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ZERO-CURVATURE FORMULATION OF ASYMPTOTIC ORDERED FLOWS OF THE EULER FLUID

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Abstract. The two-dimensional ideal incompressible (Euler) fluid evolves at relaxation from turbulent states to large scale coherent structures. We review the field theoretical formalism for the description of the continuum limit of the discrete set of point-like vortices in plane. This formalism is used in the present work to show that the equation governing the asymptotic stationary coherent flows of the Euler fluid can be derived from the condition of zero-curvature. It is suggested that the self-duality can be represented, for this particular system, also as the Hodge duality.

Key words: Euler fluids, ordered flows, fields theories, zero-curvature condition.

1. INTRODUCTION

Experimental, numerical and theoretical studies have revealed that the two-dimensional fluids and plasmas exhibit an intrinsic evolution to organization. This is most obvious at relaxation from turbulent states when the system evolves toward a reduced subset of flow patterns, characterized by a regular form of the streamfunction (*coherent structures*). Although in many situations the fluids and plasmas are subject to strong driving/dissipative processes (which may overcome and hide the self-organization tendency) the fact that the ideal system selects a restricted set of states rises the problem of *preferred states* in fluid and plasmas. This problem is common to several systems: ideal incompressible (Euler) fluid, plasma in strong magnetic field, planetary atmosphere, non-neutral plasmas, etc. *It is admitted that we do not dispose, at this moment, of a satisfactory understanding of this fundamental process.*

The conservation equations (treated as dynamical equations) are not appropriate, we need a variational approach, the preferred states being determined as configurations of flow that extremize an action functional. For the $2D$ case there exist models that are equivalent to the physical systems: the $2D$ ideal (Euler) fluid and the $2D$ magnetized plasma (and planetary atmosphere) are equivalent with discrete systems consisting of sets of point-like vortices interacting in plane by a potential (Coulombian and respectively short range). Looking for a formulation of the continuum limit of these discrete models we have been naturally led to classical field theories. The matter field (the density of the point-like objects: vortices or currents) is a complex field with a scalar self-interaction, the potential of interaction between these objects is represented by a gauge field with Chern-Simons action, the coupling is minimal.

2. THE FIELD-THEORETICAL MODEL FOR THE EULER FLUID

For a two-dimensional fluid the velocity and vorticity are expressed in terms of the streamfunction, $\psi(x, y, t)$,

$$\mathbf{v} = \nabla\psi \times \hat{\mathbf{e}}_z,$$
$$\omega = \nabla \times \mathbf{v} = -\nabla^2\psi\hat{\mathbf{e}}_z,$$

where $\hat{\mathbf{e}}_z$ is the unitary vector perpendicular to the plane. With these variables, the Euler equations for the ideal (dissipationless) incompressible fluid are

$$\nabla \cdot \mathbf{v} = 0, \quad \frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\omega = \mathbf{0}. \tag{1}$$

Numerical simulations performed by Montgomery *et al.* [1–6] have proved that the scalar stream function ψ describing the motion in two-dimensional space obeys in the far asymptotic regime the *sinh*-Poisson equation

$$\Delta\psi + \gamma \sinh(\beta\psi) = 0 \tag{2}$$

where γ and β are *positive* constants.

The fluid description is equivalent with a discrete model which consists of a system of N vorticity filaments perpendicular on plane carrying the vorticity ω_i, $i = \overline{1, N}$ [7]. The motion in plane of the k-th filament of coordinates $\mathbf{r}_k \equiv (r_k^1, r_k^2) \equiv (x_k, y_k)$ is given by

$$\frac{dr_k^i}{dt} = \varepsilon^{ij} \frac{\partial}{\partial r_k^j} \sum_{n=1, n\neq k}^{N} \omega_n G(\mathbf{r}_k - \mathbf{r}_n), \quad i, j = 1,2, \quad k = \overline{1, N} \tag{3}$$

where the summation is over all the other filaments' positions $\mathbf{r}_n$, $n \neq k$, and ε^{ij} is the antisymmetric tensor in two dimensions. The function $G(\mathbf{r}_k - \mathbf{r}_n)$ is the Green function of the Laplacian (L is the length of the square box sides)

$$G(\mathbf{r},\mathbf{r}') \approx -\frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L}\right). \tag{4}$$

The construction of Lagrangian density for the Euler fluid has been presented in a previous work [8]. The theory is in *sl*(2, **C**) which reflects the spinorial nature of the point-like vortices. The matter field nonlinear interaction is of order four.

$$\begin{aligned}\mathcal{L} = &-\varepsilon^{\mu\nu\rho} Tr\left(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho\right) + \\ &+ iTr\left(\phi^\dagger D_0 \phi\right) - \frac{1}{2} Tr\left(\left(D_i\phi\right)^\dagger D_i\phi\right) + \frac{1}{4} Tr\left(\left[\phi^\dagger,\phi\right]\right)^2.\end{aligned} \tag{5}$$

Here A^μ is the potential of the interaction between the point-like vortices, ϕ is the matter field, related to the density of point-like vortices. Applying the Bogomolnyi procedure we write the action as a sum of square terms plus (possibly) a term with topological content, and identify a new set of (self-duality, SD) equations describing the stationary states. It results at self-duality the equation *sinh*-Poisson, known to govern the asymptotic ordered states. It must be noted that the self-duality is the property which is at the origin of all known exactly integrable nonlinear equations and of their coherent nonlinear structures, like solitons, instantons, vortices of the Abelian-Higgs superconductors, Painleve transcendents, etc. We can derive the equations of motion, the expression of the (field theoretical) current and examine the states in close proximity of the self-duality. The form of the energy functional close to self-duality appears to be only weakly selective for minimal states, which suggests that reaching the relaxed states the system slows down and also can have various metastable states.

This investigation of the Euler equation is original and we have extended it in several directions. In particular it was possible to show that the stationary states obtained at relaxation from turbulence (*i.e.* solutions of the *sinh*-Poisson equation) can also be identified from a zero-curvature condition applied on a new gauge potential defined from a combination of the basic gauge and matter variables. The direct physical interpretation of these field theoretical variables is less clear.

3. CURVATURE FORMULATION

It is shown that there exists a formulation of the *self-dual* equations as equation expressing the zero-curvature condition for a fibre bundle defined with the field as a section. The particular combination of variables is

$$\begin{aligned}\mathcal{A}_+ &= A_+ - \lambda\phi, \\ \mathcal{A}_- &= A_- + \lambda\phi^\dagger,\end{aligned} \tag{6}$$

where

$$\lambda \equiv \text{real constant}, \tag{7}$$

and we calculate curvature-type fields

$$K_{\pm} \equiv \partial_{\pm}\mathcal{A}_{\mp} - \partial_{\mp}\mathcal{A}_{\pm} + \left[\mathcal{A}_{\pm}, \mathcal{A}_{\mp}\right]. \tag{8}$$

The *algebraic ansatz* which is employed in the derivation of the SD equations [8] consists of

$$\begin{aligned} \phi &= \phi_1 E_+ + \phi_2 E_-, \\ \phi^{\dagger} &= \phi_1^* E_- + \phi_2^* E_+, \end{aligned} \tag{9}$$

and

$$\begin{aligned} A_+ &= aH, \\ A_- &= -a^* H, \end{aligned} \tag{10}$$

then

$$\mathcal{A}_+ = A_+ - \lambda\phi = aH - \lambda\left(\phi_1 E_+ + \phi_2 E_-\right), \tag{11}$$

$$\mathcal{A}_- = A_- + \lambda\phi^{\dagger} = -a^* H + \lambda\left(\phi_1^* E_- + \phi_2^* E_+\right). \tag{12}$$

3.1. THE EXPRESSION OF THE CURVATURE K_+

Let us take the + sign:

$$\begin{aligned} K_+ &= \partial_+\mathcal{A}_- - \partial_-\mathcal{A}_+ + \left[\mathcal{A}_+, \mathcal{A}_-\right] = \\ &= \partial_+\left[-a^* H + \lambda\left(\phi_1^* E_- + \phi_2^* E_+\right)\right] - \partial_-\left[aH - \lambda\left(\phi_1 E_+ + \phi_2 E_-\right)\right] + \\ &\quad + \left[aH - \lambda\left(\phi_1 E_+ + \phi_2 E_-\right), -a^* H + \lambda\left(\phi_1^* E_- + \phi_2^* E_+\right)\right]. \end{aligned} \tag{13}$$

This can be written using notations

$$K_+ \equiv K_0^+ H + K_1^+ E_+ + K_2^+ E_-, \tag{14}$$

with

$$\begin{aligned} K_1^+ &\equiv \lambda\left(\partial_+\phi_2^* + \partial_-\phi_1\right) + 2\lambda\left(a\phi_2^* - a^*\phi_1\right), \\ K_2^+ &\equiv \lambda\left(\partial_+\phi_1^* + \partial_-\phi_2\right) + 2\lambda\left(-a\phi_1^* + a^*\phi_2\right), \\ K_0^+ &\equiv -\left(\partial_+ a^* + \partial_- a\right) - \lambda^2\left(\rho_1 - \rho_2\right). \end{aligned} \tag{15}$$

3.2. THE EXPRESSION OF THE CURVATURE K_-

Let us take the – sign:

$$\begin{aligned} K_- &= \partial_- \mathcal{A}_+ - \partial_+ \mathcal{A}_- + \left[\mathcal{A}_-, \mathcal{A}_+\right] = \\ &= \partial_- \left[aH - \lambda\left(\phi_1 E_+ + \phi_2 E_-\right)\right] - \partial_+ \left[-a^* H + \lambda\left(\phi_1^* E_- + \phi_2^* E_+\right)\right] + \\ &\quad + \left[-a^* H + \lambda\left(\phi_1^* E_- + \phi_2^* E_+\right), aH - \lambda\left(\phi_1 E_+ + \phi_2 E_-\right)\right]. \end{aligned} \tag{16}$$

This can be written using notations

$$K_- \equiv K_0^- H + K_1^- E_+ + K_2^- E_-, \tag{17}$$

with

$$\begin{aligned} K_1^- &\equiv -\lambda\left(\partial_+ \phi_2^* + \partial_- \phi_1\right) - 2\lambda\left(a\phi_2^* - a^* \phi_1\right), \\ K_2^- &\equiv -\lambda\left(\partial_+ \phi_1^* + \partial_- \phi_2\right) + 2\lambda\left(a\phi_1^* - a^* \phi_2\right), \\ K_0^- &\equiv \left(\partial_+ a^* + \partial_- a\right) + \lambda^2\left(\rho_1 - \rho_2\right). \end{aligned} \tag{18}$$

3.3. POWERS OF PRODUCTS OF THE CURVATURES

The fact that these three expressions are NOT zero is the signature that the fluid is NOT at self-duality.

It is interesting to calculate various powers and products of the *curvatures*, and the following formulas are useful

$$\begin{aligned} &\mathrm{tr}\left(E_+^n\right) = 0, \quad \mathrm{tr}\left(E_-^n\right) = 0, \\ &\mathrm{tr}\left(HE_+^m\right) = 0, \quad \mathrm{tr}\left(HE_-^m\right) = 0, \\ &\mathrm{tr}\left(H^n E_+^m\right) = 0, \quad \mathrm{tr}\left(H^n E_-^m\right) = 0, \\ &\mathrm{tr}\left(E_+^n E_-^m\right) = \mathrm{tr}\left(E_-^n E_+^m\right) = 1 \\ &\mathrm{tr}\left(H^{2n}\right) = 2, \quad \mathrm{tr}\left(H^{2n+1}\right) = 0 \end{aligned} \tag{19}$$

Let us consider the product of the two curvatures

$$\begin{aligned} &\mathrm{tr}\left\{\left(K_0^+ H + K_1^+ E_+ + K_2^+ E_-\right)\left(K_0^- H + K_1^- E_+ + K_2^- E_-\right)\right\} = \\ &= \mathrm{tr}\left\{K_0^+ K_0^- H^2\right\} \ \text{the trace is 2} \\ &+ \mathrm{tr}\left\{K_0^+ K_1^- HE_+\right\} \ \text{the trace is 0} \\ &+ \mathrm{tr}\left\{K_0^+ K_2^- HE_-\right\} \ \text{the trace is 0} \end{aligned} \tag{20}$$

$$+\operatorname{tr}\left\{K_1^+ K_0^- E_+ H\right\} \text{ the trace is 0}$$
$$+\operatorname{tr}\left\{K_1^+ K_1^- E_+ E_+\right\} \text{ the trace is 0}$$
$$+\operatorname{tr}\left\{K_1^+ K_2^- E_+ E_-\right\} \text{ the trace is 1}$$
$$+\operatorname{tr}\left\{K_2^+ K_0^- E_- H\right\} \text{ the trace is 0}$$
$$+\operatorname{tr}\left\{K_2^+ K_1^- E_- E_+\right\} \text{ the trace is 1}$$
$$+\operatorname{tr}\left\{K_2^+ K_2^- E_- E_-\right\} \text{ the trace is 0.}$$

Then

$$\operatorname{tr}\{K_+ K_-\} = 2K_0^+ K_0^- + K_1^+ K_2^- + K_2^+ K_1^- \tag{21}$$

or

$$\begin{aligned}\operatorname{tr}\{K_+ K_-\} = -2\Big[\big(\partial_+ a^* + \partial_- a\big) + \lambda^2 (\rho_1 - \rho_2)\Big]^2 - \\ -\lambda^2\Big[\big(\partial_+ \phi_2^* + \partial_- \phi_1\big) + 2\big(a\phi_2^* - a^*\phi_1\big)\Big]\Big[\big(\partial_+ \phi_1^* + \partial_- \phi_2\big) - 2\big(a\phi_1^* - a^*\phi_2\big)\Big].\end{aligned} \tag{22}$$

Now, since we have

$$\partial_+^* = \partial_-, \tag{23}$$

we see that the second paranthesis is the complex conjugate of the first

$$\begin{aligned}\Big[\big(\partial_+\phi_2^* + \partial_-\phi_1\big) + 2\big(a\phi_2^* - a^*\phi_1\big)\Big]^* = \big(\partial_+^*\phi_2 + \partial_-^*\phi_1^*\big) + 2\big(a^*\phi_2 - a\phi_1^*\big) = \\ = \partial_-\phi_2 + \partial_+\phi_1^* + 2\big(a^*\phi_2 - a\phi_1^*\big).\end{aligned} \tag{24}$$

Thus this relationship has been verified. We then have

$$\begin{aligned}\operatorname{tr}\{K_+ K_-\} = -2\Big[\big(\partial_+ a^* + \partial_- a\big) + \lambda^2 (\rho_1 - \rho_2)\Big]^2 - \\ -\lambda^2\Big|\big(\partial_+\phi_2^* + \partial_-\phi_1\big) + 2\big(a\phi_2^* - a^*\phi_1\big)\Big|^2.\end{aligned} \tag{25}$$

Naturally, we obtain that

$$-\operatorname{tr}\{K_+ K_-\} \geq 0 \tag{26}$$

since it is a sum of squares and the equality with zero is precisely the SD equations.

In the powers

$$\operatorname{tr}\Big(\big(K^-\big)^n\Big) \tag{27}$$

only the powers of K_0^- remains and the products $\big(K_1^-\big)^n\big(K_2^-\big)^m$. Then we have

$$\operatorname{tr}\left(\left(K^{-}\right)^{2n}\right) = \operatorname{tr}\left\{\left(\left(\partial_{+}a^{*} + \partial_{-}a\right) + \lambda^{2}\left(\rho_{1} - \rho_{2}\right)\right)H\right\}^{2n} = \\ = 2\left[\left(\partial_{+}a^{*} + \partial_{-}a\right) + \lambda^{2}\left(\rho_{1} - \rho_{2}\right)\right]^{2n} + \ldots \tag{28}$$

We note the presence of the combination that actually has led to the *sinh*-Poisson equation at self-duality [8]. We find then useful to look for the particular expressions, at self-duality.

4. THE SELF-DUALITY LIMIT OF THE CURVATURES

From the first equation at SD

$$D_{-}\phi = 0, \tag{29}$$

we derive [9, 10]

$$2\frac{\partial\phi_{1}}{\partial z} - 2a^{*}\phi_{1} = 0, \\ 2\frac{\partial\phi_{1}^{*}}{\partial z^{*}} - 2a\phi_{1}^{*} = 0, \tag{30}$$

and

$$2\frac{\partial\phi_{2}}{\partial z} + 2\phi_{2}a^{*} = 0, \\ 2\frac{\partial\phi_{2}^{*}}{\partial z^{*}} + 2a\phi_{2}^{*} = 0, \tag{31}$$

with the notations

$$\partial_{+} = \partial_{x} + i\partial_{y} = 2\frac{\partial}{\partial z^{*}}, \\ \partial_{-} = \partial_{x} - i\partial_{y} = 2\frac{\partial}{\partial z}, \tag{32}$$

from which we have

$$\partial_{+}\partial_{-} \equiv \Delta \quad \text{and} \quad 4\frac{\partial}{\partial z^{*}}\frac{\partial}{\partial z} \equiv \Delta. \tag{33}$$

We obtain the two potentials at SD

$$a^{*} = \frac{1}{2}\partial_{-}\ln\phi_{1}; \qquad a = \frac{1}{2}\partial_{+}\ln\phi_{1}^{*}; \\ a^{*} = -\frac{1}{2}\partial_{-}\ln\phi_{2}; \qquad a = -\frac{1}{2}\partial_{+}\ln\phi_{2}^{*}; \tag{34}$$

Then

$$\begin{aligned} 2a\phi_2^* &= -2\frac{\partial \phi_2^*}{\partial z^*} = -\partial_+ \phi_2^*, \\ 2a^*\phi_2 &= -2\frac{\partial \phi_2}{\partial z} = -\partial_- \phi_2, \\ 2a^*\phi_1 &= 2\frac{\partial \phi_1}{\partial z} = \partial_- \phi_1, \\ 2a\phi_1^* &= 2\frac{\partial \phi_1^*}{\partial z^*} = \partial_+ \phi_1^*. \end{aligned} \tag{35}$$

We have

$$\begin{aligned} K_1/\lambda &= \partial_+\phi_2^* + \partial_-\phi_1 + 2a\phi_2^* - 2a^*\phi_1 = \\ &= \partial_+\phi_2^* + \partial_-\phi_1 + \left(-\partial_+\phi_2^*\right) - \left(\partial_-\phi_1\right) = 0, \end{aligned} \tag{36}$$

$$\begin{aligned} K_2/\lambda &= \partial_+\phi_1^* + \partial_-\phi_2 - 2a\phi_1^* + 2a^*\phi_2 = \\ &= \partial_+\phi_1^* + \partial_-\phi_2 - \left(\partial_+\phi_1^*\right) + \left(-\partial_-\phi_2\right) = 0, \end{aligned} \tag{37}$$

$$K_0 = -\left(\partial_+ a^* + \partial_- a\right) - \lambda^2\left(\rho_1 - \rho_2\right), \tag{38}$$

where we have

$$\begin{aligned} K_0 &= -\partial_+\left[\frac{1}{2}\partial_- \ln\left(\phi_1\right)\right] - \partial_-\left[\frac{1}{2}\partial_+ \ln\left(\phi_1^*\right)\right] - \lambda^2\left(\rho_1 - \rho_2\right) = \\ &= -\partial_+\partial_- \frac{1}{2}\left[\ln\left(\phi_1\right) + \ln\left(\phi_1^*\right)\right] - \lambda^2\left(\rho_1 - \rho_2\right) = \\ &= -\frac{1}{2}\Delta \ln \rho_1 - \lambda^2\left(\rho_1 - \rho_2\right), \end{aligned} \tag{39}$$

which is real. This expression is also zero at SD since it is the equation

$$\begin{aligned} -\frac{1}{2}\Delta \ln \rho_1 - \lambda^2\left(\rho_1 - \rho_2\right) = 0, \\ \frac{1}{2}\Delta\psi + \lambda^2 2\sinh\psi = 0. \end{aligned} \tag{40}$$

The normal choice for λ is

$$\lambda^2 = \frac{1}{L^2}, \tag{41}$$

with L the length of the box.

5. CONCLUSIONS

The fact that the *sinh*-Poisson equation is derived as the zero-curvature condition may have a deep physical significance. The self-duality (which is at the origin of the *sinh*-Poisson equation, in our derivation) is in general a property of a geometric-algebraic structure, a fiber space where it is defined a differential form. For example, for the Maxwell field without sources in four dimensional space, the field tensor is the curvature two-form. The self-duality consists of the equality between the differential form and its Hodge dual, in this example: $F^{\mu\nu} dx^{\mu} \wedge dx^{\nu} = = *\left(F^{\sigma\rho} dx^{\sigma} \wedge dx^{\rho} \right)$. For the field-theoretical formulation of the Euler fluid it is not easy to identify the geometric-algebraic structure and find the Hodge dual of a differential *p*-form. The calculations shown above may be a useful step for the understanding of the full structure which is behind the ideal incompressible (Euler) fluid.

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