{\text{ex}} = \xi_0 + \ln \left(1 + e^{a-\xi_0} \right). \quad (23)$$

In the canonical ensemble I express a in terms of ξ and ξ_0 ,

$$a = \xi_0 + \ln \left(e^{\xi-\xi_0} - 1 \right), \quad (24)$$

and then I write the partition function $\beta\sigma^{-1}\mathcal{Z}_{n_0}$ in terms of ξ and ξ_0 :

$$\begin{aligned} \beta\sigma^{-1}\mathcal{Z}_{n_0} &= (1-h_0)\xi_0^2 + \xi_0 \ln \left(e^{\xi-\xi_0} - 1 \right) + \int_0^{\infty} \ln \left[1 + \left(e^{\xi-\xi_0} - 1 \right) e^{-x} \right] \\ &= (1-h_0)\xi_0^2 + \xi_0 \ln \left(e^{\xi-\xi_0} - 1 \right) - Li_2 \left(1 - e^{\xi-\xi_0} \right). \end{aligned} \quad (25)$$

Using Eq. (25) I can find the maximum of $\beta\sigma^{-1}\mathcal{Z}_{n_0}$ at fixed ξ , and for this I equate the derivative $\partial(\beta\sigma^{-1}\mathcal{Z}_{n_0})/\partial\xi_0$ to zero,

$$\frac{\partial}{\partial\xi_0}(\beta\sigma^{-1}\mathcal{Z}_{n_0}) = (1-2h_0)\xi_0 + \ln \left(1 - e^{\xi_0-\xi} \right) - \frac{\xi}{e^{\xi-\xi_0} - 1} = 0. \quad (26a)$$

To discuss the solutions of Eq. (26), it is better to write it in the form

$$(1 - 2h_0)\xi_0 = \frac{\xi}{e^{\xi-\xi_0} - 1} - \ln(1 - e^{\xi_0-\xi}) \quad (26b)$$

and denote the r.h.s. by $f(\xi, \xi_0)$. The first and the second derivative of $f(\xi, \xi_0)$,

$$\frac{\partial f(\xi, \xi_0)}{\partial \xi_0} = \frac{\xi e^{\xi-\xi_0} + e^{\xi-\xi_0} - 1}{(e^{\xi-\xi_0} - 1)^2} > 0$$

and

$$\frac{\partial^2 f(\xi, \xi_0)}{\partial \xi_0^2} = \frac{e^{\xi-\xi_0}}{(e^{\xi-\xi_0} - 1)^3} (\xi e^{\xi-\xi_0} + e^{\xi-\xi_0} + \xi - 1) > 0$$

are both bigger than zero for $\xi > 0$ and $0 \leq \xi_0 \leq \xi$; therefore $f(\xi, \xi_0)$, as a function of ξ_0 , is concave upwards.

To see if we have any solutions, I evaluate $\partial(\beta\sigma^{-1}\mathcal{Z}_{n_0})/\partial\xi_0$ at $\xi_0 = 0$ and $\xi_0 = \xi$, and I get

$$\left. \frac{\partial(\beta\sigma^{-1}\mathcal{Z}_{n_0})}{\partial\xi_0} \right|_{n_0=0} = \ln(1 - e^{-\xi}) - \frac{\xi}{e^{\xi} - 1} < 0, \quad \forall \xi > 0, \quad (27a)$$

and

$$\begin{aligned} \left. \frac{\partial(\beta\sigma^{-1}\mathcal{Z}_{n_0})}{\partial\xi_0} \right|_{n_0=N} &= (1 - 2h_0)\xi + \lim_{\xi_0 \rightarrow \xi} \left[\ln(\xi - \xi_0) - \frac{\xi}{\xi - \xi_0} \right] \\ &= -\infty, \quad \forall \xi > 0. \end{aligned} \quad (27b)$$

In other words, the concave up-wards function $f(\xi, \xi_0)$ is bigger than the linear function $(1 - 2h_0)\xi_0$ at both ends of the interval $\xi_0 \in [0, \xi]$, so, if they are not tangent, they must cross each-other either zero times or two times in this interval. Therefore the Eqs. (26) have either no solutions or have two solutions for $\xi_0 \in (0, \xi)$.

We have here the same situation as in the grandcanonical ensemble. For small ξ , Eq. (26b) have no solution. A situation like this is depicted in the left-hand plots in Fig. 1 with solid lines. As ξ increases, at a certain value ξ_t , the function $(1 - 2h_0)\xi_0$ becomes tangent to $f(\xi, \xi_0)$, and \mathcal{Z}_{n_0} , as a function of ξ_0 , has an inflection at the tangent point, as shown with dashed lines in Fig. 1. For $\xi > \xi_t$, \mathcal{Z}_{n_0} has two maxima, one at $\xi_0 = 0$ and another one at $\xi_0 = \xi_{0,2}$ (I use the same notations as in the canonical ensemble). As ξ increases above a critical value, say ξ_{cr} , the maximum formed at $\xi_{0,2}$ becomes dominant (the dotted lines in Fig. 1) and at $\xi = \xi_{cr}$ we have a phase transition.

In Fig. 2 I plot for comparison $\xi_{0,2}$ as a function of h_0 for the grandcanonical (solid line) and canonical (dashed line) ensembles. It is remarkable that although as $h_0 \rightarrow 0.5$ the Fermi gas approaches the ideal behavior, $\xi_{0,2}$ does

not converge to zero, as one would expect (i.e. no condensation in ideal, continuous systems). This means that asymptotically, in the noninteracting system a finite fraction of the gas condenses at the transition temperature [14].

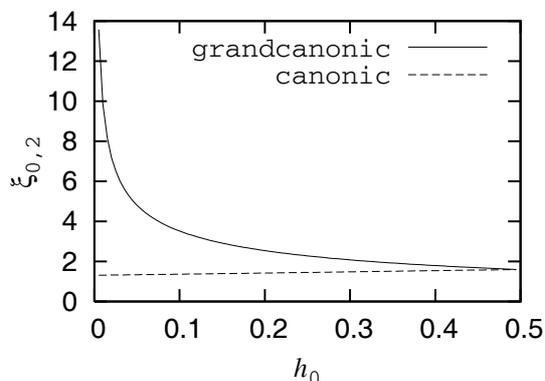


Fig. 2 – Comparison between of the jumps of the number of particles in the condensate at the transition temperature, in canonical (dashed line) and grandcanonical (solid line) ensembles.

In Section 3.2 I shall calculate the latent heat of this transition.

3. DISCONTINUITIES AT THE PHASE TRANSITION

Let us now calculate the jumps of the thermodynamic quantities that occur at the phase transition. Usually one expects that a first order phase transition in the grandcanonical ensemble is marked by discontinuities in the internal energy (i.e. a latent heat) and in the particle number. In the canonical ensemble, the particle number is held fixed and the only quantity that may vary is the internal energy. If we think of typical first order phase transitions, like liquid-vapour transition or Bose-Einstein condensation, we observe that they do not show any discontinuity in the internal energy in canonical conditions, but condense gradually as the temperature decreases. Unlike these, the transition described here, under canonical conditions, is marked by a jump of particles in the condensate, accompanied by another jump in the specific heat.

I shall explain these in the next two subsections.

3.1. GRANDCANONICAL ENSEMBLE

Let us evaluate first the latent heat of the transition in the grandcanonical ensemble. If we denote by U the internal energy of the system and introduce the adimensional quantity $\mathcal{U}(a, \xi_0) \equiv \beta^2 \sigma^{-1} U$, by standard calculations

I obtain

$$\begin{aligned}\mathcal{U}(a, \xi_0) &= h_0 \xi_0^2 + \xi_0 \ln \left(1 + e^{a-\xi_0} \right) + \int_0^\infty \ln \left(1 + e^{a-\xi_0-x} \right) dx \\ &= h_0 \xi_0^2 + \xi_0 \ln \left(1 + e^{a-\xi_0} \right) - Li_2(-e^{a-\xi_0}).\end{aligned}\quad (28)$$

Subtracting (14) from (28) and ignoring $\beta\sigma^{-1}$, I get

$$\begin{aligned}\mathcal{U}(a, \xi_0) - \beta\sigma^{-1} \ln \mathcal{Z}_{n_0} &= \xi_0 \left[\ln \left(1 + e^{\xi_0-a} \right) - (1 - 2h_0)\xi_0 \right] \\ &= -\xi_0 \frac{\partial(\beta\sigma^{-1} \ln \mathcal{Z}_{n_0})}{\partial \xi_0},\end{aligned}\quad (29)$$

so at the phase transition, $\mathcal{U}(a, \xi_0) = \beta\sigma^{-1} \ln \mathcal{Z}_{n_0}$, both at $\xi_0 = 0$ and $\xi_0 = \xi_{0,2}$ (where $\xi_{0,2}$ corresponds again to the second maximum of $\ln \mathcal{Z}_{n_0}$). But since $\ln \mathcal{Z}_{n_0}(a, \xi_0 = 0) = \ln \mathcal{Z}_{n_0}(a, \xi_{0,2})$ exactly at the phase transition, it implies that $\mathcal{U}(a, 0) = \mathcal{U}(a, \xi_{0,2})$ and the grandcanonical latent heat, $\lambda_g \equiv (k_B T)^2 \sigma [\mathcal{U}(a, \xi_{0,2}) - \mathcal{U}(a, 0)]$ is identically zero.

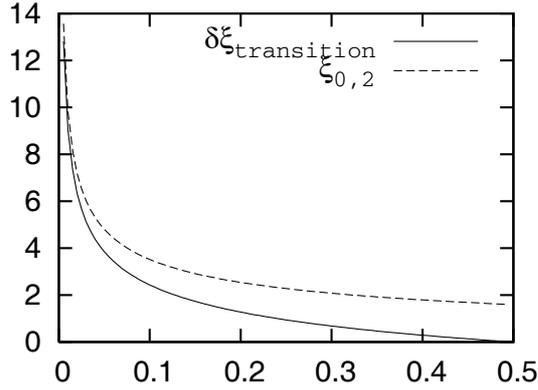


Fig. 3 – The discontinuity of the (average) total particle number in the system (continuous line) and the particle number in the condensate (dashed line) at the transition temperature in the grandcanonical ensemble, as a function of h_0 .

The quantity that varies at the grandcanonical phase transition is the particle number. From Eq. (23) I get directly

$$\delta \xi_{transition} \equiv \xi(a, \xi_{0,2}) - \xi(a, \xi_0 = 0) = \ln \left[\frac{1 + e^{\xi_{0,2}-a}}{1 + e^{-a}} \right] > 0. \quad (30)$$

The function $\delta \xi_{transition}$ is plotted in Fig. 3, together with the jump of the number of particles in the condensate, $\xi_{0,2}$, as a function of h_0 . Unlike ξ_0 , ξ converges to zero as h_0 approaches 0.5.

3.2. CANONICAL ENSEMBLE

To calculate the latent heat in the canonical ensemble, I replace a in Eq. (29) with the expression (24). In this way I get

$$\begin{aligned} \mathcal{U}(\xi, \xi_0) &= \xi_0[\xi - \xi_0(1 - h_0)] + \int_0^\infty \ln \left[1 + (e^{\xi - \xi_0} - 1) e^{-x} \right] dx \\ &= \xi_0[\xi - \xi_0(1 - h_0)] - Li_2 \left(1 - e^{\xi - \xi_0} \right). \end{aligned} \quad (31)$$

If I denote by λ_c the latent heat in the canonical ensemble, then

$$\frac{\beta^2}{\sigma} \lambda_c = \xi_0[\xi - \xi_0(1 - h_0)] + \int_0^\infty \ln \left[\frac{1 + (e^{\xi - \xi_0} - 1) e^{-x}}{1 + (e^\xi - 1) e^{-x}} \right] dx. \quad (32)$$

This adimensional latent heat is plotted in Fig. 4.

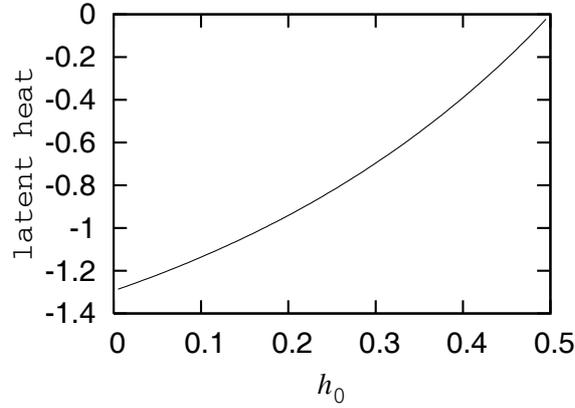


Fig. 4 – The latent heat of the canonical ensemble, in units of $(k_B T)^2 \sigma$, as a function of the interaction parameter h_0 .

Therefore the phase transition in the canonical system is marked by a discontinuity of the internal energy and a sudden condensation of a finite number of particles.

4. CONCLUSIONS

I analysed an interacting system model of constant density of single particle states, which can describe systems of generalised exclusion statistics. I showed that a first order phase transition takes place and this phase transition is the condensation of a finite number of particles on the lowest energy levels. I characterised this transition for the situation when the system can be described as a Fermi gas. In grandcanonical conditions, the phase transition has

no latent heat, but has a discontinuity in the number of particles, due to the condensation phenomenon. Unlike usual first order phase transitions, in this model we have a latent heat at the phase transition in canonical ensemble.

Another interesting fact is that at the transition temperature, the number of particles in the condensate converges to a finite, nonzero value, as the system is changed gradually into a noninteracting system. This can be put into correspondence with the condensation that occurs in ideal Fermi systems [14].

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