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RESONANCE OF THE SURFACE WAVES. THE "*H*/*V*" RATIO

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Abstract. We examine the elastic waves excited at the surface of an isotropic body by an oscillatory, localized force (Rayleigh waves). We define the "*H*/*V*"-ratio as the ratio of the intensity of the in-plane waves (horizontal waves) to the intensity of the perpendicular-to-the-plane waves (vertical waves). It is shown that this ratio exhibits a resonance at a frequency which is close to the frequency of the transverse waves. It may serve to determine Poison's ratio of the body.

Key words: surface (Rayleigh) waves, resonance, *H*/*V*-ratio.

Recently there has been a great deal of interest in the method of the "*H*/*V*"-ratio for assessing the elastic properties of soils by means of their response to external excitations [1–8]. We analyze here the surface (Rayleigh) waves excited in an elastic isotropic body by an oscillatory, localized force. The "*H*/*V*"-ratio, defined as the ratio of the intensity of the horizontal waves (in-plane waves) to the intensity of the vertical waves (perpendicular-to-the-plane waves), exhibits a resonance peak at a frequency which is close to the frequency of the transverse waves.

As is well known,[9] the equation of elastic waves in an isotropic body is given by

$$\ddot{\mathbf{u}} = c_t^2 \Delta \mathbf{u} + (c_l^2 - c_t^2) grad \cdot div \mathbf{u} + \mathbf{F} \tag{1}$$

where $\mathbf{u}$ is the local displacement, $c_{t,l}$ are the velocities of sound for transversal and, respectively, longitudinal waves and $\mathbf{F}$ is an external force (per unit mass). The sound velocities are given by

$$c_t^2 = \frac{E}{2\rho(1+\sigma)}, \quad c_l^2 = \frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)} \tag{2}$$

where E is Young's modulus, σ is Poison's ratio ($0 < \sigma < 1/2$) and ρ is the density of the body.

We consider surface waves in a half-space $z < 0$ excited by an external force

$$\mathbf{F} = -\mathbf{f}e^{-i\Omega t}\delta(\mathbf{r})e^{\kappa z} \tag{3}$$

where $\mathbf{f}$ is a force per unit superficial mass, Ω is the frequency of the force, κ is an attenuation coefficient and $\mathbf{r} = (x, y)$ are in-plane coordinates. This would correspond to surface waves excited at Earth's surface by seismic waves or other external perturbations. The localization of the force means that we detect the surface waves far away form the source of excitation.

We look for solutions of the form $\mathbf{u} \sim e^{i\mathbf{kr}}e^{\kappa z}$ and introduce the notation $\mathbf{u} = (u_l, u_t, u_v)$ and $\mathbf{k} = (k, 0)$. In addition we assume $\mathbf{f} = (f_l, 0, f_v)$. Equation (1) becomes

$$\begin{aligned}
\ddot{u}_l &= (-c_l^2 k^2 + c_t^2\kappa^2)u_l + i\kappa k(c_l^2 - c_t^2)u_v - f_l e^{-i\Omega t}, \\
\ddot{u}_v &= (-c_t^2 k^2 + c_l^2\kappa^2)u_v + i\kappa k(c_l^2 - c_t^2)u_l - f_v e^{-i\Omega t}, \\
\ddot{u}_t &= c_t^2(-k^2 + \kappa^2)u_t .
\end{aligned} \tag{4}$$

The homogeneous equations (4) have two distinct eigenfrequencies given by $\omega_{l,t}^2 = c_{l,t}^2(k^2 - \kappa^2)$ corresponding to the eigenmodes $u_l \sim ik$, $u_v \sim \kappa$ and, respectively, $u_l \sim \kappa$, $u_v \sim -ik$.

We take $\omega^2 = c_l^2(k^2 - \kappa_l) = c_t^2(k^2 - \kappa_t^2)$ and the linear combination

$$\begin{aligned}
u_l &= (ikAe^{\kappa_l z} + \kappa_t Be^{\kappa_t z})e^{ikx}, \\
u_v &= (\kappa_l Ae^{\kappa_l z} - ikBe^{\kappa_t z})e^{ikx}
\end{aligned} \tag{5}$$

in order to satisfy the boundary conditions $\sigma_{iz} = 0$ at the free surface $z = 0$, where

$$\sigma_{ij} = \frac{E}{1+\sigma}\left(u_{ij} + \frac{\sigma}{1-2\sigma}u_{ll}\delta_{ij}\right) \tag{6}$$

is the stress tensor and $u_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ is the strain tensor. These are the well-known Rayleigh waves [10]. From the boundary coonditions we get $u_t = 0$ and the equations

$$\begin{aligned}
&2i\kappa_l kA + (k^2 + \kappa_t^2)B = 0, \\
&\left[\sigma(k^2 + \kappa_l^2) - \kappa_l^2\right]A + i(1-2\sigma)\kappa_t kB = 0.
\end{aligned} \tag{7}$$

It is easy to see that the ω_l-solution corresponds to $curl\mathbf{u} = 0$ and the ω_t-solution corresponds to $div\mathbf{u} = 0$. Making use of $c_l^2(k^2 - \kappa_l^2) = c_t^2(k^2 - \kappa_t^2) = \omega^2$ and of equations (2) the second equation (7) can also be written as

$$(k^2 + \kappa_t^2)A - 2i\kappa_t kB = 0, \tag{8}$$

so that equations (7) have solutions providing

$$(k^2+\kappa_t^2)^2=4\kappa_l\kappa_t k^2. \tag{9}$$

We introduce the variable ξ defined by $\omega=c_t\xi k$ and, making use of equations (2), we get

$$\kappa_l^2=(1-c_t^2\xi^2/c_l^2)k^2=\left[1-\frac{1-2\sigma}{2(1-\sigma)}\xi^2\right]k^2,\quad \kappa_t^2=(1-\xi^2)k^2. \tag{10}$$

Equations (9) becomes

$$\xi^6-8\xi^4+8\frac{2-\sigma}{1-\sigma}\xi^2-\frac{8}{1-\sigma}=0. \tag{11}$$

This equation has a solution close to unity, $\xi\simeq 1$, for $0<\sigma<1/2$. It follows that $\kappa_l\sim k$ and $\kappa_t\sim 0$. The ratio of the two amplitudes is

$$\frac{A}{B}=2i\frac{\sqrt{1-\xi^2}}{2-\xi^2}, \tag{12}$$

so the amplitude of the κ_l-wave (*A*) is much smaller than the amplitude of the κ_t-wave (*B*). The main surface wave is a shallow wave with a large penetration depth ($\kappa_t\simeq 0$).

We pass now to solving equations (4) with the force term. The solution is of the form $\mathbf{u}\sim e^{i\mathbf{kr}}e^{\kappa z}e^{-i\Omega t}$, where κ is the attenuation coefficient in the force. We get easily

$$u_l=\frac{(\Omega^2-c_t^2k^2+c_l^2\kappa^2)f_l-i\kappa k(c_l^2-c_t^2)f_v}{\Delta},$$
$$u_v=\frac{(\Omega^2-c_l^2k^2+c_t^2\kappa^2)f_v-i\kappa k(c_l^2-c_t^2)f_l}{\Delta} \tag{13}$$

where $\Delta=\left[\Omega^2-c_l^2(k^2-\kappa^2)\right]\left[\Omega^2-c_t^2(k^2-\kappa^2)\right]$. We define the "*H/V*"-ratio as $H/V=\left|u_l\right|^2/\left|u_v\right|^2$. It is convenient to introduce the notation $s=f_l^2/f_v^2$. We get

$$H/V=\frac{(\Omega^2-c_t^2k^2+c_l^2\kappa^2)^2s+\kappa^2k^2(c_l^2-c_t^2)^2}{(\Omega^2-c_l^2k^2+c_t^2\kappa^2)^2+\kappa^2k^2(c_l^2-c_t^2)^2s}\,. \tag{14}$$

We introduce $\omega=c_t\xi k$ for $\xi\simeq 1$ and $r=c_l/c_t$. It is natural to assume $\kappa\simeq 0$, comparable with κ_t given by $\kappa_t^2=(1-\xi^2)k^2$ for $\xi\simeq 1$. Equation (14 becomes then approximately

$$H/V\simeq\frac{(\Omega^2-\omega^2)^2s}{\left(\Omega^2-r^2\omega^2\right)^2}\,. \tag{15}$$

We can see that the H/V-ratio exhibits a resonance at $\omega = \omega_0 \simeq \Omega / r = =(c_t / c_l)\Omega$. If we take $\Omega = c_l k$ this resonance is in the vicinity of the S-wave frequency $\omega \simeq c_t k$, in agreement with previous results [1]. For $s = 0$ the H/V-ratio is given by

$$H/V \simeq \frac{(1-\xi^2)(r^2-1)^2\omega^4}{\left(\Omega^2 - r^2\omega^2\right)^2} \tag{16}$$

and one can see that the resonance is rather sharp. For $s \to \infty$ the resonance disappears.

We may also use $\kappa^2 = \kappa_l^2 = (1-\xi^2/r^2)k^2$, and equation (14) becomes

$$H/V \simeq \frac{\left[\Omega^2 + (r^2-2)\omega^2\right]^2 s + (r^2-1)^3\omega^4/r^2}{\left[\Omega^2 - (r^2+1/r^2-1)\omega^2\right]^2 + s(r^2-1)^3\omega^4/r^2}. \tag{17}$$

This expression has a rather broad maximum. For $s = 0$ equation (17) exhibits a resonance at $\omega \simeq (r^2+1/r^2-1)^{-1/2}\Omega = (1+1/r^4-1/r^2)^{-1/2}\omega_0$ which is greater than ω_0 ($r^2 > 2$). For $s \to \infty$ the maximum of (17) disappears.

It is likely that the attenuation coefficient κ in the expression of the force is very small. The surface waves given by (13) and the H/V-ratio (14) acquire then a very simple expression. A small but finite value of κ shifts the resonance frequency and smooth out the resonance, giving it a small width. The frequency Ω is not necessarily related to the frequency of the elastic waves in the surface layer, but rather it is given by the frequency of the in-depth waves of excitations, or it may have other sources. If the force is a superposition of various frequencies Ω then the resonance is smoothed out and gets a finite width.

The model presented here can be extended to include damping effects and various other distributions of external forces.

REFERENCES

1. M. Nogoshi, T. Igarashi, J. Seism. Soc. Japan, **24**, 26 (1971).
2. Y. Nakamura, QR of Railway Technical Research Institute, **30**, 1 (1989).
3. Y. Nakamura, *Proc. of the X -th World Conf. Earthquake Eng., Mexico*, Elsevier (1996) p. 2134.
4. E. H. Field, K. H. Jacob, Geophys. Res. Lett. **20**, 2925 (1993).
5. J. F. Lermo, S. Francisco, J. Chavez-Garcia, Bull. Seism. Soc. Am. **83**, 1574 (1993); *ibid*, **84**, 1350 (1994).
6. C. Lachette, P.-Y. Bard, J. Phys. Earth **42**, 377 (1994).
7. H. Yamanaka, M. Dravinski, H. Kagami, Bull. Seism. Soc. Am. **63**, 1227 (1993).
8. E. Field, K. Jacob, Bull. Seism. Soc. Am. **85**, 1127 (1995).
9. L. Kandau, E. Lifshitz, *Theory of Elasticity*, Elsevier (2005).
10. Rayleigh, Lord, Proc. London Math. Soc. **17**, 4 (1885).