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Dedicated to Prof. Ioan-Iovitz Popescu's 75th Anniversary

ABOUT THE PRIMARY RAINBOW AND ITS POLARIZATION

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Abstract. The aim of this paper, referring to the polarization of the first order rainbow, is to commemorate 370 years since the publication of the first geometric theory of the rainbow (René Descartes, 1637) and 170 years since the elaboration of the first wave theory of this phenomenon (George Biddell Airy, 1838). After some historical considerations, in the first part of our paper, we propose a new manner of calculation (in a classical framework) for the Stokes parameters of the high order rainbows (corresponding to any number of reflections within the raindrops).Then, in the second part of the work, based mainly upon the Khare-Nussenzveig simplified version of Mie's theory (1908), soon centenarian old, we point out the strong dependence of the Stokes parameters for the primary rainbow on the spherical water drop dimensions, not only when the incident light is unpolarized (natural, or solar light), but also when the rainbow is generated by partially polarized (artificial) light. We disregard here the dispersive properties of the water.

Key words: Stokes parameters, polarization, primary rainbow, scattering, Mie theory.

1. INTRODUCTION AND A HISTORICAL SURVEY

The rainbow is one of the most beautiful and spectacular phenomena in nature. Along humanity's troubled history, the rainbow has impressed all generations because of its vivid colours and graceful arcs, often making a bridge between art and science, between sensorial and rational. Myth, legend, art and science have particularly approached this phenomenon according to the general level of human knowledge, sensitivity or scientific competence. Generally, for the physicist and particularly for the optician, the rainbow has, and will always be, a necessary and invigorating task out of the desire to know and better understand the causes and mechanisms that generate this strange, imposing and sublime phenomenon that nature sometimes offers.

The history of the rainbow [1, 2] is very long and abundant in significant events. Although in ancient times Aristotle tried to explain the rainbow formation, considering light reflection as responsible for what happens, it is to Descartes [3] that we owe the first pertinent theory, combining reflection with refraction. Then, during the years 1669–1672, enlarging the Cartesian study, Newton [4] gave the first correct interpretation of the rainbow arc colours. At the beginning of the 18th century, Halley [5] elaborated the first correct theory of arbitrary order rainbows, corresponding to whatever number of reflections within the raindrops.

All the above mentioned theories were founded on the straight light rays model. During the second half of the 18th century, when geometrical theories had begun to be unanimously accepted, strange things, that could no longer be explained by means of the Cartesian-Newtonian model, were furthermore discovered. We refer to the so-called supplementary or supernumerary arcs, which appeared on the lower part of the primary rainbow, under the violet arc, noticed in 1772 by Langwith [6]. It is to Young [7], that we owe the first correct interpretation of these arcs. He considered them as interference fringes generated by two rays scattered in the same direction (at minimum of the angle of deviation) by a raindrop – strictly analogous to the light passing through the two pinholes in his well-known interference experiment from 1801–1803.

In the first half of the 19th century, between 1836 and 1838, G. B. Airy [8], an distinguished english mathematician and astronomer, elaborated and published the first undulatory theory of the phenomenon, enlarging already old ideas of Young (which suggested that the supernumerary bows could not be explained unless light were thought of as a wave phenomenon) and professor Richard Potter of the Cambridge University [9], who considered that the cartesian rainbow ray can be interpreted as a *caustic*, *i.e.* the envelope of the system of rays comprising the rainbow. Airy's theory emphasize the role played by the smaller dimensions of the rain drops in the phenomenon concreteness, the interference-diffraction implications being clearly included for the explanation of the rainbow's supernumerary arcs (that are not predicted by no means by geometrical optics). According to Airy's approach, the first order rainbow is the main diffraction maximum, while supernumerary arcs are secondary maxima. Many experimenters, as Babinet, Miller, Galle, Billet, Pulfrich and others (see [1]), underlined the correctness of Airy's theory.

Paradoxically, at first, the discovery of the electromagnetic nature of the light had no good influence on the rainbow theory, although the polarization of the rainbow's arcs (discovered by Biot in 1811–1812, studied also by Brewster in 1812–1816 and by Billet between 1863 and 1868) can be correctly explained only within this framework. The electromagnetic study of the rainbow, initiated by J. W. Strutt (Lord Rayleigh) between 1879 and 1881 (before Hertz's evincing Maxwell's theoretical prediction!) had no echo among the contemporaries. Unfortunately, Rayleigh's considerations was published too late [10], only in 1910.

An important step forward was achieved in 1908 when Gustav Mie [11] elaborated the first consistent theory of electromagnetic wave diffraction on colloidal spheres with a complex refractive index. Referring to this paper, Arnold Sommerfeld [12] was writting: *The electromagnetic study of light diffraction on an object is a very complicated problem even in the case of the sphere, the simplest possible one. The field outside a sphere can be represented by series of spherical harmonics and Bessel functions of half-integer indices. These series have been discussed by G. Mie for colloidal particles of arbitrary compositions. But even there a mathematical difficulty develope which quite generally is a drawback of this method of series development: for fairly large particles* ($\beta = ka > 1$, *a=radius,* $k = 2\pi/\lambda$) *the series converge so slowly that they become practically useless. Except for this difficulty, we could, in this way, obtain a complete solution of the problem of the rainbow.*

At the beginning of the eighth decade of the former century, the physicists interested in the rainbow problem were still puzzled by a paradoxical situation: although the form of the right solution given by Mie was known, they could not make use of it in the complete description of the phenomenon. For many years this theory seemed to have little practical value because it was computationally demanding. Half seriously, having in mind the very slowly convergence of the series of partial waves, many people used to say that Mie's theory was *an elephant that would never fly.* And still, intrinsic mathematical difficulties of Mie's theory were, if not vanquished, at least avoided in 1974, and then, in 1977, when V. Khare and H. M. Nussenzveig [13] had the happy idea of rewriting Mie's series as integrals in the complex plan $\Lambda = \ell + 1/2$ (ℓ being the angular momentum). In such a way, the former *Watson transform* (method developed also by Poincaré), known eversince 1918, was modified by Khare and Nussenzveig (KN) becoming *the method of complex angular momentum* (the *CAM* method) [14]. Thus, instead of adding, in partial wave series, an immense number of terms (with a very complicated structure each), when integrating in the complex Λ plane, only some "critical points" are taken into account, namely the saddle points and the Regge poles of the integrand. This KN theory extended considerably Airy's theory and succeeded in being in a good enough agreement with experimental data for $\beta = ka \geq 30$. Furthermore, by its means, it has been demonstrated that, for TE polarization, Airy's classical theory describes the phenomenon correctly only in the main maximum zone; for secondary maxima (supernumerary arcs), the corrections applied to the classical theory are quantitatively considerable. As far as the TM component is concerned, Airy's theory is practically useless, as the diffraction maxima and minima are interchanged. On the other hand, in the recent years, stimulated by the pervasive needs of Mie computations for large spheres, many authors [15–18] have developed efficient computer algorithms up to $\beta = 5000$ (the so called *Airy's limit*).

The objective of our paper is to investigate the polarization of the rainbow light. First time, we present a new and simple manner of calculation for the Stokes parameters of the high order rainbows (corresponding to any number of reflections within the raindrops). Then, in the second part of the work, based mainly upon the

Khare-Nussenzveig simplified version of the Mie theory, we point out the strong dependence of the Stokes parameters for the primary rainbow on the spherical water drop dimensions, not only when incident light is unpolarized (natural, or solar light), but also when the rainbow is generated by partially polarized (artificial) light. Dispersive properties of the water will not be taken into consideration.

2. FUNDAMENTAL FORMULAS

Let $\xi_j^{(i)}$, $j = 1, 2, 3$ be the Stokes parameters characterizing the polarization state of incident light on spherical water raindrops. The normalized polarization (coherency) matrix of the incident light has the form [19]

$$\rho_{\mu\upsilon}^{(i)} = \frac{\left\langle E_\mu^{(i)} E_\upsilon^{(i)*} \right\rangle}{\left\langle \left| E_\perp^{(i)} \right|^2 \right\rangle + \left\langle \left| E_\parallel^{(i)} \right|^2 \right\rangle} = \frac{1}{2} \begin{pmatrix} 1+\xi_3^{(i)} & \xi_1^{(i)} - i\xi_2^{(i)} \\ \xi_1^{(i)} + i\xi_2^{(i)} & 1-\xi_3^{(i)} \end{pmatrix}, \tag{1}$$

where μ and ν can be $\perp$ and $\parallel$, and refer to the TE and respectively TM components of the electrical field of the incident wave. From here we obtain easily

$$\xi_1^{(i)} = \rho_{\parallel,\perp}^{(i)} + \rho_{\perp,\parallel}^{(i)}, \quad \xi_2^{(i)} = i(\rho_{\perp,\parallel}^{(i)} - \rho_{\parallel,\perp}^{(i)}), \quad \xi_3^{(i)} = \rho_{\perp,\perp}^{(i)} - \rho_{\parallel,\parallel}^{(i)}. \tag{2}$$

The physical significances of the various matrix elements (measurable quantities) $\rho_{\mu,\nu}^{(i)}$ are obvious from the definition (1).

Any optical system could transform the polarization state of the incident ("in") light. The Stokes'parameters of the emergent ("out") light could be easily evaluated if the elements of the system's S-matrix (the so-called scattering amplitudes) should be known. For the case of the non-absorbing systems (or media), these are defined by the relations $E_\mu^{(e)} = S_\mu E_\mu^{(i)}$, with the index $\mu = \perp$ or $\parallel$ (without summation after the lower index μ). For emergent light, the non-normalized polarization (coherency) matrix could be defined by the relation $\overline{\rho}_{\mu,\nu}^{(e)} = \left\langle E_\mu^{(e)} \cdot E_\nu^{(e)*} \right\rangle$, and we come to

$$\overline{\rho}^{(e)} = \begin{pmatrix} \left\langle \left| E_\perp^{(e)} \right|^2 \right\rangle & \left\langle E_\perp^{(e)} \cdot E_\parallel^{(e)*} \right\rangle \\ \left\langle E_\parallel^{(e)} \cdot E_\perp^{(e)*} \right\rangle & \left\langle \left| E_\parallel^{(e)} \right|^2 \right\rangle \end{pmatrix} = \begin{pmatrix} \left| S_\perp \right|^2 \cdot \overline{\rho}_{\perp,\perp}^{(i)} & S_\perp S_\parallel^* \cdot \overline{\rho}_{\perp,\parallel}^{(i)} \\ S_\parallel S_\perp^* \overline{\rho}_{\parallel,\perp}^{(i)} & \left| S_\parallel \right|^2 \cdot \overline{\rho}_{\parallel,\parallel}^{(i)} \end{pmatrix}. \tag{3}$$

The trace of this matrix is $Tr\overline{\rho}^{(e)} = \left| S_\perp \right|^2 \overline{\rho}_{\perp,\perp}^{(i)} + \left| S_\parallel \right|^2 \overline{\rho}_{\parallel,\parallel}^{(i)}$.

To be able to identify the Stokes'parameters of emergent light (the rainbow, in our case), we will have to define, for this case too, the normalized polarization (coherency) matrix. Obviously, we have

$$\rho^{(e)} = \frac{\overline{\rho}^{(e)}}{Tr\overline{\rho}^{(e)}} = \frac{1}{\left(1+\xi_3^{(i)}\right) + \frac{|S_{\parallel}|^2}{|S_{\perp}|^2}\left(1-\xi_3^{(i)}\right)} \begin{pmatrix} 1+\xi_3^{(i)} & \left(\xi_1^{(i)} - i\xi_2^{(i)}\right)\frac{S_{\parallel}^*}{S_{\perp}^*} \\ \left(\xi_1^{(i)} + i\xi_2^{(i)}\right)\frac{S_{\parallel}}{S_{\perp}} & \left(1-\xi_3^{(i)}\right)\frac{|S_{\parallel}|^2}{|S_{\perp}|^2} \end{pmatrix}. \quad (4)$$

We denote $S_{\parallel}/S_{\perp} \equiv |S_{\parallel}/S_{\perp}|\exp(i\delta)$ and we immediately obtain

$$\rho^{(e)} = \frac{1}{\left(1+\xi_3^{(i)}\right) + \frac{|S_{\parallel}|^2}{|S_{\perp}|^2}\left(1-\xi_3^{(i)}\right)} \cdot$$

$$\cdot \begin{pmatrix} 1+\xi_3^{(i)} & \left(\xi_1^{(i)} - i\xi_2^{(i)}\right)(\cos\delta - i\sin\delta)\left|\frac{S_{\parallel}}{S_{\perp}}\right| \\ \left(\xi_1^{(i)} + i\xi_2^{(i)}\right)(\cos\delta + i\sin\delta)\left|\frac{S_{\parallel}}{S_{\perp}}\right| & \left(1-\xi_3^{(i)}\right)\frac{|S_{\parallel}|^2}{|S_{\perp}|^2} \end{pmatrix}. \quad (5)$$

By writing for $\rho^{(e)}$ an expression of the form (1) and by identifying the corresponding elements from (5), we find the following expressions:

$$\xi_1^{(e)} = \frac{2\left|\frac{S_{\parallel}}{S_{\perp}}\right|\left(\xi_1^{(i)}\cos\delta - \xi_2^{(i)}\sin\delta\right)}{\left(1+\xi_3^{(i)}\right) + \left|\frac{S_{\parallel}}{S_{\perp}}\right|^2\left(1-\xi_3^{(i)}\right)}, \quad (6.a)$$

$$\xi_2^{(e)} = \frac{2\left|\frac{S_{\parallel}}{S_{\perp}}\right|\left(\xi_1^{(i)}\sin\delta + \xi_2^{(i)}\cos\delta\right)}{\left(1+\xi_3^{(i)}\right) + \left|\frac{S_{\parallel}}{S_{\perp}}\right|^2\left(1-\xi_3^{(i)}\right)}, \quad (6.b)$$

$$\xi_3^{(e)} = \frac{\left(1+\xi_3^{(i)}\right) - \left|\frac{S_{\parallel}}{S_{\perp}}\right|^2\left(1-\xi_3^{(i)}\right)}{\left(1+\xi_3^{(i)}\right) + \left|\frac{S_{\parallel}}{S_{\perp}}\right|^2\left(1-\xi_3^{(i)}\right)}. \quad (6.c)$$

On the basis of these relations, we can easily determine the polarization degree of the emergent light $P^{(e)} = +\sqrt{\xi_1^{(e)2} + \xi_2^{(e)2} + \xi_3^{(e)2}}$, as a function of the polarization degree of the incident light ($P^{(i)}$). We obtain

$$P^{(e)2}=\frac{4\left|\frac{S_{\|}}{S_{\perp}}\right|^{2}P^{(i)2}+\left(1-\left|\frac{S_{\|}}{S_{\perp}}\right|^{2}\right)\left[\left(1+\xi_{3}^{(i)2}\right)\left(1-\left|\frac{S_{\|}}{S_{\perp}}\right|^{2}\right)+2\xi_{3}^{(i)}\left(1+\left|\frac{S_{\|}}{S_{\perp}}\right|^{2}\right)\right]}{\left[\left(1+\xi_{3}^{(i)}\right)+\left|\frac{S_{\|}}{S_{\perp}}\right|^{2}\left(1-\xi_{3}^{(i)}\right)\right]^{2}}. \tag{7}$$

From here, it easily results that, when $P^{(i)}=+1$, the emergent light has the degree of polarization $P^{(e)}=+1$ for whatever value of $\xi_3^{(i)}$, a fact that shows us that, for a completely polarized incident light, the emergent light would be also entirely polarized. But, generally, when $P^{(i)}<1$ (partially polarized light) the emergent light's polarization degree depends also on the parameter $\xi_3^{(i)}$. Anyway, in such a circumstance, the formula (7) show us that $P^{(e)}<1$, that is to say the emergent light is partially polarized too.

Another useful relation which could be deduced from (6.a-c) is the one regarding the connection between $P^{(e)}$ and $\xi_3^{(e)}$, on one side, and $P^{(i)}$ and $\xi_3^{(i)}$ on the other side. We easily obtain

$$\frac{P^{(e)2}-\xi_{3}^{(e)2}}{\left(1+\xi_{3}^{(e)}\right)^{2}}=\left|\frac{S_{\|}}{S_{\perp}}\right|^{2}\cdot\frac{P^{(i)2}-\xi_{3}^{(i)2}}{\left(1+\xi_{3}^{(i)}\right)^{2}}. \tag{8}$$

3. CLASSICAL CONSIDERATIONS

When the incident light wave, with unitary amplitude, falling on a raindrop, is polarized in the plane of incidence (*i.e.* TE or $\perp$), Fresnel found that the amplitude of the reflected light wave was given by $R_{\perp}=\sin(r-i)/\sin(r+i)$ and the amplitude of the refracted light wave – by $T_{\perp}=2\cos i\sin r/\sin(i+r)$, i being the angle of incidence and r the refraction one. Also, when the incident light wave with unitary amplitude is polarized at right angles to the plane of incidence (*i.e.* TM or $\|$) the amplitude of the reflected light wave was given by $R_{\|}=\mathrm{tg}(i-r)/\mathrm{tg}(i+r)$ and the amplitude of the refracted light wave–by $T_{\|}=2\cos i\sin r/\sin(i+r)\cos(i-r)$.

In the cartesian theory of the rainbow with N internal reflections and two refractions, the total angular deviation is $D=2(i-r)+N(\pi-2r)$. From the condition of extreme deviation $dD/di=0$, having in mind the law of refraction

$\sin r = (1/n)\sin i,$ we find $\sin i_c = \left[(m^2 - n^2)/(m^2 - 1)\right]^{1/2}$, (c – as below index, means "*cartesian*"), $m = N+1$, so that, finnally

$$S_{\perp} = \frac{\sin 2i_c \sin 2r_c \sin^N (i_c - r_c)}{\sin^{N+2}(i_c + r_c)} = \frac{4(N+1)N^N}{(N+2)^{N+2}}, \tag{9}$$

$$S_{\parallel} = \frac{\sin 2i_c \sin 2r_c tg^N (r_c - i_c)}{\sin^2(i_c + r_c)\cos^2(i_c - r_c) tg^N (i_c + r_c)} = \frac{4n^2(N+1)(N+1-n^2)^N}{(N+1+n^2)^{N+2}}, \tag{10}$$

and consequently

$$\frac{S_{\parallel}}{S_{\perp}} = \frac{(N+2)^{N+2}}{N^N} \cdot \frac{n^2(N+1-n^2)^N}{(N+1+n^2)^{N+2}}. \tag{11}$$

If $n < \sqrt{N+1}$, from the relations (6) with $\delta = 0$, after a reasonable amount of algebra, we find

$$\xi_{1,2}^{(e)} = \frac{2n^2(N+2)^{N+2}N^N(N+1-n^2)^N(N+1+n^2)^{N+2}}{(1+\xi_3^{(i)})N^{2N}(N+1+n^2)^{2N+4} + (1-\xi_3^{(i)})n^4(N+2)^{2N+4}(N+1-n^2)^{2N}}\xi_{1,2}^{(i)}, \tag{12}$$

$$\xi_3^{(e)} = \frac{(1+\xi_3^{(i)})N^{2N}(N+1+n^2)^{2N+4} - (1-\xi_3^{(i)})n^4(N+2)^{2N+4}(N+1-n^2)^{2N}}{(1+\xi_3^{(i)})N^{2N}(N+1+n^2)^{2N+4} + (1-\xi_3^{(i)})n^4(N+2)^{2N+4}(N+1-n^2)^{2N}}. \tag{13}$$

For incident natural (unpolarized) light, when $\xi_j^{(i)} = 0,$ for j = 1, 2 and 3, we have

$$\xi_{1,2}^{(e)} = 0 \tag{14.a,b}$$

and

$$\xi_3^{(e)} = \frac{N^{2N}(N+1+n^2)^{2N+4} - n^4(N+2)^{2N+4}(N+1-n^2)^{2N}}{N^{2N}(N+1+n^2)^{2N+4} + n^4(N+2)^{2N+4}(N+1-n^2)^{2N}} \equiv P_N^{(e)}. \tag{14.c}$$

Now, we can remark that the extreme values of the degree of polarization (for $N = 1$ and $N \to 8$) are

$$P_1^{(e)} = \frac{\left(2+n^2\right)^6 - 729n^4\left(2-n^2\right)^2}{\left(2+n^2\right)^6 + 729n^4\left(2-n^2\right)^2} \quad \text{and} \quad P_{\infty}^{(e)} = \frac{1-\left(ne^{1-n^2}\right)^4}{1+\left(ne^{1-n^2}\right)^4}. \tag{15}$$

When n = 4/3 (water drops), $P_1^{(e)} = 0.9247,$ respectively $P_{\infty}^{(e)} = 0.7532.$

Strong polarization is the result of the path that the beams of light generating the rainbows must follow through the drops. In the case of the primary rainbow, the internal reflection occurs close (with a good enough approximation!) to the

Brewster angle [arctg$\left(1/n\right) \approx 37°$, and $r_c \approx 40°$ respectively] so that the light is very strongly polarized in the tangential direction.

4. KHARE-NUSSENZVEIG VERSION OF THE MIE SCATTERING THEORY

As in the papers [13], we denote by $S_j(\beta, \theta)$, $j = 1$ and 2, the scattering amplitudes of the TE component ($j = 1$) and of the TM component ($j = 2$), depending on the adimensional size parameter β and on the scattering angle θ-measured between the direction of the incident light wave and the one of the emergent light wave. According to Debye's method [20], the scattering amplitude can be expanded in series of the form $S_j = S_j^{(0)} + S_j^{(1)} + S_j^{(2)} + ...$ Here, $S_j^{(0)}$ correspond to the direct reflection of incident light on the dielectric sphere, $S_j^{(1)}$ - to the emergent ray after two refractions, $S_j^{(2)}$ -to the emergent ray after one refraction, one internal reflection and another refraction, a.s.o. Of course, according to cartesian theory, the term responsible for the generation of the primary rainbow is $S_j^{(2)}$, having a complicate dependence on the size parameter β and on the scattering angle θ. The other Debye terms ($S_j^{(0)}$, $S_j^{(1)}$, $S_j^{(3)}$, …) determine only a background illumination (of the sky-in natural situations). In the KN theory, the scattering amplitudes $S_j^{(2)}$ are given by

$$S_j^{(2)}\left(\beta,\theta\right) = -n\exp\left(i\pi/4\right)\left(\pi\sin\theta\right)^{-1/2} K^{3/2} F_j\left(\beta,\theta\right), \tag{16}$$

where

$$\begin{aligned} F_j\left(\beta,\theta\right) &= \left(2\pi i\right)K^{-1/3}\exp\left(KA(\varepsilon)\right)\times \\ &\times\Big\{\Big[p_{0j}(\varepsilon) - \Big(q_{1j}(\varepsilon) + 2\zeta(\varepsilon)q_{2j}(\varepsilon)\Big)K^{-1} + O(K^{-2})\Big]Ai(x) - \\ &-K^{-1/3}\Big[q_{0j}(\varepsilon) - 2p_{2j}(\varepsilon)K^{-1} + O(K^{-2})\Big]Ai'(x)\Big\}, \quad j = 1 \text{ and } 2. \end{aligned} \tag{17}$$

Here, the following denotations were used: $K = 2\beta = 4\pi a/\lambda$, $x = K^{2/3}\zeta(\varepsilon)$, $\varepsilon = \theta - \theta_c$, θ_c being the scattering angle of the primary rainbow, calculated according to cartesian theory, n is the refractive index of the dielectric (water) sphere. For the water droplets with $n = 4/3$, the value of the cartesian angle is

$$\theta_c = D(i_c) = 2\arccos\left[\frac{1}{n^2}\left(\frac{4-n^2}{3}\right)^{3/2}\right] \cong 137.5°. \tag{18}$$

The explicit form of the functions $F_j(\beta,\theta)$, as well as of the coefficients $p_{ij}(\varepsilon)$ and $q_{ij}(\varepsilon)$, was determined by means of the Chester-Friedman-Ursell (CFU) method [21] in terms of the exactly known saddle points, but only in the limit $|\varepsilon| << K^{-1/3}$. In accordance with the papers [13, 14], we summarize the final form of the coefficients $p_{ij}(\varepsilon)$ and $q_{ij}(\varepsilon)$ in Table 1.

Table 1

CFU coefficients $p_{ij}(\varepsilon)$ and $q_{ij}(\varepsilon)$, for $n = 4/3$

Coefficient	$j = 1$	$j = 2$
$p_{0j}(\varepsilon)$	$i[0.0381 - 0.031\varepsilon - 0.19\varepsilon^2]$	$i[0.00786 + 0.046\varepsilon - 0.078\varepsilon^2]$
$q_{0j}(\varepsilon)$	$0.0227 - 0.15\varepsilon - 0.59\varepsilon^2$	$0.108 - 0.015\varepsilon - 0.43\varepsilon^2$
$q_{1j}(\varepsilon)$	$0.40 + 3.0\varepsilon$	$0.042 + 2.3\varepsilon$
$p_{2j}(\varepsilon)$	$-1.4i$	$-0.64i$
$q_{2j}(\varepsilon)$	-4.1	-3.1

On the other hand, the functions $A(\varepsilon)$ and $\zeta(\varepsilon)$ – corresponding to the half sum and half difference of the optical path length through the spherical water droplet, are given by

$$A(\varepsilon) = i\left[1.519 + 0.431\varepsilon - 0.115\varepsilon^2 + O(\varepsilon^3)\right], \tag{19}$$

$$\zeta(\varepsilon) = -0.369\varepsilon - 0.0745\varepsilon^2 + O\left(\varepsilon^3\right). \tag{20}$$

$Ai(x)$ and $Ai'(x)$ designate Airy's function and its derivative with respect to x. Their values were taken from the known book of Abramowitz and Stegun [22].

In order to determine the Stokes parameters as well as the polarization degree of the emergent light (main rainbow) we are interested to find numerical values for the quantity $|S_{\parallel}/S_{\perp}|^2$ and for the phase δ of the complex number $S_{\parallel}/S_{\perp}$. We have

$$\left|\frac{S_2^{(2)}(\beta,\theta)}{S_1^{(2)}(\beta,\theta)}\right|^2 = \frac{\left[\operatorname{Re} F_2(\beta,\theta)\right]^2 + \left[\operatorname{Im} F_2(\beta,\theta)\right]^2}{\left[\operatorname{Re} F_1(\beta,\theta)\right]^2 + \left[\operatorname{Im} F_1(\beta,\theta)\right]^2} =$$

$$= \frac{\left\{p''_{02}(\varepsilon)Ai(x) - 2|p_{22}|\frac{Ai'(x)}{K^{4/3}}\right\}^2 + \left\{\frac{1}{K}\left[q_{12}(\varepsilon) + 2\zeta(\varepsilon)q_{22}\right]Ai(x) + q_{02}(\varepsilon)\frac{Ai'(x)}{K^{1/3}}\right\}^2}{\left\{p''_{01}(\varepsilon)Ai(x) - 2|p_{21}|\frac{Ai'(x)}{K^{4/3}}\right\}^2 + \left\{\frac{1}{K}\left[q_{11}(\varepsilon) + 2\zeta(\varepsilon)q_{21}\right]Ai(x) + q_{01}(\varepsilon)\frac{Ai'(x)}{K^{1/3}}\right\}^2} \tag{21}$$

and

$$\mathrm{tg}\delta = \frac{(\mathrm{Re}\,F_2)(\mathrm{Im}\,F_1) - (\mathrm{Re}\,F_1)(\mathrm{Im}\,F_2)}{(\mathrm{Re}\,F_1)(\mathrm{Re}\,F_2) + (\mathrm{Im}\,F_1)(\mathrm{Im}\,F_2)}. \tag{22}$$

Here $p''_{0j}(\varepsilon)$ designate the imaginary part of the function $p_{0j}(\varepsilon)$ and $x = K^{2/3}\zeta(\varepsilon)$.

In the case of the first order rainbow, for the center of the main diffraction peak, in the works [23] the following relation is deduced

$$\left|\frac{S_{\parallel}}{S_{\perp}}\right|^2 = \frac{\left(\frac{0.0149111}{K} - \frac{0.0279524}{K^{1/3}}\right)^2 + \left(2.79052\cdot 10^{-3} + \frac{0.3312888}{K^{4/3}}\right)^2}{\left(\frac{0.1420112}{K} - \frac{5.8752003\cdot 10^{-3}}{K^{1/3}}\right)^2 + \left(0.0135265 + \frac{0.7246943}{K^{4/3}}\right)^2}. \tag{23}$$

Table 2

Values of some important parameters, evaluated through the KN theory

β	$\frac{\lvert S_{\parallel}\rvert^2}{\lvert S_{\perp}\rvert^2}$	$\frac{\lvert S_{\parallel}\rvert}{\lvert S_{\perp}\rvert}$	$\delta(°)$	$\sin\delta$	$\cos\delta$	$\xi^{(e)}_{3,natural}$
50	0.20544	0.45325	–59.763	–0.86395	+ 0.50358	+ 0.6591
75	0.19334	0.43970	–57.521	–0.84359	+ 0.53699	+ 0.6760
100	0.15778	0.39721	–55.652	–0.82562	+ 0.56422	+ 0.7274
125	0.14389	0.37933	–54.075	–0.80977	+ 0.58672	+ 0.7484
150	0.13344	0.36530	–52.717	–0.79565	+ 0.60575	+ 0.7645
175	0.12530	0.35398	–51.525	–0.78288	+ 0.62217	+ 0.7773
200	0.11870	0.34453	–50.464	0.77122	+ 0.63656	+ 0.7878
250	0.10871	0.32971	–48.638	–0.75055	+ 0.66081	+ 0.8039
300	0.10144	0.31850	–47.406	–0.73262	+ 0.68064	+ 0.8158
350	0.09587	0.30963	–45.788	–0.71676	+ 0.69732	+ 0.8250
400	0.09144	0.30240	–44.632	–0.70255	+ 0.71163	+ 0.8324
450	0.08784	0.29638	–43.604	–0.68968	+ 0.72412	+ 0.8385
500	0.08482	0.29124	–42.681	–0.67791	+ 0.73514	+ 0.8436
750	0.07493	0.27373	–39.110	–0.63081	+ 0.77594	+ 0.8606
1 000	0.06933	0.26331	–36.591	–0.59610	+ 0.80291	+ 0.8703
1 500	0.06302	0.25104	–33.116	–0.54634	+ 0.83756	+ 0.8814
2 500	0.05713	0.23902	–28.947	–0.48400	+ 0.87507	+ 0.8919
5 000	0.05174	0.22746	–23.804	–0.40361	+ 0.91493	+ 0.9016
10 000	0.04834	0.21986	–19.351	–0.33136	+ 0.94350	+ 0.9078
50 000	0.04454	0.21105	–11.645	–0.20184	+ 0.97942	+ 0.9147
100 000	0.04381	0.20931	–09.296	–0.16153	+ 0.98687	+ 0.9161
∞	0.04256	0.20630	00.000	–0.00000	+ 1.00000	+ 0.9184

Considering the size parameter β between 50 and +∞, the data presented in Table 2 were obtained.

The last column in this table presents the values of the polarization degree for the primary rainbow generated by a totally unpolarized (natural, solar) incident light (with $\xi^{(i)}_{1,2,3} = 0$), calculated through the well-known formula

$$P^{(e)} = \left|\xi^{(e)}_{3,natural}\right| = \left[1 - \left|S_{\parallel} / S_{\perp}\right|^2\right] \Big/ \left[1 + \left|S_{\parallel} / S_{\perp}\right|^2\right]. \quad (24)$$

With the help of the parameters from the other columns of this table, using the general formulas (6)–(8), we can approach now the study of rainbows generated by artificial (partially polarized) light. From the multitude of possibilities of this kind, we will deal first with a particular case, namely the one where all three Stokes'parameters of incident light have the same value +1/2. In this case $P^{(i)} = 0.8660$. Using the formulas (6) and (7), we obtain the data from Table 3.

Table 3

The Stokes'parameters and the polarization degree for a primary artificial rainbow generated by incident light partially polarized with $\xi^{(i)}_1 = \xi^{(i)}_2 = \xi^{(i)}_3 = 1/2$

β	$\xi^{(e)}_1$	$\xi^{(e)}_2$	$\xi^{(e)}_3$	$P^{(e)}$
50	0.3867	–0.1019	0.8718	0.9592
75	0.3639	–0.0808	0.8789	0.9547
100	0.3497	–0.0658	0.9001	0.9678
125	0.3370	–0.0538	0.9085	0.9704
150	0.3268	–0.0443	0.9148	0.9724
175	0.3182	–0.0364	0.9198	0.9740
200	0.3110	–0.0298	0.9239	0.9753
250	0.2994	–0.0269	0.9301	0.9774
300	0.2903	–0.0107	0.9346	0.9787
350	0.2829	–0.0039	0.9381	0.9798
400	0.2767	+0.0018	0.9408	0.9807
450	0.2714	+0.0066	0.9431	0.9814
500	0.2668	+0.0108	0.9450	0.9820
750	0.2505	+0.0258	0.9513	0.9840
1000	0.2400	+0.0355	0.9548	0.9852
1 500	0.2268	+0.0477	0.9589	0.9865
2 500	0.2125	+0.0011	0.9626	0.9877
5 000	0.1966	+0.0762	0.9661	0.9888
10 000	0.1839	+0.0883	0.9683	0.9895
50 000	0.1638	+0.1078	0.9707	0.9903
100 000	0.1579	+0.1135	0.9712	0.9905
∞	0.1356	+0.1356	0.9720	0.9908

The last column of the table shows that the polarization degree of the emergent (rainbow) light is sensitively higher than the one of the incident light, a fact that we might have expected anyway. We also remark that, for a certain value of the parameter β (between 350 and 400), the sign of the Stokes' parameter $\xi_2^{(e)}$ changes, which implies the modification of the light's helicity. This happens when $\mathrm{tg}\delta = -\xi_2^{(i)}/\xi_1^{(i)}$. Generally, from (6.a), we notice that when $\mathrm{sgn}\,\xi_1^{(i)} = -\mathrm{sgn}\,\xi_2^{(i)}$ it is also possible that $\xi_1^{(e)}$ should modify its sign too; this happens when $\mathrm{tg}\delta = +\xi_1^{(i)}/\xi_2^{(i)}$. As for the sign of $\xi_3^{(e)}$, it always coincides with the sign of $\xi_3^{(i)}$.

Among many other possibilities, two interesting situations, with incident light partially polarized too, are presented in Table 4. The same polarization degree of the incident light, for example $P^{(i)} = 0{,}9354$, can be obtained in two "complementary" and artificial states of polarization: A) $\xi_1^{(i)} = 1/4$, $\xi_2^{(i)} = 1/2$, $\xi_3^{(i)} = 3/4$ and B) $\xi_1^{(i)} = 3/4$, $\xi_2^{(i)} = 1/2$, $\xi_3^{(i)} = 1/4$.

Table 4

Complementary partially polarized incident light (A and B)

(A) [$\xi_{(1)}^{(i)} = 1/4$; $\xi_{(2)}^{(i)} = 1/2$; $\xi_{(3)}^{(i)} = 3/4$]; **(B)** [$\xi_{(1)}^{(i)} = 3/4$; $\xi_{(2)}^{(i)} = 1/2$; $\xi_{(3)}^{(i)} = 1/4$]

β	$P^{(i)} = 0.9354$	$\xi_1^{(e)}$	$\xi_2^{(e)}$	$\xi_3^{(e)}$	$P^{(e)}$
50	A	0.2807	0.0180	0.9430	0.9840
	B	0.5227	–0.2558	0.7805	0.9736
75	A	0.2719	0.0282	0.9462	0.9849
	B	0.5198	–0.2296	0.7921	0.9748
100	A	0.2459	0.0336	0.9559	0.9876
	B	0.4854	–0.1957	0.8270	0.9787
125	A	0.2343	0.0386	0.9597	0.9888
	B	0.4721	–0.1754	0.8411	0.9803
150	A	0.2250	0.0426	0.9626	0.9895
	B	0.4611	–0.1590	0.8517	09815
175	A	0.2174	0.0459	0.9648	0.9901
	B	0.4520	–0.1454	0.8602	0.9825
200	A	0.2109	0.0486	0.9667	0.9906
	B	0.4441	–0.1339	0.8670	0.9833
250	A	0.2005	0.0530	0.9694	0.9914
	B	0.4313	–0.1152	0.8775	0.9846
300	A	0.1925	0.0564	0.9714	0.9919
	B	0.4212	–0.1005	0.8853	0.9855
350	A	0.1860	0.0592	0.9730	0.9924
	B	0.4129	–0.0885	0.8912	0.9862

Table 4 (continued)

β	$P^{(i)} = 0.9354$	$\xi_1^{(e)}$	$\xi_2^{(e)}$	$\xi_3^{(e)}$	$P^{(e)}$
400	A	0.1805	0.0615	0.9742	0.9927
	B	0.4059	–0.0785	0.8960	0.9868
450	A	0.1759	0.0634	0.9752	0.9930
	B	0.4000	–0.0699	0.8999	0.9872
500	A	0.1719	0.0651	0.9761	0.9932
	B	0.3948	–0.0625	0.9032	0.9876
750	A	0.1577	0.0713	0.9788	0.9940
	B	0.3761	–0.0357	0.9140	0.9890
1000	A	0.1486	0.0752	0.9804	0.9944
	B	0.3641	–0.0350	0.9201	0.9902
1 500	A	0.1372	0.0802	0.9822	0.9949
	B	0.3489	+0.0035	0.9271	0.9906
2 500	A	0.1249	0.0858	0.9838	0.9954
	B	0.3322	+0.0276	0.9337	0.9914
5 000	A	0.1111	0.0920	0.9853	0.9958
	B	0.3135	+0.0546	0.9398	0.9922
10 000	A	0.1002	0.0971	0.9863	0.9961
	B	0.2986	+0.0763	0.9436	0.9927
50 000	A	0.0829	0.1053	0.9874	0.9964
	B	0.2748	+0.1113	0.9479	0.9932
100 000	A	0.0779	0.1077	0.9876	0.9965
	B	0.2679	+0.1215	0.9488	0.9933
∞	A	0.0586	0.1172	0.9879	0.9966
	B	0.2414	+0.1609	0.9502	0.9935

From the last column of the table we see that, in both cases, the degree of polarization of the generated rainbow $P^{(e)}$ is with 4–5 percents, at least, higher than $P^{(i)}$. We also see that, in the case B), the parameter $\xi_2^{(e)}$ change the sign for a value of the adimensional parameter β beween 1000 and 1500. This happens where $\text{tg}\delta = -\xi_2^{(i)} / \xi_1^{(i)} = -2/3 = -0.66(6)$ *i.e.* for $\delta = -33.69°$.

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