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CONDENSED MATTER

A NEW-IMPROVED HSDT DEFORMATION HYPOTHESIS USED TO MODEL THE ELASTIC BEHAVIOR OF THE PRISMATIC COMPOSITE BARS HAVING ELASTIC SYMMETRY

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Abstract. This paper presents an original HSDT deformation hypothesis, which satisfies the Saint-Vénaut correlation, some of the Cauchy conditions and –for the very first time- the Gay conditions. In some cases – when the bar is thin enough – this deformation theory can be simply reduced at the well-known Bernoulli hypothesis.

Key words: composite bars, HSDT deformation theory, conditions of the Theory of Elasticity (Saint-Vénaut, Cauchy).

1. INTRODUCTION

It is known that the success in action of modeling the elastic behavior of composite materials depends mainly on quality of the elastic deformation hypothesis that is used in order to reach this purpose. This paper presents a hypothesis like this based on a HSDT (third order) field of elastic displacements. This field presents also an advantage: it can take a very simple form in case of thin bars: the well – known Bernoulli hypothesis.

2. THEORETICAL CONSIDERATIONS

We will consider a right composite bar having a rectangular section. Let's note the length of the bar with *L*, the bar's section having the dimensions *B* and *H*. We'll consider that the bar has a constant section and – very important – it remains right in its non deformed status. The bar has its own reference system $Ox_1x_2x_3$

positioned the way that the plane Ox_2x_3 is parallel to the plane of the section the Ox_1, axis being positioned on the length of the bar.

If the bar is not a homogeneous one, we can apply a homogenization theory like the one that had been presented in [1]. In this case the Ox_1 axis can be considered as describing a masic and elastic symmetry.

In conditions like those presented we'll consider an elastic displacements field:

$$\vec{w} = \{w\}^t \{\vec{i}\}, \tag{1}$$

where $\{\vec{i}\} = \{\vec{i}_1; \vec{i}_2; \vec{i}_3\}^t$ is the unit vector basis of the $Ox_1x_2x_3$ reference system and $\{w\} = \{w_1, \cdot w_2, \cdot w_3\}$ is the column vector of $\vec{w}$.

Our assumption is that:

$$\vec{w} = \vec{u} + \vec{\theta} \times \vec{r} - \vec{r}_I \times [\vec{r} \times (\vec{r} \times \vec{\varphi})] - \vec{r}_{III} \times \vec{\varphi}, \tag{2}$$

where:

1. $\vec{u} = \{u\}^t \{\vec{i}\} = \{u_1; u_2; u_3\}\{\vec{i}\}$ characterizes the displacement of the elastic center of the bar section;

2. $\vec{\theta} = \{\theta\}^t \{\vec{i}\} = \{\theta_1; \theta_2; \theta_3\}^t \{\vec{i}\}$ characterizes the total rotation of the section of the bar. We have: $\vec{u} = \vec{u}(x_1, t)$ and $\vec{\theta} = \vec{\theta}(x_1, t)$;

3. $\vec{r} = \{0; x_2; x_3\}\{\vec{i}\}$ describes the position of a point of the bar considered in its own section, which section is characterized by its x_1 coordinate.

4. $\vec{r}_I = \{0; \alpha x_2; \beta x_3\}^t \{\vec{i}\}$ is a vector, very much like $\vec{r}$, where α, β are *numbers* depending on the *form of the bar section*;

5. $\vec{\varphi} = \{0; \varphi_2; \varphi_3\}\{\vec{i}\}^t$ is a vector that considers the rotations of the bar sections only around the x_2 and x_3 axis:

$$\varphi_2 = \theta_2 + \frac{\partial u_3}{\partial x_1}; \ \varphi_3 = \theta_3 - \frac{\partial u_2}{\partial x_1}, \tag{3}$$

where:

$u_{3,1} = \dfrac{\partial u_3}{\partial x_1}$ and $u_{2,1} = \dfrac{\partial u_2}{\partial x_1}$ describes the rotations of the sections due to simple bending (Bernoulli); we have: $\vec{\varphi} = \vec{\varphi}(x_1, t)$.

6. $\vec{r}_{III} = \left\{0; k\alpha x_2^3; k\beta x_3^3\right\}\left\{\vec{i}\right\}$ is a correcting vector, k being a corrector (scalar) having no physical dimension.

The vectorial relations (2) produces [2], elementarely, the following relations:

$$\begin{cases} w_1 = u_1 - \theta_3 x_2 + \theta_2 x_3 - \alpha\varphi_2 x_2^2 x_3 + \beta\varphi_3 x_2 x_3^2 + \\ \qquad + (1-k)\alpha\varphi_3 x_2^3 + (k-1)\beta\varphi_2 x_3^3 \\ w_2 = u_2 - \theta_1 x_3 \\ w_3 = u_3 + \theta_1 x_2. \end{cases} \tag{4}$$

In case we are considering a homogeneous bar (or a "homogenized" bar) [3, 4] we put:

$$\alpha = \frac{4}{B^2};\ \beta = \frac{4}{H^2};\ k = \frac{14}{9}; \tag{5}$$

So, (4) becomes, with (3):

$$\begin{cases} w_1 = u_1 - \theta_3 x_2 + \theta_2 x_3 + \dfrac{4}{H^2}(\theta_3 - \dfrac{\partial u_2}{\partial x_1}) x_2 x_3^2 - \dfrac{4}{B^2}(\theta_2 + \dfrac{\partial u_3}{\partial x_1}) x_2^2 x_3 - \\ \qquad - \dfrac{4}{B^2}(\theta_3 - \dfrac{\partial u_2}{\partial x_1}) x_2^3 + \dfrac{5}{9}\dfrac{4}{H^2}(\theta_2 + \dfrac{\partial u_3}{\partial x_1}) x_3^3 \\ w_2 = u_2 - \theta_1 x_3 \\ w_3 = u_3 + \theta_1 x_2. \end{cases} \tag{4'}$$

It can be elementarely proved that, using a matrix formalism (4) becomes, taking account of (3):

$$\{w\} = \{u\} - [r]\{\theta\} - \left[[r][T][r][r] + \frac{14}{9}[r][\Delta][\Delta][T]\right]\{\varphi\}, \tag{5}$$

where:

$$[r] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{bmatrix};\ [\Delta] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_3 & 0 \\ 0 & 0 & x_2 \end{bmatrix};\ [T] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{4}{H^2} & 0 \\ 0 & 0 & \dfrac{4}{B^2} \end{bmatrix}. \tag{6}$$

It can be elementarely verified that the Saint-Vénant conditions are respected:

$$[\mathrm{SV}]([\mathrm{D}]\{w\}) = \{0\}\,, \tag{7}$$

where:

$$[\mathrm{D}] = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \tag{8}$$

$$[\mathrm{SV}] = \begin{bmatrix} 0 & \frac{\partial^2}{\partial x_3^2} & \frac{\partial^2}{\partial x_2^2} & -\frac{\partial^2}{\partial x_2 \partial x_3} & 0 & 0 \\ \frac{\partial^2}{\partial x_3^2} & 0 & \frac{\partial^2}{\partial x_1^2} & 0 & -\frac{\partial^2}{\partial x_1 \partial x_3} & 0 \\ \frac{\partial^2}{\partial x_2^2} & \frac{\partial^2}{\partial x_1^2} & 0 & 0 & 0 & -\frac{\partial^2}{\partial x_1 \partial x_2} \\ -\frac{\partial^2}{\partial x_2 \partial x_3} & 0 & 0 & -\frac{1}{2}\cdot\frac{\partial^2}{\partial x_1^2} & \frac{1}{2}\cdot\frac{\partial^2}{\partial x_1 \partial x_2} & \frac{1}{2}\cdot\frac{\partial^2}{\partial x_1 \partial x_3} \\ 0 & -\frac{\partial^2}{\partial x_1 \partial x_3} & 0 & \frac{1}{2}\cdot\frac{\partial^2}{\partial x_1 \partial x_2} & -\frac{1}{2}\cdot\frac{\partial^2}{\partial x_2^2} & \frac{1}{2}\cdot\frac{\partial^2}{\partial x_2 \partial x_3} \\ 0 & 0 & -\frac{\partial^2}{\partial x_1 \partial x_2} & \frac{1}{2}\cdot\frac{\partial^2}{\partial x_1 \partial x_3} & \frac{1}{2}\cdot\frac{\partial^2}{\partial x_2 \partial x_3} & -\frac{1}{2}\cdot\frac{\partial^2}{\partial x_3^2} \end{bmatrix} \tag{9}$$

are the operator [D] of the linear-elastic deformations and [SV] is the Saint-Vénant operator.

Considering an ortothropic elastic symmetry the constitutive equation of the material of the bar is:

$$\{\sigma\} = [C]([\mathrm{D}]\{w\})\,; \tag{10}$$

where [C] is the rigidy matrix:

$$[C]=\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix} \tag{11}$$

where:

$$C_{ijkl}=C_{jikl}=C_{ijlk}=C_{klij}, \quad i,j,k=\overline{1,3} \tag{12}$$

and:

$$\{\sigma\}=\{\sigma_{11};\sigma_{22};\sigma_{33};\sigma_{23};\sigma_{13};\sigma_{12}\}^t \tag{13}$$

is the very well known strain column vector.

Using (3) in (8) it result a very important issue:

$$\sigma_{23}=\sigma_{32}=0 \text{ (Cauchy).} \tag{13'}$$

The Gay conditions [5] for a bar like this which is studied take the following particular form:

$$\iint_S \left[[r][T][r][r]+\frac{14}{9}[r][\Delta][\Delta][T]\right]\{\varphi\}\mathrm{d}S=\{0\},$$

$$\iint_S [r]\left[[r][T][r][r]+\frac{14}{9}[r][\Delta][\Delta][T]\right]\{\varphi\}\mathrm{d}S=\{0\}, \tag{14}$$

the bar being considered a homogeneous one. In (14) we noted S as the bar section surface.

The fact of the matter is that the Gay conditions [5] are taking the next form, in case of homogeneous bars, that elementarely results from the matrix form (14):

$$\begin{gathered}\iint_S \Delta w_1 \mathrm{d}S=0; \quad \iint_S \Delta w_2 \mathrm{d}S=0; \quad \iint_S \Delta w_3 \mathrm{d}S=0; \\ \iint_S \Delta w_1 x_2 \mathrm{d}S=0; \quad \iint_S \Delta w_1 x_3 \mathrm{d}S=0,\end{gathered} \tag{15}$$

where we have been noted:

$$\begin{aligned}\Delta w_1 &= -\frac{4}{B^2}\varphi_2 x_2^2 x_3 + \frac{4}{H^2}\varphi_3 x_2 x_3^2 - \\ &\quad -\frac{5}{9}\cdot\frac{4}{B^2}\varphi_3 x_2^3 + \frac{5}{9}\cdot\frac{4}{H^2}\varphi_2 x_3^3, \\ \Delta w_2 &= -\theta_1 x_3, \\ \Delta w_3 &= \theta_1 x_2.\end{aligned} \tag{16}$$

So, the relations (14) or (15) are, in fact, the following:

$$\iint_S \Delta w_2 \mathrm{d}S = \iint_S \Delta w_2 \mathrm{d}x_2 \mathrm{d}x_3 = -\theta_1 \int_{-\frac{B}{2}}^{\frac{B}{2}} \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3 \mathrm{d}x_3 = 0, \tag{17}$$

$$\iint_S \Delta w_3 \mathrm{d}S = \iint_S \Delta w_3 \mathrm{d}x_2 \mathrm{d}x_3 = \theta_1 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} \mathrm{d}x_3 = 0, \tag{18}$$

$$\begin{aligned}&\iint_S \Delta w_1 x_2 \mathrm{d}S = \iint_S x_2 \Delta w_1 \mathrm{d}x_2 \mathrm{d}x_3 = +\frac{4}{H^2}\varphi_3 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2^2 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3^2 \mathrm{d}x_3 - \\ &\quad -\frac{4}{B^2}\varphi_2 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2^3 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3 \mathrm{d}x_3 - \frac{5}{9}\cdot\frac{4}{B^2}\varphi_3 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2^4 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} \mathrm{d}x_3 + \\ &\quad +\frac{5}{9}\cdot\frac{4}{H^2}\varphi_2 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3^3 \mathrm{d}x_3 = \frac{4\varphi_3}{H^2}\cdot \left.\frac{x_2^3}{3}\right|_{-\frac{B}{2}}^{\frac{B}{2}} \cdot \left.\frac{x_3^3}{3}\right|_{-\frac{H}{2}}^{\frac{H}{2}} - \\ &\quad -\frac{5}{9}\cdot\frac{4\varphi_3}{B^2}\cdot \left.\frac{x_2^5}{5}\right|_{-\frac{B}{2}}^{\frac{B}{2}} \cdot \left. x_3\right|_{-\frac{H}{2}}^{\frac{H}{2}} = \frac{4\varphi_3}{H^2}\cdot\frac{1}{3}\left(\frac{B^3}{8}+\frac{B^3}{8}\right)\left(\frac{H^3}{8}+\frac{H^3}{8}\right)\cdot\frac{1}{3} - \\ &\quad -\frac{5}{9}\cdot\frac{4\varphi_3}{B^2}\cdot\frac{1}{5}\left(\frac{B^5}{32}+\frac{B^5}{32}\right)\cdot H = \frac{\varphi_3 H B^3}{36} - \frac{\varphi_3 H B^3}{36} = 0,\end{aligned} \tag{19}$$

$$\begin{aligned}
&\iint_S x_3 \Delta w_1 \mathrm{d}S = \iint_S x_3 \Delta w_1 \mathrm{d}x_2 \mathrm{d}x_3 = \frac{4}{H^2}\varphi_3 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3^3 \mathrm{d}x_3 - \\
&\quad - \frac{4}{B^2}\varphi_2 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2^2 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3^2 \mathrm{d}x_3 - \frac{5}{9}\cdot\frac{4}{B^2}\varphi_3 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2^3 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3 \mathrm{d}x_3 + \\
&\quad + \frac{5}{9}\cdot\frac{4}{H^2}\varphi_2 \int_{-\frac{B}{2}}^{\frac{B}{2}} \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3^4 \mathrm{d}x_3 = -\frac{4\varphi_2}{B^2}\cdot \left.\frac{x_2^3}{3}\right|_{-\frac{B}{2}}^{\frac{B}{2}} \cdot \left.\frac{x_3^3}{3}\right|_{-\frac{H}{2}}^{\frac{H}{2}} + \qquad (20)\\
&\quad + \frac{5}{9}\cdot\frac{4\varphi_2}{B^2}\cdot \left. x_2 \right|_{-\frac{B}{2}}^{\frac{B}{2}} \cdot \left.\frac{x_3^5}{5}\right|_{-\frac{H}{2}}^{\frac{H}{2}} = -\frac{4\varphi_2}{H^2}\cdot\frac{1}{3}\left(\frac{B^3}{8}+\frac{B^3}{8}\right)\cdot\frac{1}{3}\left(\frac{H^3}{8}+\frac{H^3}{8}\right) + \\
&\quad + \frac{5}{9}\cdot\frac{4\varphi_2}{B^2}\cdot B\cdot\frac{1}{5}\left(\frac{H^5}{32}+\frac{H^5}{32}\right) = -\frac{\varphi_3 B H^3}{36} + \frac{\varphi_3 B H^3}{36} = 0
\end{aligned}$$

$$\begin{aligned}
&\iint_S \Delta w_1 \mathrm{d}S = \iint_S \Delta w_1 \mathrm{d}x_2 \mathrm{d}x_3 = \frac{4}{B^2}\varphi_2 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2^2 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3 \mathrm{d}x_3 + \\
&\quad + \frac{4}{H^2}\varphi_3 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3^2 \mathrm{d}x_3 - \frac{5}{9}\cdot\frac{4}{B^2}\varphi_3 \int_{-\frac{B}{2}}^{\frac{B}{2}} x_2^3 \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} \mathrm{d}x_3 + \qquad (21)\\
&\quad + \frac{5}{9}\cdot\frac{4}{H^2}\varphi_2 \int_{-\frac{B}{2}}^{\frac{B}{2}} \mathrm{d}x_2 \int_{-\frac{H}{2}}^{\frac{H}{2}} x_3^3 \mathrm{d}x_3 = 0.
\end{aligned}$$

3. CONCLUSIONS

We conclude that for an orthotropic homogeneous right composite bar, free of tensions on its frontier, we have at our disposal a displacements field of third order (HSDT) that satisfies some very important issues:

– it satisfies the Saint-Vénant, conditions of compatibility;

– $\sigma_{23} = \sigma_{32} \equiv 0$, on the frontier;

– the Gay conditions are satisfied, so we can assimilate the deformed section with a plane one;

– the displacements field, in its matrix form, is written using some significant issues concerning the form of bar section, issues that have important geometrical interpretation.

This paper is only a very small contribution added to the generally resuming effort to step beyond the Bernoulli-hypothesis which simply doesn't work in case of non-homogenous (composite) materials.

The work is far from being finished.

The theoretical research must be continued, at least for homogenous materials, in order to reach a deformations hypothesis that could satisfy more in terms of the condition to be the strain-stress status is zero on the frontier (here we have only $\sigma_{23} = \sigma_{32} = 0$).

In case of non-homogenous materials we shall use a "homogenization method" combined with this (or a better one) displacements field.

Elsewhere, we must be fair and recognize that all considerations we made are available only in case of right bars. The case of curved bars raises some particular difficult problems and must be treated separately. Also the bars (whatever they are right or curved) which are not tensions free on the frontier rises some enormous difficulties of studying them.

Finally, we have to add that in case of non-homogenous bars is not appropriate to combine whatever homogenization method with no matter HSDT deformation theory; it must be verified the compatibility of strain-stress conditions between whatever two constituents of the composite material and the strain-stress conditions on the frontier of the material.

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