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COSMOLOGY WITH A FRACTIONAL ACTION PRINCIPLE

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Abstract. The Friedmann-Robertson-Walker universe is re-examined in the context of fractional action-like variational approach or fractionally differentiated Lagrangian function depending on a parameter α recently introduced by the author to model nonconservative and weak dissipative classical and quantum dynamical systems. It is non-standard only in that within the framework of fractional action principle the gravitational constant G was proved to be perturbed by a certain decaying factor given by $\Delta G = 3(1-\alpha)H/4\pi G\rho T$, H is the Hubble parameter, ρ is the matter-density and T is the cosmic time. A suggestion is made to replace the "cosmological constant" by "dissipative force".

Key words: fractional action principle, perturbed gravity, dissipative force.

1. INTRODUCTION

Fractional calculus plays an important and leading role in the understanding of complex classical and quantum (conservative and nonconservative) dynamical systems with holonomic as well as with nonholonomic constraints [1-2 and references therein]. Its origin goes back more than three centuries, when in 1695 L'Hopital made some remarks to Leibniz about the mathematical meaning of a fractional derivative of order 1/2. Leibniz's response was "*an apparent paradox, from which one day useful consequences will be drawn*". In these words fractional calculus was born. After that, many famous mathematicians (J. Fourier, N.H. Abel, J. Liouville, B. Riemann, etc.) contributed strongly to the fractional analysis program, e.g. fractional derivatives and integrals. Although the fractional theory is very rich, it was considered for more than three centuries as a theoretical mathematical field with no physical interests. In the last few decades, fractional theory was proved to be very useful and important in various fields of science including classical and quantum physics, field theory, solid state physics, fluid dynamics, turbulence, chemistry in general, nonlinear biology, stochastic analysis, nonlinear control theory, image processing [1, 2]. While various fields of application of fractional derivatives and integrals are already well done, some others have just started in particular the study of fractional problems of the Calculus of Variations (COV) and respective Euler-Lagrange type equations is a subject of current strong research and investigations [4–8]. The physical reasons for the appearance of fractional equations are, in general, long-range dissipation and non-conservatism. For this reason, it seems of interest to study the fractional Hamiltonian of nonconservative dynamical systems.

In a recent work we developed a novel approach known as the fractional action-like variational approach (FALVA) or fractionally differentiated Lagrangian function (FDLF) to model and describe nonconservative Lagrangian dynamical systems within the framework of fractional differential calculus [9, 10, 11]. In our proposed method fractional time integral introduces only one parameter a while in other models an arbitrary number of fractional parameters (*orders of derivatives*) appears. The FALVA is based on the following concept: consider a smooth manifold M and denote $L:\mathsf{R}\times TM\to\mathsf{R}$ be the smooth Lagrangian function. For any piecewise smooth path $g:[t_0,t_1]\to M$ we define the fractional action by:

$$S_L[g]\,\square\,\frac{1}{\Gamma(a)}\int_{t_0}^{t}L\big(\dot{g}(t),g(t),t\big)(t-t)^{a-1}dt=\int_{t_0=0}^{t}L(\dot{g},g,t)\,dg_t(t)\to\min,\quad(1)$$

where $L(\dot{g},g,t)$ is the Lagrangian weighted with $(t-t)^{a-1}\big/\Gamma(a)$ and $\Gamma(1+a)g_t(t)=t^a-(t-t)^a$ with the scaling properties $g_{mt}(mt)=m^a g_t(t)$, $m>0$. In reality, we considered a smooth action integral (a time smeared measure $dg_t(t)$ on the time interval $[0,t]\in\mathsf{R}^+$) which can be rewritten as the strictly singular Riemann-Liouville type fractional derivative Lagrangian

$$S_{b\in(0,1)}[q]=D_t^{-1+b}L\big(\dot{q}(t),q(t),t\big)=$$
$$=\int_0^t L\big(\dot{q}(t),q(t),t\big)\frac{dt}{(t-t)^b}\overset{b\to0}{\to}\int_0^t L\big(\dot{q}(t),q(t),t\big)dt,$$

and thereby retrieved the standard action integral or functional integral. In this work, we have $b=1-a, a\in(0,1)$. Such type of functional is known in mathematical economy, describing, for instance, a so called "discounting" economical dynamics.

Let $L:\mathsf{R}\times TM\to\mathsf{R}$ be the Lagrangian map, (p_0,p_1) are two fixed points and $g:[t_0,t_1]\to M$ be a smooth path such that $g_{t_i}=p_i, i=0,1$ and $S_L[g]\le S_L[\tilde{g}]$

for any smooth path $\tilde{g}:[t_0,t_1]\to M$ joining $p_0 \to p_1$. Then, g satisfies the fractional or modified Euler-Lagrange equation:

$$\frac{\partial L}{\partial g}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{g}}\right)=\frac{1-a}{t-t}\frac{\partial L}{\partial \dot{g}}\equiv \bar{F}\,, \tag{2}$$

$\bar{F}$ is the modified frictional force, a common type of non conservative force. When $a=1$, we find the standard Euler-Lagrange equation and at late times, that is $t\to\infty$, $\bar{F}\to 0$. The FALVA or FDLF was proved to have consequences and applications in various fields of science ranging from classical dynamical systems to differential geometry and modern cosmology [12–16]. We will discuss in the next section the important implications of the fractional action integral formalism in Riemann geometry and cosmology.

2. FALVA, RIEMANN GEOMETRY AND PERTURBED GRAVITY

An important implication of our variational principles is within the context of Riemann geometry. In our previous works, we considered the following Lagrangian:

$$L=L\left(x,\dot{x};l\right)=g_{mn}\left(x,\dot{x}\right)\dot{x}^m\dot{x}^n \tag{3}$$

and its corresponding modified geodesic equation

$$\ddot{x}^m+\frac{a-1}{T}\dot{x}^m+\Gamma^m_{nd}\dot{x}^n\dot{x}^d=0\,, \tag{4}$$

with solutions $x^m=x^m\left(l\right)$, Γ^m_{nd} is the Christoffel symbol. The second term is the predicted time-decaying friction term. It was showed in [12] that equation (4) will modify the General Relativity by perturbing the gravitational constant "G" by a certain decaying factor given by:

$$\Delta G=\frac{3\left(1-a\right)}{4pGrT}\frac{\dot{R}}{R} \tag{5}$$

at late times, r being the fluid density and $R\left(T\right)$ is the scale factor of the universe. We expect now some important consequences in Friedmann-Robertson-Walker (FRW) cosmology with the presence of the cosmological constant Λ. In the traditional Einstein General Relativity, Λ is fixed, and it serves as the source for the metric field: in other words the input in the Einstein field equation is Λ, the output is de Sitter expansion, if matter is absent. The Einstein field equation does not allow us to obtain the time dependence of the cosmological constant, because of the Bianchi identities $G^{n}_{m;n} = 0, G_{mn} \equiv R_{mn} - (1/2)\, g_{mn} R$ and covariant conservation law of matter $T^{n\,(matter)}_{m;n} = 0$ both leading to $\partial_m \Lambda = 0$. But they allow us to obtain the value of the cosmological constant in different static universes, such as the Einstein closed Universe, where the cosmological constant is obtained as a function of the curvature and matter density [17]. In what follows, we make use of the Raychaudhuri expansion scalar factor $\partial_i \mathrm{v}_i \equiv 3\dot{q} = 3\dot{R}/R$. In FRW cosmology, the term ΔG modifies the Friedman equations in the absence of the cosmological constant as follows [18]:

$$\frac{\dot{R}^2}{R^2} + \frac{2(a-1)}{T}\frac{\dot{R}}{R} + \frac{k}{R^2} = \frac{8pGr}{3}, \tag{6}$$

$$\frac{\ddot{R}}{R} + \frac{a-1}{T}\frac{\dot{R}}{R} = -\frac{4pGr}{3} \tag{7}$$

for zero pressure (dust), while in case of radiation $(p = r/3)$, we get the differential system :

$$\frac{\dot{R}^2}{R^2} + \frac{2(a-1)}{T}\frac{\dot{R}}{R} + \frac{k}{R^2} = \frac{8pGr}{3}, \tag{8}$$

$$\frac{\ddot{R}}{R} + \frac{2(a-1)}{T}\frac{\dot{R}}{R} = -\frac{8pGr}{3}, \tag{9}$$

$k = -1, 0, +1$ for open, flat and closed spacetime respectively. Equations (6) and (8) can in fact be written in an alternative forms like:

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8pG}{3}\left[r + \frac{(1-a)}{T}\frac{3}{4pG}\frac{\dot{R}}{R} \right] \equiv \frac{8pG}{3}\left[r + r^{a}_{decaying} \right], \tag{10}$$

where $r^{a}_{decaying} \equiv 3(1-a)\dot{R}/4pGRT$ is the fractional decaying density. This is to say that the density is perturbed. This perturbation can be viewed as a decaying vacuum density or a decaying cosmological constant explaining the small value of Einstein lambda and become suppressed for $T \to \infty$.

3. THE VACUUM CASE

Let us now consider the very early universe and choose for simplicity the spatially flat solution ($k = 0$). Consequently, equations (6) and (8) yields with $p = -r$:

$$\frac{\dot{R}^2}{R^2}+\frac{2(a-1)}{T}\frac{\dot{R}}{R}=0 \tag{11}$$

and two possible solutions exist: $R=$ constant and $R\propto T^{2(1-a)}$. The first solution corresponds to a static empty universe and the second one corresponds to an accelerated expansion for $0<a<1/2$, to an eternal expansion for $1/2<a<1$ and to a decelerating expansion for $a>1$. An empty universe with zero energy density and zero cosmological constant can accelerates with time. In case $k\neq 0$, equations (6) and (8) are written in the following form:

$$\dot{R}^2+\frac{2(a-1)}{T}\dot{R}R+k=0\,. \tag{12}$$

A realistic solution corresponds to $R\propto T$ with $-1<k=1-2a<0$ for $0<a<1$, yielding that the universe will expand forever. In other case, only open modified spacetime are acceptable and realistic. Note that for $a=1/2$, $k=0$, e.g. we find the previous solution. It is interesting to have an accelerating vacuum or empty space-time in the absence of the cosmological constant where only decaying friction force exists [19, 20, 21].

4. THE INFLATIONARY SOLUTION

We choose again for simplicity the spatially flat solution ($k=0$). From equation (6), we obtain:

$$\frac{\dot{R}^2}{R^2}+\frac{2(a-1)}{T}\frac{\dot{R}}{R}=\frac{8pGr}{3}\,. \tag{13}$$

In the absence of the gravity perturbations, the solution of equation (11) is given by the familiar de-Sitter inflationary solution $R\propto e^{HT}, H=\sqrt{8pGr/3}$ =constant. r and G are constants. In the presence of the perturbed gravity, a possible inflationary solution is given by:

$$R=\sqrt{\frac{8pGr}{3}}\frac{1}{2(1-a)}e^{a-1}T^{1-a}e^{(1-a)\left[\frac{(1-a)^2+\frac{8pGr}{3}T^2+(1-a)\sqrt{(1-a)^2+\frac{8pGr}{3}T^2}}{(1-a)^2-\frac{8pGr}{3}T^2+(1-a)\sqrt{(1-a)^2+\frac{8pGr}{3}T^2}}-\ln\left(\frac{1-a}{\sqrt{\frac{8pGr}{3}}T}+\sqrt{\frac{(1-a)^2}{\frac{8pGr}{3}T^2}+1}\right)\right]} \tag{14}$$

where $0<a<1$. The first constant terms are set for dimensional reasons. It is easy to check from Friedman equations (6) that the classical inflation with the equation state $p=-r$ is permitted.

5. ACCELERATION OF THE LATE MATTER-DOMINATED UNIVERSE

Now suppose the pressure $p = 0$ and $k = 0$, we get from equations (6) and (8):

$$2\frac{\ddot{R}}{R} + \frac{4(a-1)}{T}\frac{\dot{R}}{R} + \frac{\dot{R}^2}{R^2} = 0 \tag{15}$$

A possible solution is given by $R = T^m$, where $m = (6-4a)/3$, $m > 1$ for $a < 3/4$, and the acceleration of the universe may be attributed to the fractional dissipative force. Consequently, the matter-density of the universe decays as $r \propto 2a(6-4a)/9T^2$ independently of whether the gravitational constant is constant or variable and in agreement with most of the theoretical cosmological scenarios described in literature [22–39]. For $a < 3/4$, the deceleration parameter $q \equiv -\ddot{R}R/\dot{R}^2 = (1-m)/m = (4a-3)/(6-4a)$ is negative, the age of the universe is $H = m/T = (6-4a)/3T$ and consequently $1/T < H < 2/T$ in agreement with recent observations [19, 20, 21]. The density parameter of the universe is given by $\Omega^m = r/r_c = 2a/(6-4a)$ and $\Omega^{Total} = 1$ if there exist Ω^f such that $\Omega^f = 1 - \Omega^m = 3(1-a)/(3-2a)$. In other words, we shall define a fractional decaying friction force $r_f \propto (1-a)(3-2a)/T^2$. Moreover, $\Omega^{Total} = 1$ for $a = 1$. In most of the theoretical models described in literature, the universe must be accelerated if there is a positive cosmological constant. This was been interpreted by postulating an increasing gravity and as a result, the universe has to increase its expansion rate to escape the future collapse or alternatively, the decayed vacuum energy is given as a kinetic energy to accelerate the expansion of the universe. In our model, the cosmological constant is equal to zero and the accelerated expansion is attributed to the presence of a friction decaying force.

6. THE RADIATION-DOMINATED EPOCH

This is characterized by the equation of state $p = r/3$ and is modeled by equations (8) and (9) combined in the following form $(k = 0)$:

$$\frac{\ddot{R}}{R} + \frac{4(a-1)}{T}\frac{\dot{R}}{R} + \frac{\dot{R}^2}{R^2} = 0\,. \tag{16}$$

A possible solution is given also by the power-law $R = T^p$, where $p = (5-4a)/2$. $p > 1$ again for $0 < a < 3/4$ and the acceleration of the universe may be attributed again to the presence of the fractional dissipative force. Consequently, the radiation-density of the universe decays as $r \propto (5-4a)/4T^2$. For $a < 3/4$, the deceleration parameter $q = (1-p)/p = (4a-3)/(5-4a)$ is negative, the age of the universe is $H = p/T = (5-4a)/2T$ and consequently $1/T < H < 2/T$ in agreement again with recent observations. The density parameter in the radiation epoch of the universe is given by $\Omega^r = r/r_c = 1/(5-4a)$ and $\Omega^{Total} = 1$ if $\Omega^f = 1-\Omega^r = 4(1-a)/(5-4a)$ and consequently, the fractional decaying friction force in the radiation epoch is given by $r_f \propto (1-a)(5-4a)/T^2$. It is also easy to verify that $\Omega^{Total} = 1$ for $a = 1$.

7. PARTICLE CREATION IN THE MATTER AND RADIATION DOMINATED EPOCH

We now turn to calculate the rate of particle creation (annihilation) which is defined as $n_p = \left(1/R_0^3\right)\left(d\left(r R^3\right)/dT\right)_0$, where R_0 is the actual value of the scale factor [18]. In the matter-dominated epoch, simple mathematical manipulation gives $n_p = 6(1-a)\, r_0 H_p/(3-2a)$, with $0 < a < 3/4$. H_p is the present value of the Hubble parameter. Note that for $a = 1$, there is no particle creation or annihilation as in the standard model. For the restricted value of the fractional parameter, $r_0 H_0 < n_p < 2 r_0 H_0$ and this rate is less than that of the Steady State Cosmology which is $3 r_0 H_0$. This is to say that the decaying fractional force introduced from the fractional function action is responsible of the creation of particle in the universe. In the radiation-dominated epoch, we find $n_p = r_0 H_p (11-12a)/(5-4a)$ which for the restricted value of a yields $r_0 H_p < n_p < 11 r_0 H_p/5$ less again than that of the Steady State Cosmology and in contrast to the Standard Model, where there are nor creation neither annihilation.

8. CONCLUSIONS

While vast literature exists to address the observational fact of the current expansion and evolution of the universe, we are not aware of models similar to the one developed in this paper. In summary, we have discussed through this work a modified cosmology from fractional action integral approach with its main feature a "perturbed gravity" and consequently a "dissipative cosmological constant". Our approach is an extension of the Einstein's General Relativity. We have showed that the modified fractional cosmology enrich the Big-Bang cosmology. The cosmological constant in our formalism is zero and the accelerated expansion of the universe may be attributed to the fractional dissipative force with no need to introduce any kind of exotic matters or scalar field. The age, horizon and flatness problems are better solved.

Further consequences are in progress, in particular the presence of ultra-light masses in the theory which were proved recently to play an important role in many cosmological scenarios [40–52]. Although, the presence of friction in General Relativity and vacuum needs more investigation, the model described in this paper reveals nice interesting features and could have important consequences in quantum field theory, quantum gravity and modern astrophysics. Further consequences and details are in progress.

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