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ENTANGLEMENT IN OPEN QUANTUM SYSTEMS

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Abstract. In the framework of the theory of open systems based on quantum dynamical semigroups, we solve the master equation for two independent bosonic oscillators interacting with an environment in the asymptotic long-time regime. We give a description of the continuous-variable entanglement in terms of the covariance matrix of the quantum states of the considered system for an arbitrary Gaussian input state. Using the Peres–Simon necessary and sufficient condition for separability of two-mode Gaussian states, we show that the two non-interacting systems immersed in a common environment and evolving under a Markovian, completely positive dynamics become asymptotically entangled for certain environments, so that their non-local quantum correlations exist in the long-time regime.

Key words: quantum information theory, open systems, quantum entanglement, inseparable states.

1. INTRODUCTION

Quantum entanglement represents the physical resource in quantum information science which is indispensable for the description and performance of such tasks as teleportation, superdense coding, quantum cryptography and quantum computation [1]. Therefore the generation, detection and manipulation of the entanglement continues to be presently a problem of intense investigation.

When two systems are immersed in an environment, then the decoherence phenomenon usually occurs. At the same time, an external environment can also generate a quantum entanglement of the two systems and therefore an additional mechanism to correlate them [2–4]. In certain circumstances, the environment enhances entanglement and in others it suppresses the entanglement and the state describing the two systems becomes separable. Suppose the two systems are prepared in an initial separable state without either classical or quantum correlations and put into an environment. The structure and properties of the environment may be such that not only the two systems become entangled, but also such that a certain amount of entanglement survives in the asymptotic long-time regime. The reason is that even if not directly coupled, the two systems immersed in the same environment can interact through the environment itself and it depends on how strong this indirect interaction is with respect to the decoherence whether entanglement can be generated at the beginning of the evolution and, in the case of an affirmative answer, if it can be maintained for a definite time or it survives indefinitely in time.

In the present work we investigate the existence of the continuous variable entanglement for a subsystem composed of two identical bosonic oscillators interacting with an environment. Their evolution is described by a Markovian dynamics in the framework of the theory of open quantum systems, based on completely positive dynamical semigroups. We are interested in discussing the correlation effect of the environment, therefore we assume that the two systems are independent, *i.e.* they do not interact directly. The initial state of the subsystem is taken of Gaussian form and the evolution under the quantum dynamical semigroup assures the preservation in time of the Gaussian form of the state. We only investigate here the asymptotic behaviour of the subsystem states. The time evolution of the entanglement, in particular the possibility of the so-called "entanglement sudden death", that is suppression of the entanglement at a certain finite moment of time, is discussed elsewhere.

In Sec. 2 we write the equations of motion in the Heisenberg picture for the considered open system. With these equations we derive in Sec. 3 the asymptotic values of the variances and covariances of the coordinates and momenta which enter the asymptotic covariance matrix. In Sec. 4, by using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states [5, 6], we investigate the behaviour of the environment induced entanglement in the limit of long times. We show that for certain classes of environments the initial state evolves asymptotically to an equilibrium state which is entangled, while for other values of the parameters describing the environment, the entanglement is suppressed and the asymptotic state is separable. The existence of the quantum correlations between the two systems in the asymptotic long-time regime is the result of the competition between entanglement and decoherence. A summary is given in Sec. 5.

2. EQUATIONS OF MOTION IN HEISENBERG PICTURE

We are interested in the generation of entanglement between two harmonic oscillators due to the back-action of the environment on the subsystem. Since the two harmonic oscillators interact with a common environment, there will be induced coupling between the two oscillators even when initially they are uncoupled. Thus, the master equation for the two bosonic oscillators must account for their mutual interaction by their coupling to the common environment. We shall study the dynamics of the subsystem composed of the two identical non-interacting (independent) oscillators in weak interaction with a large environment, so that their reduced time evolution can be described by a Markovian, completely positive quantum dynamical semigroup.

If $\tilde{\Phi}_t$ is the dynamical semigroup describing the time evolution of the open quantum system in the Heisenberg picture, then the master equation is given for an operator A as follows [7, 8]:

$$\frac{d\tilde{\Phi}_t(A)}{dt}=\frac{i}{\hbar}\left[H,\tilde{\Phi}_t(A)\right]+\frac{1}{2\hbar}\sum_j\left(V_j^\dagger\left[\tilde{\Phi}_t(A),V_j\right]+\left[V_j^\dagger,\tilde{\Phi}_t(A)\right]V_j\right). \quad (1)$$

Here, H denotes the Hamiltonian of the open quantum system and V_j are operators defined on the Hilbert space of H. These operators represent the interaction of the open system with the environment and can be chosen freely. Being interested in the set of Gaussian states, we introduce those quantum dynamical semigroups that preserve that set. Therefore H is taken to be a polynomial of second degree in the coordinates x, y and momenta p_x, p_y of the two quantum oscillators and V_j, $V_j^\dagger$ are taken polynomials of only first degree in these canonical observables. Then in the linear space spanned by the coordinates and momenta there exist only four linearly independent operators $V_{j=1,2,3,4}$ [9]:

$$V_j=a_{xj}p_x+a_{yj}p_y+b_{xj}x+b_{yj}y, \quad (2)$$

where $a_{xj},a_{yj},b_{xj},b_{yj}\in\mathbf{C}$ and

$$V_j^\dagger=a_{xj}^*p_x+a_{yj}^*p_y+b_{xj}^*x+b_{yj}^*y, \quad (3)$$

where * denotes the complex conjugation.

The Hamiltonian H of the two uncoupled identical bosonic oscillators of mass m and frequency ω is chosen of the form

$$H=\frac{1}{2m}(p_x^2+p_y^2)+\frac{m\omega^2}{2}(x^2+y^2). \quad (4)$$

The fact that $\tilde{\Phi}_t$ is a dynamical semigroup implies the positivity of the following matrix formed by the scalar products of the four vectors $\mathbf{a}_x$, $\mathbf{b}_x$, $\mathbf{a}_y$, $\mathbf{b}_y$:

$$\frac{1}{2}\hbar\begin{pmatrix}(\mathbf{a}_x\mathbf{a}_x) & (\mathbf{a}_x\mathbf{b}_x) & (\mathbf{a}_x\mathbf{a}_y) & (\mathbf{a}_x\mathbf{b}_y)\\ (\mathbf{b}_x\mathbf{a}_x) & (\mathbf{b}_x\mathbf{b}_x) & (\mathbf{b}_x\mathbf{a}_y) & (\mathbf{b}_x\mathbf{b}_y)\\ (\mathbf{a}_y\mathbf{a}_x) & (\mathbf{a}_y\mathbf{b}_x) & (\mathbf{a}_y\mathbf{a}_y) & (\mathbf{a}_y\mathbf{b}_y)\\ (\mathbf{b}_y\mathbf{a}_x) & (\mathbf{b}_y\mathbf{b}_x) & (\mathbf{b}_y\mathbf{a}_y) & (\mathbf{b}_y\mathbf{b}_y)\end{pmatrix}. \quad (5)$$

For simplicity we take this matrix of the following form, where all coefficients $D_{..}$ and λ are real quantities:

$$\begin{pmatrix} D_{xx} & -D_{xp_x} - i\hbar\lambda/2 & D_{xy} & -D_{xp_y} \\ -D_{xp_x} + i\hbar\lambda/2 & D_{p_xp_x} & -D_{yp_x} & D_{p_xp_y} \\ D_{xy} & -D_{yp_x} & D_{yy} & -D_{yp_y} - i\hbar\lambda/2 \\ -D_{xp_y} & D_{p_xp_y} & -D_{yp_y} + i\hbar\lambda/2 & D_{p_yp_y} \end{pmatrix}. \tag{6}$$

It follows that the principal minors of this matrix are positive or zero. From the Cauchy-Schwarz inequality the following relations for the coefficients defined in Eq. (6) hold (from now on we put, for simplicity, $\hbar = 1$):

$$D_{xx}D_{yy} - D_{xy}^2 \geq 0, \quad D_{xx}D_{p_xp_x} - D_{xp_x}^2 \geq \frac{\lambda^2}{4}, \quad D_{xx}D_{p_yp_y} - D_{xp_y}^2 \geq 0, \tag{7}$$

$$D_{yy}D_{p_xp_x} - D_{yp_x}^2 \geq 0, \quad D_{yy}D_{p_yp_y} - D_{yp_y}^2 \geq \frac{\lambda^2}{4}, \quad D_{p_xp_x}D_{p_yp_y} - D_{p_xp_y}^2 \geq 0. \tag{8}$$

These inequalities are constraints imposed on the phenomenological constants by the fact that $\tilde{\Phi}_t$ is a dynamical semigroup.

The matrix of the coefficients (6) can be conveniently written as

$$\begin{pmatrix} C_1 & C_3 \\ C_3^\dagger & C_2 \end{pmatrix}, \tag{9}$$

in terms of 2×2 matrices $C_1 = C_1^\dagger$, $C_2 = C_2^\dagger$ and C_3. This decomposition has a direct physical interpretation: the elements containing the diagonal contributions C_1 and C_2 represent diffusion and dissipation coefficients corresponding to the first, respectively the second, system in absence of the other, while the elements in C_3 represent environment generated couplings between the two, initially independent, oscillators.

The variance and covariance of self-adjoint operators A_1 and A_2 can be written with the density operator ρ, describing the initial state of the quantum system, as follows:

$$\sigma_{A_1A_2}(t) = \frac{1}{2}\mathrm{Tr}(\rho\tilde{\Phi}_t(A_1A_2 + A_2A_1)). \tag{10}$$

We introduce the following 4×4 covariance matrix:

$$\sigma(t) = \begin{pmatrix} \sigma_{xx} & \sigma_{xp_x} & \sigma_{xy} & \sigma_{xp_y} \\ \sigma_{xp_x} & \sigma_{p_xp_x} & \sigma_{yp_x} & \sigma_{p_xp_y} \\ \sigma_{xy} & \sigma_{yp_x} & \sigma_{yy} & \sigma_{yp_y} \\ \sigma_{xp_y} & \sigma_{p_xp_y} & \sigma_{yp_y} & \sigma_{p_yp_y} \end{pmatrix}. \tag{11}$$

By direct calculation we obtain [9]:

$$\frac{d\sigma}{dt} = Y\sigma + \sigma Y^{\mathrm{T}} + 2D, \tag{12}$$

where

$$Y = \begin{pmatrix} -\lambda & 1/m & 0 & 0 \\ -m\omega^2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1/m \\ 0 & 0 & -m\omega^2 & -\lambda \end{pmatrix}, \tag{13}$$

D is the matrix of the diffusion coefficients

$$D = \begin{pmatrix} D_{xx} & D_{xp_x} & D_{xy} & D_{xp_y} \\ D_{xp_x} & D_{p_x p_x} & D_{yp_x} & D_{p_x p_y} \\ D_{xy} & D_{yp_x} & D_{yy} & D_{yp_y} \\ D_{xp_y} & D_{p_x p_y} & D_{yp_y} & D_{p_y p_y} \end{pmatrix} \tag{14}$$

and Y^{T} is the transposed matrix of Y. The time-dependent solution of Eq. (12) is given by [9]

$$\sigma(t) = M(t)(\sigma(0) - \sigma(\infty))M^{\mathrm{T}}(t) + \sigma(\infty), \tag{15}$$

where $M(t) = \exp(tY)$. The matrix $M(t)$ has to fulfil the condition $\lim_{t\to\infty} M(t) = 0.$ In order that this limit exists, Y must only have eigenvalues with negative real parts. The values at infinity are obtained from the equation [9]

$$Y\sigma(\infty) + \sigma(\infty)Y^{T} = -2D. \tag{16}$$

3. COVARIANCE MATRIX

The two-mode Gaussian state is entirely specified by its covariance matrix σ (11), which is a real, symmetric and positive matrix with the following block structure:

$$\sigma = \begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}, \tag{17}$$

where A, B and C are 2×2 matrices. Their entries are correlations of the canonical operators x, y, p_x, p_y, A and B denote the symmetric covariance matrices for the individual reduced one-mode states, while the matrix C contains the crosscorrelations between modes. The entries of the covariance matrix depend on Y and D and can be calculated from Eq. (16). To simplify further the calculations, we shall consider environments for which the two diagonal submatrices in Eq. (9) are equal: $C_1 = C_2$, so that $D_{xx} = D_{yy}$, $D_{xp_x} = D_{yp_y}$, $D_{p_xp_x} = D_{p_yp_y}$. In addition, in the matrix C_3 we take $D_{xp_y} = D_{yp_x}$. Then both unimodal covariance matrices are equal, $A = B$ and the entanglement matrix C is symmetric. With the chosen coefficients, we obtain the following elements of the asymptotic entanglement matrix C:

$$\sigma_{xy}(\infty) = \frac{m^2(2\lambda^2+\omega^2)D_{xy} + 2m\lambda D_{xp_y} + D_{p_xp_y}}{2m^2\lambda(\lambda^2+\omega^2)}, \tag{18}$$

$$\sigma_{xp_y}(\infty) = \sigma_{yp_x}(\infty) = \frac{-m^2\omega^2 D_{xy} + 2m\lambda D_{xp_y} + D_{p_xp_y}}{2m(\lambda^2+\omega^2)}, \tag{19}$$

$$\sigma_{p_xp_y}(\infty) = \frac{m^2\omega^4 D_{xy} - 2m\omega^2\lambda D_{xp_y} + (2\lambda^2+\omega^2)D_{p_xp_y}}{2\lambda(\lambda^2+\omega^2)}. \tag{20}$$

The entries of the matrices A and B are:

$$\sigma_{xx}(\infty) = \sigma_{yy}(\infty) = \frac{m^2(2\lambda^2+\omega^2)D_{xx} + 2m\lambda D_{xp_x} + D_{p_xp_x}}{2m^2\lambda(\lambda^2+\omega^2)}, \tag{21}$$

$$\sigma_{xp_x}(\infty) = \sigma_{yp_y}(\infty) = \frac{-m^2\omega^2 D_{xx} + 2m\lambda D_{xp_x} + D_{p_xp_x}}{2m(\lambda^2+\omega^2)}, \tag{22}$$

$$\sigma_{p_xp_x}(\infty) = \sigma_{p_yp_y}(\infty) = \frac{m^2\omega^4 D_{xx} - 2m\omega^2\lambda D_{xp_x} + (2\lambda^2+\omega^2)D_{p_xp_x}}{2\lambda(\lambda^2+\omega^2)}. \tag{23}$$

With these quantities we calculate the determinant of the entanglement matrix:

$$\det C = \frac{1}{4\lambda^2(\lambda^2+\omega^2)}\left[\left(m\omega^2 D_{xy} + \frac{1}{m}D_{p_xp_y}\right)^2 + 4\lambda^2(D_{xy}D_{p_xp_y} - D_{xp_y}^2)\right]. \tag{24}$$

It is very interesting that the general theory of open quantum systems allows couplings via the environment between uncoupled oscillators. According to the definitions of the environment parameters, the diffusion coefficients above can be different from zero and can simulate an interaction between the uncoupled oscillators. Indeed, the Gaussian states with $\det C \geq 0$ are separable states, but for $\det C < 0$, it may be possible that the asymptotic equilibrium states are entangled, as it will be shown in the next Section.

4. ENVIRONMENT INDUCED ENTANGLEMENT

On general grounds, one expects that the effects of decoherence, counter-acting entanglement production, be dominant in the long-time regime, so that no quantum correlation (no entanglement) is expected to be left at infinity. Nevertheless, there are situations in which the environment allows the presence of entangled asymptotic equilibrium states. In order to investigate whether an external environment can actually entangle the two independent systems, we can use the partial transposition criterion [5, 6]: a state results entangled if and only if the operation of partial transposition does not preserve its positivity. Simon [6] obtained the following necessary and sufficient criterion for separability: $S < 0$, where

$$S \equiv \det A \det B + \left(\frac{1}{4} - |\det C|\right)^2 - \mathrm{Tr}[AJCJBJC^{\mathrm{T}}J] - \frac{1}{4}(\det A + \det B) \tag{25}$$

and J is the 2×2 symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{26}$$

In order to analyze the possible persistence of the environment induced entanglement in the asymptotic long-time regime, we consider the environment characterized by the following values of its parameters: $m^2\omega^2 D_{xx} = D_{p_x p_x}$, $D_{xp_x} = 0$, $m^2\omega^2 D_{xy} = D_{p_x p_y}$. In this case the Simon expression (25) takes the form:

$$S = \left(\frac{m^2\omega^2(D_{xx}^2 - D_{xy}^2)}{\lambda^2} + \frac{D_{xp_y}^2}{\lambda^2 + \omega^2} - \frac{1}{4}\right)^2 - 4\frac{m^2\omega^2 D_{xx}^2 D_{xp_y}^2}{\lambda^2(\lambda^2 + \omega^2)}. \tag{27}$$

For environments characterized by such coefficients that the expression (27) is negative, the asymptotic final state is entangled. In particular, if $D_{xy} = 0$, we obtain that $S < 0$, *i.e.* the asymptotic final state is entangled, for the following range of values of the coefficient D_{xp_y} characterizing the environment:

$$\frac{m\omega D_{xx}}{\lambda} - \frac{1}{2} < \frac{D_{xp_y}}{\sqrt{\lambda^2 + \omega^2}} < \frac{m\omega D_{xx}}{\lambda} + \frac{1}{2}, \tag{28}$$

where the coefficient D_{xx} satisfies the condition $m\omega D_{xx}/\lambda \geq 1/2$, equivalent with the unimodal uncertainty relation. If the coefficients do not fulfil the inequalities (28), then $S \geq 0$ and therefore the asymptotic final state of the considered bipartite system is separable.

5. SUMMARY

In the framework of the theory of open quantum systems, based on completely positive dynamical semigroups, we investigated the existence of the quantum entanglement for a subsystem composed of two uncoupled identical bosonic oscillators interacting with an environment. By using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states, we have shown that for certain classes of environments the initial state evolves asymptotically to an equilibrium state which is entangled, *i.e.* there exist non-local quantum correlations for the bipartite states of the considered open system, while for other values of the coefficients describing the environment, the asymptotic state is separable. Due to the increasing interest manifested towards the continuous variables approach [10] to the theory of quantum information, the obtained results, in particular the possibility of maintaining a bipartite entanglement in a diffusive-dissipative environment even for asymptotic long times, could be useful for both phenomenological and experimental applications in the field of quantum information processing and communication.

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