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RIEMANN SURFACE APPROACH TO NATURAL MODES IN QUANTUM PHYSICS

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Abstract. An approach to natural modes in the scattering by a central potential of complex strength g with a barrier, based on the global method for all S-matrix poles analysis is presented. The Riemann surface R_g over the g-plane, on which the pole function k = k(g) is single-valued and analytic is constructed. By using the Riemann surface approach we treat in a unified way all the natural modes (bound and resonant states) of the system particle + potential. New classes of poles and resonant states (exotic) are identified and their properties are studied.

Key words: natural modes, bound and resonant states.

1. INTRODUCTION

The natural modes of the system (particle + potential) are defined as those regular solutions of the 3D Schrödinger equation that satisfy the so-called Sommerfeld radiation condition at infinity [1]. The natural modes are radiative (in which case the energy is complex) or nonradiative (in which case the energy is real). The radiative natural modes are identified with the resonant states and the nonradiative ones with bound states of the system [1]. By using the Riemann surface approach to bound and resonant states we treat in a unified way all the natural modes of the system. The resonant structures in the cross section of many interesting problems, such as field ionization, photo-ionization, electron scattering on atoms and molecules, and nuclear scattering are, in most cases, ascribed to resonant states of the quantum system. The most fundamental approach to resonant scattering is through the analytical properties of the S-matrix [2–4]. Let us consider the nonrelativistic scattering of a particle by a central potential $V(r) = gV_n(r) +$ + $V_{bar}(r)$, where the short range complex nuclear potential V_n of strength $g \in C$ has a square or a Woods-Saxon form factor, and V_{bar} is a potential barrier. The S-matrix poles are the solutions k(g) of the equation $\mathcal{F}_+(g,k) = 0$, where $\mathcal{F}_+(g,k)$ is the Jost function, provided that $\mathcal{F}_{-}(g,k) \neq 0$. Here $\mathcal{F}_{+}(g,k)$ and $\mathcal{F}_{-}(g,k)$ are the denominator and the numerator, respectively of the S-matrix element. Let k(g) be the function which gives the S-matrix pole position as a function of the potential strength g. The function k(g) is a many-valued function of the complex variable g. The S-matrix poles distribution in the k-plane has been extensively studied by using the *pole trajectory method*: a particular path in the complex g-plane is chosen and the corresponding trajectory of the S-matrix poles in the k-plane is determined. The pole trajectory method does not provide all S-matrix poles, some important poles being lost, and one can never be sure that the same pole is followed. In order to have a complete description of the S-matrix poles in the k-plane, the *Riemann surface approach* based on a global analysis of the function k(g) must be used.

2. RIEMANN SURFACE APPROACH TO BOUND AND RESONANT STATES

The Riemann surface approach to S-matrix poles consists in constructing the Riemann surface R_g over the g-plane, on which the pole function k(g) is single valued and analytic. This implies the division of the Riemann surface R_g into sheets Σ_n and the construction of the Riemann sheet images Σ'_n in the k-plane. If g takes values on a given Riemann sheet Σ_n , then the pole k(g) belongs to the k-plane Riemann sheet image Σ'_n . The number n that labels the Riemann sheet Σ_n and the k-plane Riemann sheet image Σ'_n is used as a new quantum number for this pole and for the corresponding state of the quantum system. A schematic illustration of the method is given in Fig. 1.

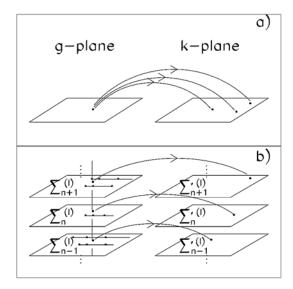


Fig. 1 – a) The many-valued function k(g) defined on the complex g-plane and its distinct values in the k-plane. b) The single-valued function k(g) defined on the Riemann surface over the complex g-plane. The branch-points indicated by ★ and the branch-lines that join the sheets are shown.

The method has been used for various shapes of potentials: a rectangular or Woods-Saxon nuclear potential plus a barrier which can be a rectangular or Coulomb barrier, as well as a centrifugal component resulted from partial waves decomposition [5–9]. For each of these shapes of potentials the singular points have been determined as solutions g_i of the system

$$\mathcal{F}_{+}(g,k) = 0, \tag{1}$$

$$\partial \mathcal{F}_{+}(g,k)/\partial k = 0.$$
 (2)

From among these singular points those that are branch points may be found by permitting the variable g to describe successive small circuits round each singular point g_i , and by observing whether the function k(g) returns to its initial value. Let m > 1 be the smallest number of rotations after which one obtains again the initial value of the function k(g). Then g_i is a branch point of order m-1 and m sheets of the Riemann surface are joined at this point. The border of any k-plane image of a Riemann sheet is obtained by letting g trace a path along the cuts on the corresponding Riemann sheet, without crossing them, and along a large radius circle joining the cuts.

3. MERITS OF THE RIEMANN SURFACE APPROACH TO BOUND AND RESONANT STATES

The Riemann surface approach to bound and resonant states, based on the global method for all S-matrix poles analysis has several merits:

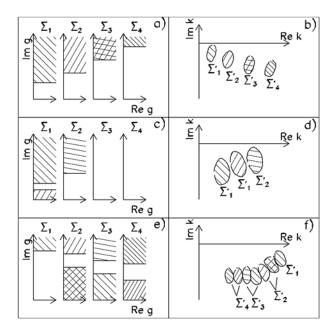
- 1. Instead of analyzing an infinity of poles in the k-plane it allows us to analyze the single pole on each Riemann sheet image Σ'_n in the k-plane. In this way no pole can be lost and one can be sure that the same pole is studied.
- 2. By the Riemann surface approach to a given state n of the quantum system one associates a sheet Σ_n of the Riemann surface R_g . This approach allows not only studying each state n, but also understanding the transition from a state n to another state m as a result of the potential strength variation. Indeed, let us suppose that g describes a closed contour which starts from a point on sheet Σ_n and encloses the branch-point joining the sheets Σ_n and Σ_m . Then the pole passes from the sheet image Σ'_n to the sheet image Σ'_n , i.e. the system makes a transition from the state n to the state m, as a result of potential strength variation. Here the states n and m can be either bound or resonant.
- 3. For a given potential form factor V(r) the well and barrier with absorption and emission are treated simultaneously, which allows a smooth transition from one case to the other. Indeed, as g covers all the complex plane, each Riemann sheet

- Σ_n for the potential $gV(r) + V_{bar}(r)$ contains a wall and a barrier with absorption or emission. If g follows a continuous path on a given sheet Σ_n , then the corresponding pole follows a continuous path in the k-plane sheet image Σ'_n .
- 4. The global method for all S-matrix poles analysis is stable under the potential strength variation. Indeed, one cannot create or destroy S-matrix poles by varying the strength of the potential in the analyticity domain of the pole function k(g). The poles can be created or destroyed only at the branch-point g = 0.
- 5. A new quantum number n with topological meaning (label of the Riemann sheet Σ_n and of its k-plane image Σ'_n) is introduced in order to label the pole belonging to Σ'_n and the corresponding state.
- 6. The Riemann surface approach allows the identification of a new class of S-matrix poles (exotic poles) with properties that differ from those of the usual S-matrix poles. While the usual resonant state poles move towards the imaginary k-axis and become bound $(\Im k > 0)$ or virtual $(\Im k < 0)$ state poles as the strength of the potential well is increased, the exotic resonant state poles remain in some bound regions of the lower k half-plane, even for $g \to \infty$.

4. PROPERTIES OF THE EXOTIC RESONANT POLES AND STATES

1. The exotic resonant state poles remain in the neighborhood of some special points called "stable points" which act as "attractors" when the strength of the potential well increases indefinitely. These exotic resonant state poles are insensitive to the behavior of the potential in the region of the well, and are almost completely determined by the geometric shape of the potential barrier. The number and position of the bound regions in the k-plane where the exotic poles are located depends on the shape and height of the barrier. For a rectangular barrier there are exotic poles only for strong absorptive potentials. There is an infinite number of Riemann sheet images on which there are situated exotic poles. On each Riemann sheet image there is only one bound region for the exotic resonant pole. In the case of a rectangular or Woods-Saxon well with centrifugal barrier there are exotic poles on a finite number of Riemann sheet images, the number of these sheet images increasing as the orbital angular momentum l increases. The exotic poles occur for either strong or week absorption. In the case of a rectangular or Woods-Saxon well with Coulomb and centrifugal barrier there is an infinite number of Riemann sheet images where the exotic poles are situated, and the exotic poles occur for either strong or week absorption. The situation is illustrated in Fig. 2.

Fig. 2 – The first four sheets Σ_n and the aggregates of their k-plane images Σ'_n . In Figs. (a) and (b) the case of a rectangular well followed by a rectangular barrier is shown; in Figs. (c) and (d) the case of a rectangular or Woods-Saxon well with centrifugal barrier is illustrated; in Figs. (e) and (f) the case of a rectangular or Woods-Saxon well with Coulomb and centrifugal barriers given.



2. The way the exotic resonant state poles move in the *k*-plane as the strength of the potential well increases is shown in Fig. 3. One can see that there are some "attractors" in the *k*-plane towards which the pole move.

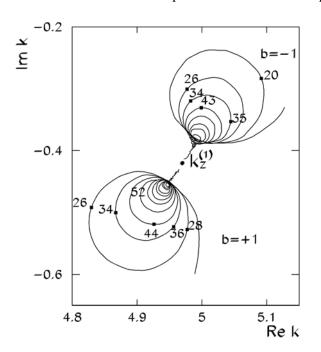


Fig. 3 – The pole trajectories on the sheet image Σ_1' for the potential made of a rectangular well followed by a rectangular barrier. The strength of the potential well varies as $g = \lambda(a+ib)$ with a=20, $b=\pm 1$ and the strength of the barrier is $V_1=16$. By $k_z^{(1)}$ the attractor is marked. The numbers besides the curves gave the values of the parameter λ .

3. The wave functions of the resonant states corresponding to poles situated in the neighborhood of the attractors are almost completely confined to the region of the barrier. This is illustrated in Fig. 4 for the potential made of a rectangular well followed by a rectangular barrier. As a consequence of the localization of their wave functions the exotic resonant states are almost insensitive to the behavior of the potential inside the barrier region. This fact is very important, due to the fact that in the nuclear physics the potential is much better known in the barrier region than in the well region. The small amplitude of the wave function of an exotic resonant state inside the well leads to a small overlap with the adjacent resonant states of the continuum, that are mostly confined to the compound nucleus radius. This small overlap confers stability to the exotic resonant states against the dissolution into the complex neighboring compound nuclear states. The exotic resonant states are a new kind of doorway states, whose stability is a consequence of the localization of the wave function, rather than of a symmetry. In the case of a nuclear well with Coulomb and centrifugal barrier it has been shown [6, 11] that the parent quasimolecular resonant states excited in heavy ion reactions are exotic resonant states. In this case it has been finally understood how a short-range nuclear attraction could be operative in order to generate resonant states having the wave function localized outside the potential well. Moreover, the properties of the parent quasimolecular resonant states (excitation energies, deviation from the linear dependence of excitation energies on l(l+1), widths, stability, observability) result naturally from the general properties of the exotic resonant states.

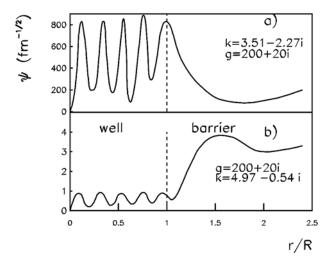


Fig. 4 – The moduli of the wave function of an usual resonant state (a) and of an exotic resonant state (b) for the potential made of a rectangular well followed by a rectangular barrier. The usual pole is situated on Σ_1' , while the exotic pole is situated on Σ_5' . The same radii for the well and the barrier are taken. The values of the potential well depth g and the corresponding poles are given.

The Riemann surface approach to all natural modes of the quantum system has been generalized to the case of a two channel model [12]. It has been shown that besides the branch-points which originate from the branch-points for the

uncoupled channels there are new branch-points which are intrinsic to the coupling. The classification of resonant states as Feshbach or Fonda-Newton resonant states [13, 14] has been clarified by studying the behaviour of the corresponding resonant state poles as the strength of the coupling $V_{12} \rightarrow 0$. Taking into account the analysis of the branch-points four classes of Riemann sheets have been identified: on the images of the first class Riemann sheets there are poles that correspond to bound or shape resonant states, on the images of the second class there are poles corresponding to bound or Feshbach resonant states, and on the images of the third and fourth classes of Riemann sheets there are poles corresponding to bound or Fonda-Newton resonant states. In this case two new quantum numbers (m, n) with topological meaning, which label each bound or resonant state, have been introduced. The first indicates the class of the Riemann sheet, while the second labels the sheet for each class. The approach provides a new insight into the nature of the resonant states for two coupled channels. The distinct origin and the relationship of the Feshbach and Newton-Fonda resonant states have been clarified.

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