CHAOS ANALYSIS OF NUCLEAR STABILITY USING A CLASSICAL BILLIARD MODEL*

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Abstract. We consider several non interacting nucleons moving in a 2D Woods-Saxon type potential well and hitting the vibrating surface. The Hamiltonian has a coupling term between the particle motion and the collective coordinate which generates a self consistent dynamics. The numerical simulation is based on the solutions of the Hamilton equations which was solved using an algorithm of Runge-Kutta type (order 4–5) having an optimized step size, taking into account that the absolute error for each variable is less than 10^{-6}. Total energy is conserved with high accuracy, *i.e.* approx. 10^{-6} in absolute value. We analyze the chaotic behavior of the nonlinear dynamical system using phase-space maps, autocorrelation functions, power spectra, Lyapunov exponents and Kolmogorov-Sinai entropy. In this frame we analyse the onset of chaotical behavior in different dynamical regimes prior to nuclear fragmentation. A qualitative and quantitative picture of the achievement of soft chaos is shown for a comparative study between different stages of the nuclear interaction.

Key words: Lyapunov exponent, Kolmogorov-Sinai entropy, multifragmentation, power spectrum, autocorrelation, phase portrait.

1. INTRODUCTION

Over the last two decades an increasing number of papers have treated the study of the deterministic chaotic behavior of Fermi nuclear systems. The order to chaos transition in the dynamics of independent classical particles in a container was first studied using computer simulations by Blocki *et al.* [1]. They analyzed the behavior of a gas of classical noninteracting particles enclosed in a multipole-deformed container which undergoes periodic shape oscillations and showed that higher multipolarities lead to chaotic motion. The destruction of order is paralleled by a transition from rubber-like to honey-like behavior of the independent particle

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nuclear model. Another step in this direction was done by Bauer *et al.* in [2]. They performed self-consistent calculations in semi-classical approximation using a multipole-multipole interaction of the Bohr-Mottelson type for quadrupole and octupole deformations. In both cases the dynamical evolution showed a regular undamped collective motion which coexists with a weakly chaotic single-particle dynamics. Then Burgio *et al.* [3, 4] considered a number of nucleons without spin and charge and with no internal structure which are moving in a 2D deep Woods-Saxon potential well and hitting the oscillating surface with a certain frequency. They discuss the dissipative behavior of the wall motion and its relation with the order to chaos transition in the dynamics of the microscopic degrees of freedom. We developed those studies in [5] using informational Shannon entropy in order to study the nuclear fragmentation in the presence of dissipation. The parameters of the Woods-Saxon potential for different multipole deformations and their time dependence were taken into account in both adiabatic and resonant regimes, in order to obtain information on nuclear fragmentation throughout a chaotic process. Thus, our numerical simulations have shown that every new interaction that allows supplementary couplings between particles decreases the onset of the nuclear fragmentation toward realistic nuclear interaction time scale.

2. THE TOY MODEL

We chose the system under study as an ensemble with N nucleons moving in a 2D Woods-Saxon potential well and hitting periodically the oscillating surface with a certain frequency. The ensemble of nucleons is considered as a Hamiltonian system therefore the total energy is conserved and this means that the walls can give energy to nucleons. The Hamiltonian of this kind of interaction in polar coordinates is [3, 4]:

$$H(r_i, \theta_i, \alpha) = \sum_{i=1}^{N} \left(\frac{p_{r_i}^2}{2m} + \frac{p_{\theta_i}^2}{2mr_i^2} + V\left(r_i, R(\theta_i)\right) \right) + \frac{p_\alpha^2}{2M} + \frac{1}{2}M\Omega^2\alpha^2, \qquad (1)$$

where $\left\{ p_{r_i}, p_{\theta_i}, p_\alpha \right\}$ are the conjugate momenta of the collective coordinates of the i[(th)] particle $\{r_i, \theta_i, \alpha\}$, $m = 938.5$ MeV, Ω is the oscillating frequency of the collective variable α and $M = mNR_o^2$ is the Inglis mass. The Woods-Saxon potential is given by the function:

$$V\left(r_i, R(\theta_i)\right) = \frac{V_o}{1 + e^{\frac{r_i - R(\theta_i)}{a}}}, \qquad (2)$$

with $V_o = -1500$ MeV and $a = 0.5$ fm. The potential is zero inside the billiard and rises very quickly to the surface. The nucleons were initially distributed randomly in the billiard having R_0 radius and the corresponding momenta were generated according to Maxwell distribution with a temperature T = 36MeV. This value of temperature was chosen in order to mimic the Fermi motion of the particles [4].

The vibrating surface of the potential well can be written as usual depending on the collective variable and Legendre polynomials $P_L\left(\cos(\theta)\right)$ [6]:

$$R\left(\theta_i, \alpha\right) = R_o\left[1 + \alpha \cdot P_L\left(\cos(\theta_i)\right)\right], \tag{3}$$

where L is the order of multipole deformation of the potential well. In this way the potential couples the collective variable motion to nucleons dynamics. The calculations presented in this paper are done for $L = 1$ (dipole multipolarity). The numerical simulations are based on the solution of a system having $4N + 2$ nonlinear Hamilton equations coupled by the collective variable:

$$\begin{cases} \dot{r}_i = \dfrac{p_{r_i}}{m} \\[2mm] \dot{p}_{r_i} = \dfrac{p_{\theta_i}^2}{mr_i^3} - \dfrac{\partial V}{\partial r_i} \\[2mm] \dot{\theta}_i = \dfrac{p_{\theta_i}}{mr_i^2} \\[2mm] \dot{p}_{\theta_i} = -\dfrac{\partial V}{\partial R} \cdot \dfrac{\partial R}{\partial \theta_i} \\[2mm] \dot{\alpha} = \dfrac{p_\alpha}{M} \\[2mm] \dot{p}_\alpha = -M\Omega^2\alpha - \sum_i \dfrac{\partial V}{\partial R_i} \cdot \dfrac{\partial R_i}{\partial \alpha} \end{cases} \tag{4}$$

3. NUMERICAL STUDY

First we choose a wall frequency smaller than the single particle one $\Omega_{ad} = 0.05\,\mathrm{c/fm}$, which corresponds to an oscillation period equal to:

$$\tau_{ad} = \frac{2\pi}{\Omega_{ad}} = 125.66\,\mathrm{fm/c}.$$

The maximum particle speed is: $v = \sqrt{\dfrac{2T}{m}}$. We choose $R_0 = 6.0$ fm, $T = 36$ MeV

and the single particle period is $\tau_p = \dfrac{2R_0}{v} = 43.33$ fm/c.

In addition to [3, 4] we introduced a physical constraint and continued the study of the dynamical system. Thus, we first considered a physical situation [7] and we took instead a static vibrating nuclear billiard, a projectile nucleus having the same proprieties and we impinged it on a giving target nucleus. It is well know that the nuclear interaction at bombarding energy from MeV to GeV scale can have as result a multitude of processes from nuclear evaporation to complete fragmentation or multifragmentation according to the impact parameters of a given collision. It was shown elsewhere [8, 9] that during this kind of processes a non-negligible amount of energy is transferred to the nucleons of the projectile and the traverse and longitudinal momentum distribution as measured in the projectile rest frame can reveal the centrality status of the interaction. This can offer, too, a hint on the apparent temperature of a Fermi gas which was found to be near the isotopic temperatures, *i.e.*, several MeV [10, 11]. We gradually increased the wall frequency to resonant frequency $\Omega_{res} = 0.145$ c/fm and beyond because it was suggested that the onset of nuclear fragmentation is linked to the resonance stage of interaction, [5, 7].

We wrote our codes using Scilab 3.1.1 (Inria, Enpc). Scilab [12], developed by researchers from INRIA and ENPC, is a scientific software package for numerical computations providing a powerful open computing environment for engineering and scientific applications. We chose Scilab, because it provides various functions for ordinary differential equation solving, Fast Fourier Transform, autocorrelation and excellent 2D and 3D graphical capabilities. The *ode* function provided by Scilab (ordinary differential equation solver) is an interface to various solvers. We used *rkf* option which corresponds to Fehlberg's Runge-Kutta method of order 4 and 5 (RKF45). This is for non-stiff and mildly stiff problems. Alternatively we used *stiff* option. In this case *lsode* solver of package ODEPACK is called and it uses the stiff Backward Differentiation Formula (BDF) method. The solving of the system of ordinary differential equations was done taking into account that the absolute error for each variable is less than 10^{-6}. Total energy is conserved with high accuracy, *i.e.* approx. 10^{-6} in absolute value. The chaos analysis we performed include phase portraits, power spectra, autocorrelation functions, Lyapunov exponents and Kolmogorov-Sinai entropy [13–17]. We studied systems having the number of nucleons between two and ten.

The transition to chaos can be characterized by the autocorrelation function of a variable $X(t)$ corresponding to the nonlinear dynamical system. This reflects the extent to which a variable correlates with itself over time.

$$C(\tau) = \lim_{T \to \infty} \int_0^T dt \, X(t) \, X(t + \tau). \tag{5}$$

In order to compute the autocorrelation we used *corr* function provided by Scilab which calculates the correlation between two variables over time. In our case we computed the correlation of a variable with itself. The existence of chaos is confirmed by the observation of rapidly decreasing autocorrelation functions to zero (Fig. 1).
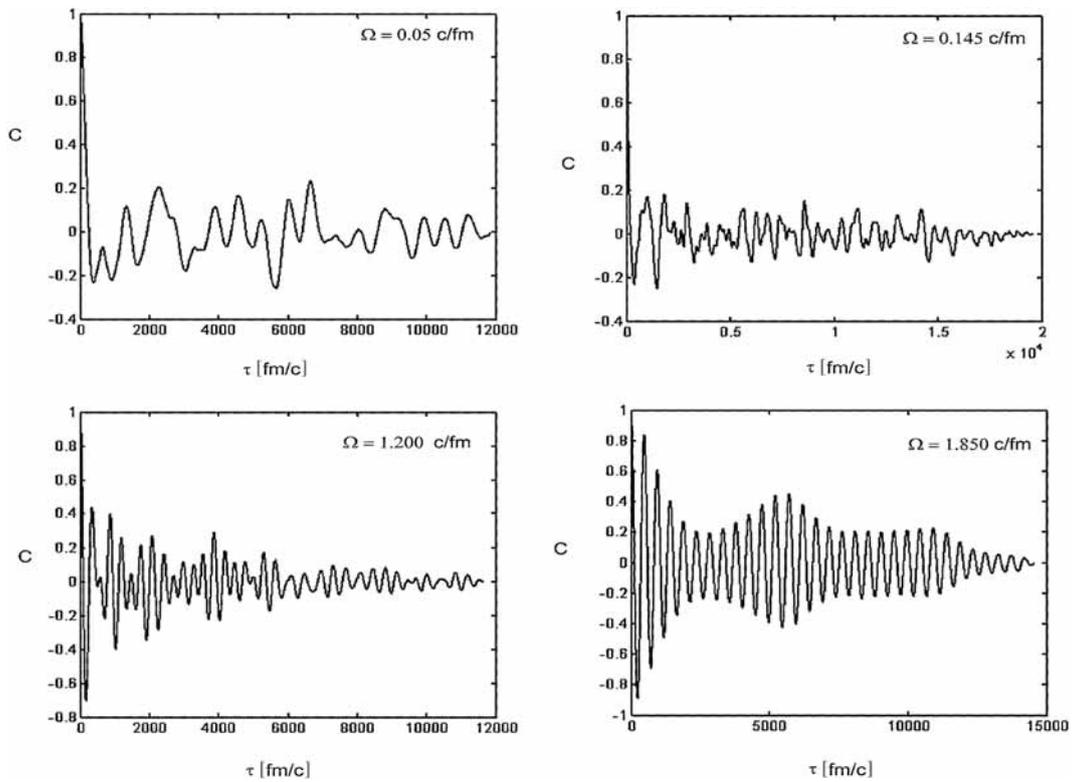

Fig. 1 – Autocorrelation function versus time for the radius of first particle corresponding to frequency values $\Omega = 0.050, 0.145, 1.200$ and 1.850 c/fm.

The power spectrum $P(\omega)$ of a variable $X(t)$ is defined as the square of its Fourier amplitude:

$$P(\omega) = \lim_{T \to \infty} \frac{1}{T} \left| \int_0^T dt \, e^{i\omega t} X(t) \right|^2. \tag{6}$$

In order to obtian the power spectrum we used *fft* function provided by Scilab. This routine computes the fast Fourier transform of the variable. The power spectrum allows us to distinguish between quasiperiodicity and chaotic behavior. In the first case the power spectrum contains only the discrete lines corresponding to basic frequencies and harmonics and in the second case a continuous broadband spectrum is present. The power spectrum changes from discrete lines to a broadband spread when chaos sets in (Fig. 2).
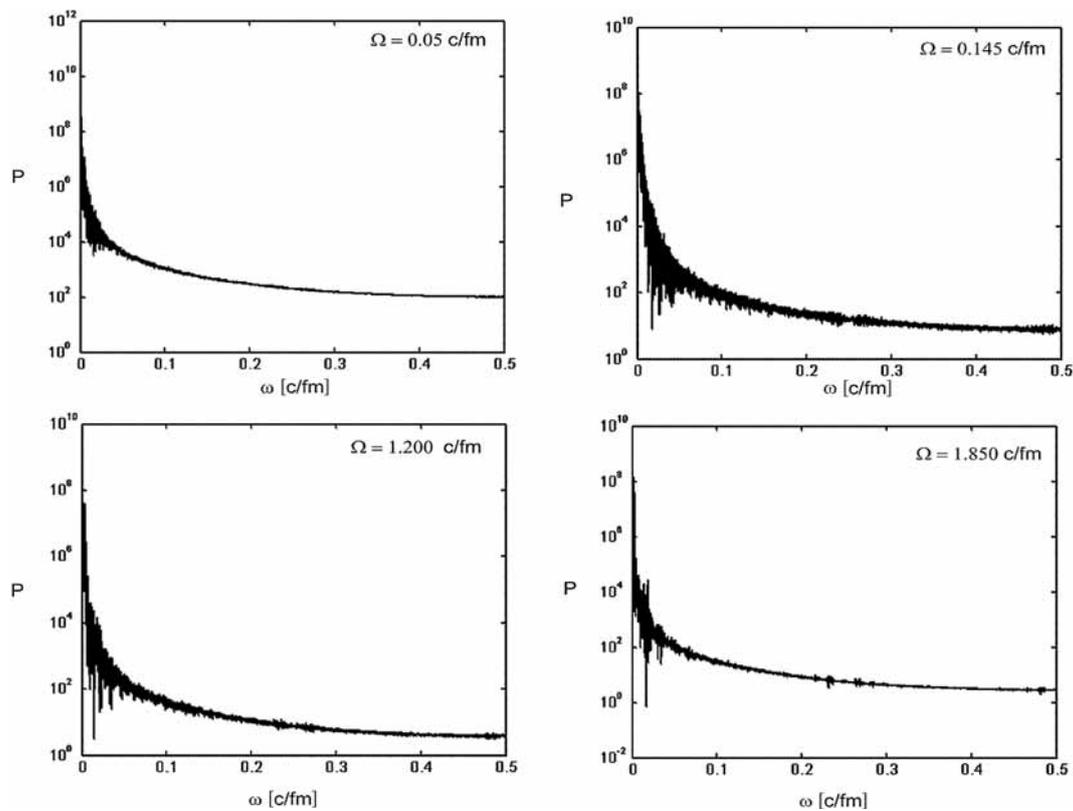

Fig. 2 – Power spectrum for the radius of first particle corresponding to frequency values $\Omega = 0.050$, 0.145, 1.200 and 1.850 c/fm.

Also the phase space maps $(r \leftrightarrow P_r)$ and $(\alpha \leftrightarrow P_\alpha)$ reveal an increasing filling degree of the phase space once the frequency is increasing and also the appearance of some attraction basins around some standard orbits (Fig. 3 and Fig. 4).

One of the important ways for studying the dynamics of the system is to calculate the largest Lyapunov exponent (LLE) which is a measure of the sensitivity of the system to initial conditions.
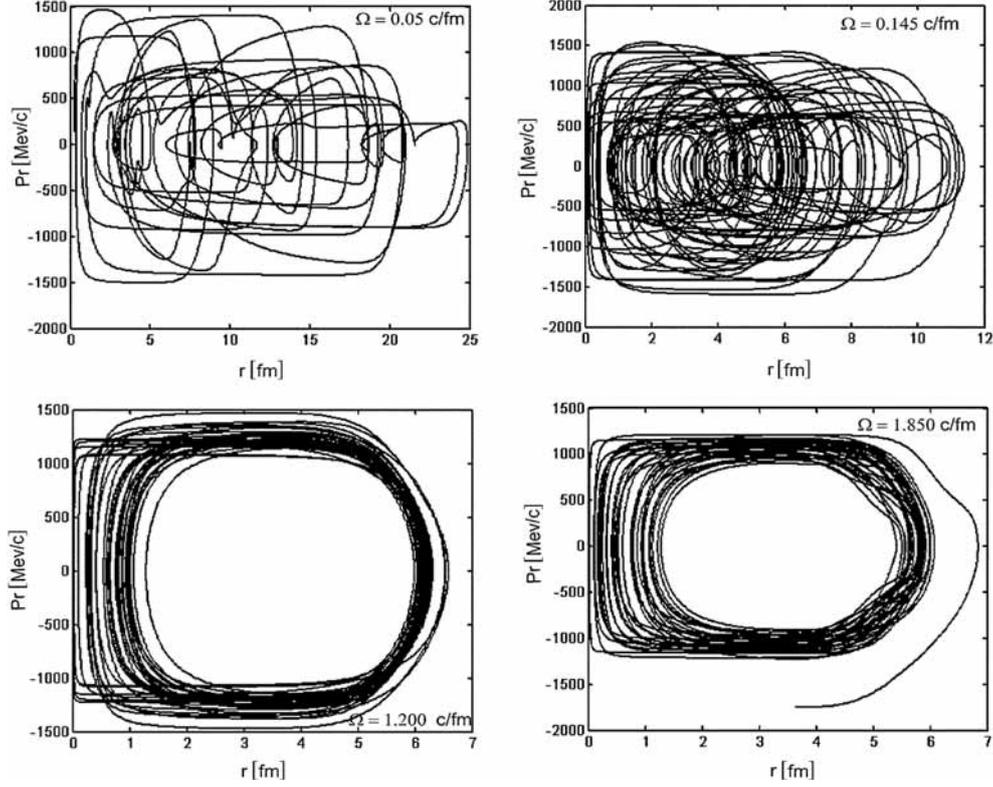
Fig. 3 – Phase space diagram of the first nucleon degree of liberty: $(r \leftrightarrow P_r)$ corresponding to
frequency values $\Omega = 0.050, 0.145, 1.200$ and 1.850 c/fm.

The LLE was previously used to study the solid to liquid phase transitions in
Lennard-Jones drops by Dorso and Bonasera in reference [18]. They showed the
presence of a maximum in LLE which indicates the transition from a chaotic
regime to a more regular one. In the chaotic regime the system is composed mainly
of a liquid drop while the regular one corresponds to almost freely flowing
particles and small clusters.

We calculated the multi-dimensional Lyapunov exponents using the formula:

$$\lambda(t) = \lim_{d_0 \to 0} \ln \frac{d}{d_0} = \lim_{d_0 \to 0} \ln \frac{\sqrt{\sum_{i=1}^{N}[(r - r_0)^2 + (\theta - \theta_0)^2 + (p_r - p_{r0})^2 + (p_\theta - p_{\theta 0})^2]_i}}{d_0}, (7)$$

where the sums are taken over all N nucleons of the system and the quantities from
the parentheses refer to two trajectories in phase-space which differ only by an
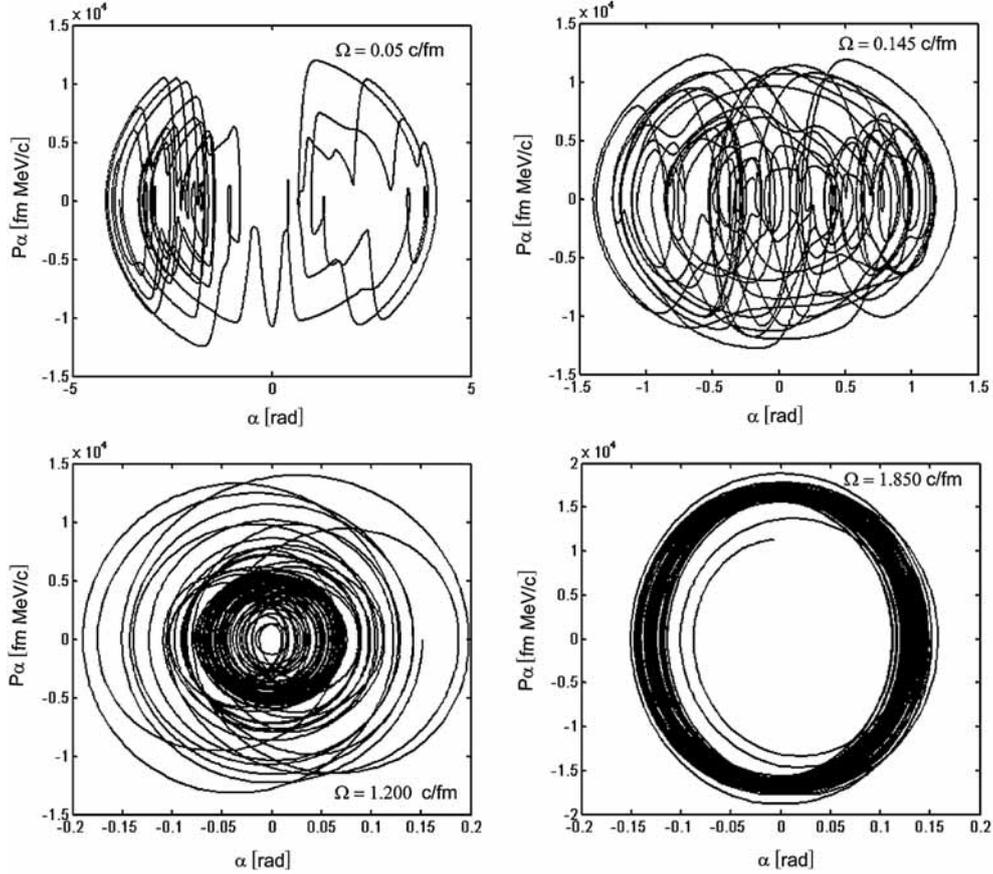infinitesimal quantity $d_0 = 10^{-6}$. We took integrations times about to 500 fm/c

Fig. 4 – Phase space diagram of the collective degree of liberty: $(\alpha \leftrightarrow P_\alpha)$, corresponding to frequency values $\Omega = 0.050$, 0.145, 1.200 and 1.850 c/fm.

sufficiently large to avoid errors on the calculation of the Lyapunov exponents. We calculated the largest Lyapunov exponent from the dependence of multi-dimensional Lyapunov exponent *versus* time.

From the Lyapunov spectrum we can compute Kolmogorov-Sinai entropy, as the sum of the positive Lyapunov exponents [15, 16]:

$$h_{KS} = \sum_{i=1}^{N} \lambda_i. \tag{8}$$

The Kolmogorov-Sinai entropy measures at which rate information about the initial state of a system is lost. A chaotic system continously evolves with time. Compared to any chosen time, the system at a later time is in an unpredictable different state. That unpredictable evolution provides a steady supply of new information. Consequently, the Kolmogorov-Sinai entropy for chaotic systems must be positive.

The preliminary results for ten nucleon systems are presented in Fig. 5 and Fig. 6. All the calculations are done for $L = 1$ (dipole). It can be seen that LLEs have positive values, which is evidence of a chaotic behavior of the system. Moreover we believe that the dependence of the Lyapunov exponents *versus* frequency can give us qualitative and quantitative information about the transition to chaos. Thus, the presence of the maximum in the LLE and Kolmogorov-Sinai entropy signals a transition from a chaotic motion to a more ordered behavior of the system. Also it can be seen that the maximum is located near the resonance frequency $\Omega_{res} = 0.145$ c/fm.

A similar correlation was made, by using quantum molecular dynamics numerical simulations for nucleonic systems, by Zhang, Wu and Lee [19]. Accordingly, the temperature corresponding to the maximal LLE is named "critical temperature" which signals a transition from a chaotic to a more ordered motion. The behavior of the LLE as a function of wall frequency can be understood as follows: the raising branch is due to the increase of fluctuations, and the presence of the maximum in the LLE, results from multifragmention; the behavior of the decreasing branch can be traced track to the fact that in this region the system breaks up very soon and the ordered expansion collective motion dominates the evolution of the system.
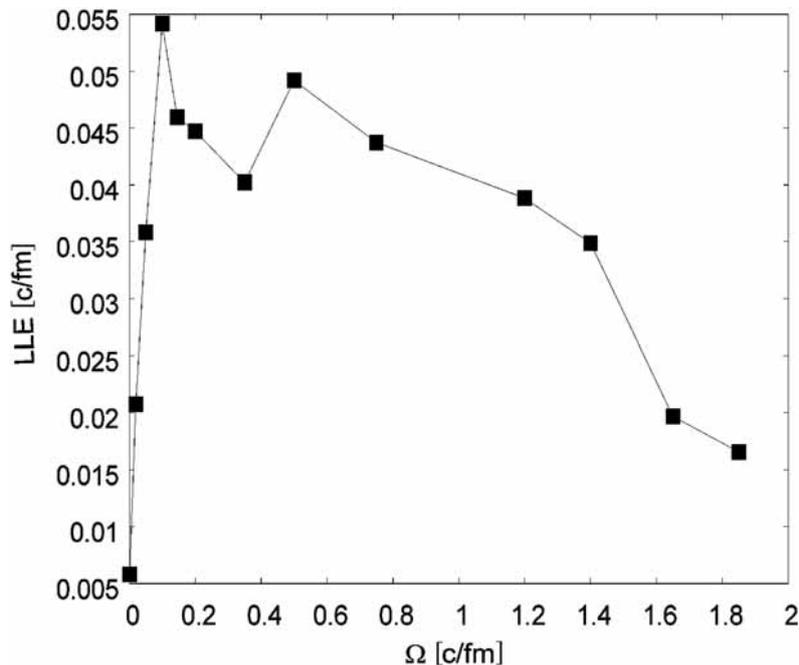

Fig. 5 – The largest Lyapunov exponent (LLE) as a function of wall frequency for ten nucleon system. (The lines which connect points are only for eye orientation).
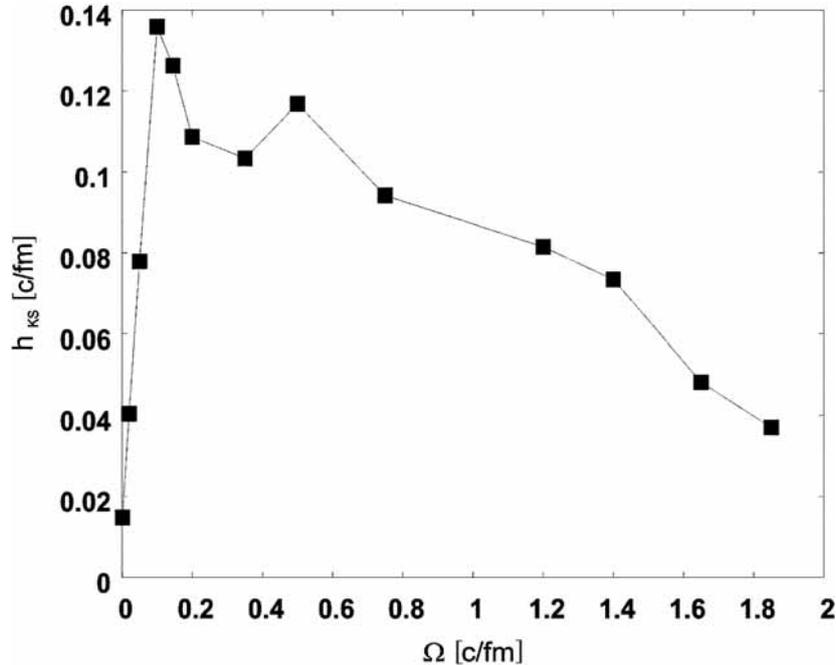
Fig. 6 – The Kolmogorov-Sinai entropy (h_{KS}) as a function of wall frequency for ten
nucleon system. (The lines which connect points are only for eye orientation).

4. CONCLUSIONS

We studied the one particle dynamics in a Woods-Saxon potential and the coupling between individual and collective degrees of freedom which was shown to generate different paths to chaos taking account that during the interaction the nuclear billiard passes through different physical regimes from adiabatic to resonance. The onset of nuclear fragmentation would be linked to the resonance stage of interaction. We believe that future works considering an increasing number of nucleons and taking into account the spin and isospin degrees of freedom of the nucleons will offer valuable information about multifragmentation.

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