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THE MOMENTUM REPRESENTATION OF THE DIRAC SPHERICAL SPINORS IN MOVING CHARTS OF THE de SITTER SPACETIME

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Abstract. The spherical spinors in momentum representation are derived using Fourier transformations as in Minkowski case. In this way one obtains the massless spinors in the chiral representation of the Dirac matrices.

Key words: Dirac spherical spinors, de Sitter spacetime.

The Dirac equation on de Sitter spacetime (dS) (whose cosmological constant Λ_c gives the radius $R = 1/\omega = \sqrt{3/\Lambda_c}$) has been studied in moving or static local charts (*i.e.* natural frames) suitable for separation of variables leading to significant analytical solutions [1–4].

The analytical solutions of the Dirac equation in moving charts with Cartesian coordinates $\{t, \vec{x}\}$ of the dS spacetime can be expressed in terms of space-like momentum [3]. The corresponding spherically symmetric solutions were found by Shishkin [1] in moving charts with spherical coordinates $\{t, r, \theta, \phi\}$ associated to the Cartesian ones $\{t, \vec{x}\}$ (with $r = |\vec{x}|$). Moreover, recently we have shown that these solutions are related among themselves, establishing the definitive form of the normalized Shiskin spherical spinors in coordinate representation [4]. Our purpose in this paper is to calculate the form of these spherical spinors in the momentum representation using the Fourier transformation as in Minkowski case [5].

The particular solutions of the Dirac equation derived in Ref. [4] in the chart $\{t, r, \theta, \phi\}$ are denoted by $\psi = \psi_{p,k_j,m_j}$ since these are common eigenspinors of the complete set of commuting operators $\{E_D, \mathbf{P}^2, \mathbf{J}^2, K, J_3\}$ corresponding to the eingenvalues $\{m, p^2, j(j+1), -k_j, m_j\}$. We use the standard notation [6] with

$k_j = \pm\left(j+\frac{1}{2}\right)$ while p is the value of the scalar momentum. These solutions are normalized with respect to the time-independent relativistic scalar product defined as

$$\left\langle \psi, \psi' \right\rangle = \int d^3 x e^{3\omega t} \overline{\psi}(t,\vec{x}) \gamma^0 \psi'(t,\vec{x}), \tag{1}$$

in the chart $\{t,\vec{x}\}$. The standard form of the the normalized spinors of positive frequencies is

$$\begin{aligned} \psi_{p,k_j,m_j}(x) = \frac{p}{2}\sqrt{\frac{\pi}{\omega r}} e^{-2\omega t} \Bigg[& e^{\pi k/2} H^{(1)}_{\nu_-}\left(\frac{p}{\omega}e^{-\omega t}\right) J_{|k_j+\frac{1}{2}|}(pr) \Phi^+_{m_j,k_j}(\theta,\phi) + \\ & + e^{-\pi k/2} H^{(1)}_{\nu_+}\left(\frac{p}{\omega}e^{-\omega t}\right) J_{|k_j-\frac{1}{2}|}(pr) \Phi^-_{m_j,k_j}(\theta,\phi) \Bigg]. \end{aligned} \tag{2}$$

The normalized spinors of negative frequencies can be obtained via the charge conjugation operation as in Refs. [2, 3],

$$\begin{aligned} \chi_{p,k_j,m_j}(x) &= (\psi_{p,k_j,m_j}(x))^C = \\ &= \frac{p}{2}\sqrt{\frac{\pi}{\omega r}} e^{-2\omega t} \Bigg[e^{\pi k/2} H^{(2)}_{\nu_+}\left(\frac{p}{\omega}e^{-\omega t}\right) J_{|k_j+\frac{1}{2}|}(pr) \Phi^+_{m_j,k_j}(\theta,\phi) - \\ &\quad - e^{-\pi k/2} H^{(2)}_{\nu_-}\left(\frac{p}{\omega}e^{-\omega t}\right) J_{|k_j-\frac{1}{2}|}(pr) \Phi^-_{m_j,k_j}(\theta,\phi) \Bigg]. \end{aligned} \tag{3}$$

The properties of the Hankel functions $H^{(1,2)}$ are given in Ref. [7] and briefly reviewed in Ref. [3]. The spherical spinors we use here have the form [6],

$$\Phi^+_{m_j,k_j} = \begin{pmatrix} i\Psi^{m_j}_{j\mp\frac{1}{2}} \\ 0 \end{pmatrix}, \qquad \Phi^-_{m_j,k_j} = \begin{pmatrix} 0 \\ i\Psi^{m_j}_{j\pm\frac{1}{2}} \end{pmatrix}, \tag{4}$$

where $\Psi^{m_j}_{j-\frac{1}{2}}$ and $\Psi^{m_j}_{j+\frac{1}{2}}$ read

$$\Psi^{m_j}_{j-\frac{1}{2}} = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j}\, Y^{m_{j-\frac{1}{2}}}_{j-\frac{1}{2}} \\ \sqrt{j-m_j}\, Y^{m_j+\frac{1}{2}}_{j-\frac{1}{2}} \end{pmatrix} \tag{5}$$

$$\Psi^{m_j}_{j+\frac{1}{2}} = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j+1-m_j}\, Y^{m_{j-\frac{1}{2}}}_{j+\frac{1}{2}} \\ -\sqrt{j+1+m_j}\, Y^{m_j+\frac{1}{2}}_{j+\frac{1}{2}} \end{pmatrix}. \tag{6}$$

We remind the reader that for $\Psi^{m_j}_{j-\frac{1}{2}}$ the orbital angular momentum and spin are parallel, $l = j - \frac{1}{2} = 0, 1, \ldots,$ and $k_j = -l - 1 = -\left(j + \frac{1}{2}\right)$. For $\Psi^{m_j}_{j+\frac{1}{2}}$ we have $l = j + \frac{1}{2} = 1, 2, \ldots,$ and $k_j = l = \left(j + \frac{1}{2}\right)$, , hence the spin and orbital angular momentum are antiparallel.

As mentioned before, our purpose is to calculate the spherical spinors in momentum representation. To this end, we use the Fourier transformation

$$\psi(\vec{p}\,') = \int \psi(\vec{x}) e^{-i(p'\cdot\vec{x})} d^3x, \tag{7}$$

as in Minkowski case [5].

This integral can be calculated with the help of the well-known expansion

$$e^{-i(p'\cdot\vec{x})} = \sum_{l=0}^{\infty} \sum_{m_j'=-j'}^{j'} \frac{(2\pi)^{\frac{3}{2}} i^{-l}}{p'} R_{p'l}(p'r) Y_{l,m_j'}\left(\frac{\vec{p}\,'}{p'}\right) Y^{\star}_{l,m_j'}\left(\frac{\vec{x}}{r}\right), \tag{8}$$

where $R_{p'l}(p'r) = \sqrt{\frac{p'}{r}}\left[J_{|k_j+\frac{1}{2}|}(p'r) + J_{|k_j-\frac{1}{2}|}(p'r)\right]$. Since the angles θ and ϕ define the direction of $\vec{x}/r$ we can use the compact notation $\Phi(\theta,\phi) = \Phi(\vec{x}/r)$. Then, introducing Eqs. (2) and (8) in Eq. (7) we can solve the radial integrals as in Ref. [4] remaining with the angular integral

$$\begin{aligned}\psi(\vec{p}\,') = {} & \frac{(2\pi)^{3/2} i^{-l} e^{-2\omega t}}{\sqrt{p\omega}} \delta(p - p') \times \\ & \times [e^{\pi k/2} H^{(1)}_{\nu_-}\left(\frac{p}{\omega} e^{-\omega t}\right) \int \Phi^{+}_{m_j,k_j}\left(\frac{\vec{x}}{r}\right) Y^{\star}_{l,m_j'}\left(\frac{\vec{x}}{r}\right) d\Omega + \\ & + e^{-\pi k/2} H^{(1)}_{\nu_+}\left(\frac{p}{\omega} e^{-\omega t}\right) \int \Phi^{-}_{m_j,k_j}\left(\frac{\vec{x}}{r}\right) Y^{\star}_{l,m_j'}\left(\frac{\vec{x}}{r}\right) d\Omega] Y_{l,m_j'}\left(\frac{\vec{p}\,'}{p'}\right)\end{aligned}$$

where $Y(\vec{p}\,'/p')$ depend on the angular coordinates of the unit vector $\vec{p}\,'/p'$. Solving this integral we arrive at the final result obtaining the spherical spinors in the momentum representation,

$$\begin{aligned}\psi_{p,k_j,m_j}(\vec{p}\,') = {} & \frac{(2\pi)^{3/2} \sqrt{\pi} i^{-l} e^{-2\omega t}}{2\sqrt{p\omega}} \delta(p - p') \left[e^{\pi k/2} H^{(1)}_{\nu_-}\left(\frac{p}{\omega} e^{-\omega t}\right) \Phi^{+}_{m_j,k_j}\left(\frac{\vec{p}\,'}{p'}\right) + \right. \\ & \left. + e^{-\pi k/2} H^{(1)}_{\nu_+}\left(\frac{p}{\omega} e^{-\omega t}\right) \Phi^{-}_{m_j,k_j}\left(\frac{\vec{p}\,'}{p'}\right) \right],\end{aligned} \tag{9}$$

and

$$\chi_{p,k_j,m_j}(\vec{p}\,') = \frac{(2\pi)^{3/2}\sqrt{\pi}i^{-l}e^{-2\omega t}}{2\sqrt{p\omega}}\delta(p-p')\left[e^{\pi k/2}H^{(2)}_{\nu_+}\left(\frac{p}{\omega}e^{-\omega t}\right)\Phi^{+}_{m_j,k_j}\left(\frac{\vec{p}\,'}{p'}\right)-\right.$$
$$\left.-e^{-\pi k/2}H^{(2)}_{\nu_-}\left(\frac{p}{\omega}e^{-\omega t}\right)\Phi^{-}_{m_j,k_j}\left(\frac{\vec{p}\,'}{p'}\right)\right]. \tag{10}$$

In this way the spherical spinors in momentum representation were derived using the same method as in Minkowski space.

In the case of $m=0$ (when $k=0$) it is convenient to use the chiral representation of the Dirac matrices [6] (with diagonal γ^5):

$$\psi^{0}_{p,k_j,m_j}(x) = \lim_{k\to 0}\frac{1-\gamma^5}{2}\psi_{p,k_j,m_j}(x). \tag{11}$$

Then, using the formulas

$$\left(\frac{1-\gamma^5}{2}\right)\Phi^{+}_{m_j,k_j} = \Phi^{+}_{m_j,k_j}, \quad \left(\frac{1-\gamma^5}{2}\right)\Phi^{-}_{m_j,k_j} = 0 \tag{12}$$

and taking into account that $\nu_\pm = \frac{1}{2}\pm ik$, we obtain, $H^{(1)}_{\nu_-}\left(\frac{p}{\omega}e^{-\omega t}\right) = = H^{(1)}_{\nu_+}\left(\frac{p}{\omega}e^{-\omega t}\right) = H^{(1)}_{1/2}\left(\frac{p}{\omega}e^{-\omega t}\right)$. Furthermore, according to the definition of the Hankel functions [7], we find that the massless spherical spinors read

$$\psi^{0}_{p,k_j,m_j}(x) = (-\omega t_c)^{3/2}\sqrt{\frac{p}{2r}}\frac{e^{-ipt_c}}{i}J_{|k_j+\frac{1}{2}|}(pr)\Phi^{+}_{m_j,k_j}(\theta,\phi). \tag{13}$$

For the Dirac antiparticle we have to use $H^{(2)}_{1/2}\left(\frac{p}{\omega}e^{-\omega t}\right)$ obtaining

$$\chi^{0}_{p,k_j,m_j}(x) = (-\omega t_c)^{3/2}\sqrt{\frac{p}{2r}}\frac{e^{ipt_c}}{-i}J_{|k_j+\frac{1}{2}|}(pr)\Phi^{+}_{m_j,k_j}(\theta,\phi). \tag{14}$$

These results are also similar to those known from special relativity.

We hope that our results obtained here will be useful for calculating scattering processes in momentum representation on dS spacetime.

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