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NONCOMMUTATIVITY VERSUS HIGHER-ORDER DERIVATIVE ACTIONS – A SHORT REVIEW

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Abstract. We describe how by integrating over the momenta in a phase-space path integral an arbitrary quantum mechanical systems with noncommuting coordinates can be transformed into one described by a constrained Lagrangian. The only correction with respect to the commutative case is provided by an universal term depending on velocities and accelerations, previously studied by Lukierski et al [1]. The case in which the system is in addition minimally coupled to a magnetic field is also briefly discussed. A discussion of the rather exceptional case of the harmonic oscillator, and a preliminary formulation of a potential-independent perturbation theory around the commutative case close this biased overview.

Key words: noncommutative mechanics, higher-order derivative actions, path integrals.

1. INTRODUCTION AND OUTLOOK

In the last years quantum mechanics with noncommuting coordinates attracted much attention [1]-[14]. In its simplest form noncommutative (NC) mechanics follows the structure of ordinary mechanics, but allows in addition for nonzero commutators among the coordinate operators.

In the NC classical version one similarly generalizes the symplectic structure, by allowing further nonvanishing Poisson brackets among coordinates.

The resulting equations of motion do not admit a standard Lagrangian formulation [14, 2].

Correspondingly, the NC quantum mechanical theory admits a first principles path integral formulation only in phase space [13]. No simple configuration space formulation is available for generic potentials. Nevertheless, one may search for an effective Lagrangian theory in configuration space, by integrating over the momenta in the phase space path integral.

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In view of the above remarks, the resulting effective Lagrangian can not be a standard one, depending only on coordinates and velocities. Given the nonlocalities introduced by noncommutativity, one may expect a quite involved function, perhaps nonlocal, with a form a priori depending on the potential function of the system under study. Strikingly enough, it turns out that the effective Lagrangian is just the usual one, plus an universal correction depending also on the particle accelerations. In (2+1) dimensions the additional term reads (for each particle)

$$\Delta L = -\frac{1}{2}\theta m^2 \left(v_1 a_2 - v_2 a_1 \right). \tag{1}$$

Above θ is the noncommutative scale, whereas m, v_i and a_j are the mass, velocity along i-axis, and acceleration along the j-axis, respectively, for a given particle. Subsequently, we will use the standard notation $v_i \equiv \dot{q}_i$, $a_j = \ddot{q}_j$.

The term (1) was previously studied in detail in [1] starting from different considerations, and in fact its appearance can be traced back to earlier developments (cf. [2, 4]). Lukierski et al. [1] added (1) to a free Lagrangian $\frac{m}{2}\vec{v}^2$ to provide a dynamical realization of a centrally extended (2+1)-dimensional Galilean algebra. Upon constrained quantization of the resulting higher order action (which circumvents the no-go theorem of [14]) noncommutative dynamics was shown to emerge for appropriate choices of canonical variables. The negative energy resulting from two "internal modes" posed no problem, since they were easily shown to decouple from the four relevant degrees of freedom. Interactions were subsequently introduced in a constrained way, in order to keep the ghosts harmless.

In this paper we go the opposite way; we start from arbitrary systems with Heisenberg noncommutativity of coordinates, and obtain the correction term via path integral methods. To our knowledge a direct derivation of a higher order Lagrangian from the extended Hamiltonian formalism was never presented before, though the inverse (Lagrangian to Hamiltonian) analysis of [1] indeed suggests (1) as an interesting possibility. It does not single it out however, the maximal order of the time derivatives appearing in the effective Lagrangian being not fixed apriori. A direct derivation is thus desirable; it is provided here within a 'Wilsonian' approach, We obtain the additional acceleration-dependent term of [1] up to coefficients, and such correction turns out (somehow surprisingly?) to be the only possibility available for NC systems of Heisenberg type and Hamiltonians of the form $H = \frac{1}{2m}\left(p_1^2 + p_2^2 \right) + V\left(q_1, q_2 \right)$. The path integral derivation gives a technical understanding of why higher order terms are forbidden in the Lagrangian. The price to be paid for the initial noncommutativity of the coordinates is the appearance of second order time derivatives in the effective action, and the ensuing

lack of appropriate boundary/initial conditions for the two (fortunately ghost-like) additional degrees of freedom. This last issue is not a serious problem, since the "internal modes" have to be eliminated anyway [1]. We stress that our derivation starts ab initio with arbitrary potentials $V(q_1, q_2)$, in contrast to the opposite route taken in Ref. [1], where one has to carefully pin down the (in the end NC) variables on which interactions must depend.

The agreement between the two approaches is remarkable, the more so as it is not a priori necessary. In fact, it does not extend to the situation in which a gauge field is minimally coupled [3], as will also be discussed briefly.

It might be interesting to try to extend the approach of [1] to nonconstant θ (it seems to be tied, at least technically, to constant θ). The path integral approach is in principle valid for any $\theta(p)$; in practice however one cannot perform the integration over the momenta explicitely.

Section 2 reviews a few facts about NC mechanics which will be put to use later. Section 3 derives the effective Lagrangian, by integrating out the momenta in the path integral. Section 4 discusses the peculiar case of the harmonic oscillator in this context, and makes some comments of wider applicability. An external arbitrary magnetic field is also coupled to the theory in two different manners, and the results compared with those of [3].

2. NC MECHANICS – A BRIEF REVIEW

A NC system is defined at the quantum level by its Hamiltonian and the following commutation relations

$$\left[\hat{q}^i\ \hat{q}^j\right] = i\theta^{ij} \qquad \left[\hat{q}^i, \hat{p}_j\right] = i\delta^i_j \qquad \left[\hat{p}_i\ \hat{p}_j\right] = iF_{ij}\,. \tag{2}$$

We will restrict here to the case of constant θ_{ij} (and F_{ij}, when non-zero), the so-called noncommutative mechanics – a restricted set of references is given at the end ([1] to [14]). For results concerning nonconstant θ one may consult for instance [15].

At the classical level one has instead the generalized Poisson brackets

$$\left\{q^i, q^j\right\} = \theta^{ij} \qquad \left\{q^i, p_j\right\} = \delta^i_j \qquad \left\{p_i, p_j\right\} = F_{ij}\,. \tag{3}$$

For simplicity in notation we will subsequently work in (2+1)-dimensions; the extension to higher dimensionalities is straightforward. We denote by x_a, $a = 1, 2, 3, 4$ the phase space coordinates, $x_{1,2,3,4} = q_1, q_2, p_1, p_2$. Since no risk of confusion exists, all indices are put down. Eqs. (3) can then be rewritten as $\{x_a, x_b\} = \Theta_{ab}$, where

$$\Theta = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & \sigma \\ 0 & -1 & -\sigma & 0 \end{pmatrix} \quad \text{i.e.} \quad \omega = \frac{1}{1-\theta\sigma} \begin{pmatrix} 0 & -\sigma & 1 & 0 \\ \sigma & 0 & 0 & 1 \\ -1 & 0 & 0 & -\theta \\ 0 & -1 & \theta & 0 \end{pmatrix}. \tag{4}$$

Above, $\Theta_{ab} = (\omega^{-1})_{ab}$, and ω is the symplectic form, which enters the action

$$S = \int dt \left(\frac{1}{2} \omega_{ab} x_a \dot{x}_b - H(x) \right). \tag{5}$$

Independent variation of S along each x_a provides the equations of motion

$$\dot{x}_a = \{ x_a, H \} = \Theta_{ab} \frac{\partial H}{\partial x_b}, \tag{6}$$

more explicitely $\dot{q}_i = \frac{\partial H}{\partial p_i} + \theta \varepsilon_{ij} \frac{\partial H}{\partial q_j}$, $\dot{p}_i = \frac{\partial H}{\partial q_i} + \sigma \varepsilon_{ij} \frac{\partial H}{\partial p_j}$, with i = 1, 2 and $\varepsilon_{12} = -\varepsilon_{21} = 1$. If θ = 0 and σ = 0, one recovers the usual Hamilton equations. For Hamiltonians of the form

$$H = \frac{1}{2} \left(p_1^2 + p_2^2 \right) + V(q_1, q_2), \tag{7}$$

the momenta are easily eliminated and the coordinate equations of motion are:

$$\ddot{q}_i = -(1 - \theta\sigma) \frac{\partial V}{\partial q_i} + \sigma \varepsilon_{ij} \dot{q}_j + \theta \varepsilon_{ij} \frac{d}{dt} \frac{\partial V}{\partial q_j}, \qquad i = 1, 2. \tag{8}$$

For $\theta \neq 0$, the equations (8) are not derivable from a Lagrangian [14], if the potential V is higher than quadratic in the coordinates.

Operatorial quantization is trivially implemented using Eqs (2,4):

$$\frac{d}{dt} \hat{x}_a = i [\hat{x}_a, H] = i [\hat{x}_a \hat{x}_b] \frac{\partial H}{\partial \hat{x}_b} = \Theta_{ab} \frac{\partial H}{\partial \hat{x}_b}. \tag{9}$$

The equations of motion (9) are an extension of the usual Heisenberg ones. They are the same as (6), with the coordinates becoming operators. A Schrödinger (wave function) formulation can easily be constructed (see e.g. [14]), once an appropriate basis is chosen in the Hilbert space on which the operators $\hat{x}_a$ act.

A path integral for systems with arbitrary Hamiltonian $H(x)$ obeying the commutation relations (2) is readily constructed [13]. Since equations (6) do not admit a Lagrangian formulation, one can at best hope for a phase-space path integral formulation of the quantum theory, which is provided by

$$Z = \int \prod_{k=1}^{4} Dx_k e^{iS} = \int \prod_{k=1}^{4} Dx_k e^{i\int dt\left(\frac{1}{2}\omega_{ij}x_i\dot{x}_j - H(x)\right)}. \tag{10}$$

The prescription is simple: if $\left[\hat{x}_i, \hat{x}_j\right] = i\Theta_{ij}$ then $Z = \int Dx e^{i\int_t \Theta_{ij}^{-1}\frac{x_i\dot{x}_j}{2} - H}$, and general: it applies to any Hamiltonian H. Integration of the momenta is particularly transparent in the above path integral, and the result – detailed in the next section – will be a simple and universal (the correction term is system independent) effective Lagrangian: the simplest one not excluded by the no-go argument in [14].

3. EFFECTIVE LAGRANGIAN

We path-integrate over the momenta in (10), to obtain the effective Lagrangian. Starting from the partition function

$$\int Dq_1 Dq_2 Dp_1 Dp_2 e^{iS}\ , \tag{11}$$

with action

$$S = \int dt\left[p_1\dot{q}_1 + p_2\dot{q}_2 - \frac{\theta}{2}\left(p_1\dot{p}_2 - p_2\dot{p}_1\right) - \frac{p_1^2}{2m} - \frac{p_2^2}{2m} - V\left(q\right)\right], \tag{12}$$

we wish to integrate over the momenta p_1, p_2. The potential part $V(q)$ depends only on q_1 and q_2 and plays no role in what follows (the method is valid for any $V(q)$, or Hamiltonian with separate quadratic dependence upon momenta). We divide the time interval T in n subintervals $\varepsilon = \frac{T}{n}$ ($n \to \infty$ achieves the continuum limit), and choose for simplicity the discrete derivative $\dot{x}_k \equiv \frac{x_{k+1} - x_k}{\varepsilon}$ (no issues requiring symmetric operations appear in the following)). The relevant part of the discretized action (excluding $V(q)$ for now) becomes

$$\tilde{S} = \sum_{k=0}^{n}\left[\varepsilon p_1^{(k)}v_1^{(k)} + p_2^{(k)}v_2^{(k)} + \frac{\theta}{2}\left(p_1^{(k)}p_2^{(k+1)} - p_2^{(k)}p_1^{(k+1)}\right) - \frac{\varepsilon}{2m}\left(p_1^{(k)}\right)^2 - \frac{\varepsilon}{2m}\left(p_2^{(k)}\right)^2\right]. \tag{13}$$

The clearest way to proceed with the coupled Gaussian integrals is to introduce matrix notation. Define the column vectors

$$\varepsilon\left(v_1^{(0)}, v_1^{(1)}, \ldots, v_1^{(n)} \ldots v_2^{(0)}, v_2^{(1)}, \ldots, v_2^{(n)} \ldots\right)^T, \tag{14}$$

$$P \equiv \left(p_1^{(0)}, p_1^{(1)}, \ldots p_1^{(n)} \ldots p_2^{(0)}, p_2^{(1)}, \ldots, p_2^{(n)} \ldots \right)^T , \tag{15}$$

and the matrix

$$J = -a \begin{pmatrix} 1 & 0 & 0 & . & . & 0 & b & 0 & . & . \\ 0 & 1 & 0 & . & . & 0 & 0 & b & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & -b & 0 & . & . & 1 & 0 & 0 & . & . \\ 0 & 0 & -b & . & . & 0 & 1 & 0 & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix},$$

where $a = \dfrac{\varepsilon}{2m}$, $b = \dfrac{m\theta}{\varepsilon}$. Its inverse J^{-1} has the same form as above, but with different entries a', b', namely $a' = 1/a$ and $b' = -b$ (the off diagonal part changes sign and the overall factor is reversed). In matrix notation the discrete action becomes

$$\tilde{S} = P^T V + P^T J P. \tag{16}$$

The coordinate transformation

$$\overline{P} \equiv P + \frac{1}{2} J^{-1} V \tag{17}$$

does not change the path integral measure $\left(D\overline{P} = DP \right)$, and leads to

$$\tilde{S} = \overline{P}^T J \overline{P} - \frac{1}{4} V^T J^{-1} V. \tag{18}$$

The first term is now integrated out – and no more dependency on momenta appears, whereas the second term leads to an exponent of the form (modulo a factor of i)

$$-\frac{1}{4} V^T J^{-1} V = \sum_{k=0}^{n} \left[\varepsilon \frac{m}{2} \left(v_1^{(k)} \right)^2 + \varepsilon \frac{m}{2} \left(v_2^{(k)} \right)^2 - \frac{\theta m^2}{2} \left(v_1^{(k)} v_2^{(k+1)} - v_2^{(k)} v_1^{(k+1)} \right) \right]. \tag{19}$$

Upon taking the continuum limit $\varepsilon \to 0$ our main result follows:

$$\int Dq_1 Dq_2 Dp_1 Dp_2 e^{iS} = N \int Dq_1 Dq_2 e^{i \int_0^T dt L_{eff}(q_i, v_i, a_i)} , \tag{20}$$

with

$$L_{eff} = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{\theta m^2}{2}\left(\dot{q}_1\ddot{q}_2 - \dot{q}_2\ddot{q}_1\right) - V\left(q_1, q_2\right) \tag{21}$$

and N a constant not depending on the q's. We have reintroduced the potential term. The second term is the correction due to noncommutativity, and has an universal character. Its relative simplicity is striking, and one is reconforted to find that exactly this Lagrangian was studied by Lukierski et al. [1] and shown to engender a NC structure (the potential V was contrained in their analysis to depend only on the "would-be" NC coordinates).

The equations of motion engendered by (21) are of third order in time derivatives,

$$\varepsilon_{ij}\theta m^2 \frac{d^3 q_j}{dt^3} + m\ddot{q}_i + \partial_{qi} V = 0\,. \tag{22}$$

No fourth-order time derivatives arise for q_1, q_2, and this leads to two constraints in the Hamiltonian formulation. Six constants (BC/IC) are still required, two more in comparison with the commutative case. We are not able to provide them, since we can at the very beginning start with only four constants (for instance the initial and final values of q_1 and p_2). This apparent indeterminacy is a consequence of the initial nonocommutativity of q_1 and q_2, but poses no serious problem, since the analysis in [1] shows that exactly those constants are needed for the two "internal" modes. Now these modes must be eliminated for consistency (an elementary analysis appears in [4], which we see no reason to duplicate).

In consequence we showed that all NC systems with Hamiltonians of the form $p^2 + V(q)$ can be treated in configuration space via higher-order Lagrangians. We went in opposite direction with respect to Lukierski et al., and showed that a large class of NC Hamiltonian systems are bound (if enforced into configuration space) by their higher order-Lagrangian.

4. HARMONIC OSCILLATOR AND θ *VERSUS* B

The harmonic oscillator case,

$$V\left(q_1, q_2\right) = \frac{m\omega^2}{2}\left(q_1^2 + q_2^2\right) \tag{23}$$

is special since the integral $\int DQ$ can be easily performed as well. This allows us to perform first either the coordinate path integral $\int DQ$ or the momentum path integral $\int DP$ and to compare the results.

The harmonic oscillator was also the only situation (except for constant forces) in which a Lagrangian formulation existed from the beginning [14].

Here it will allow us to compare in a familiar setting the two ways of introducing a magnetic field: via minimal coupling and via extra commutators.

As already argued, if perform first $\int DP$, the partition function for the harmonic oscillator becomes

$$Z_{h.o.} = N_1 \int Dq_1 Dq_2 e^{iL_1\left(q, \frac{dq}{dt}, \frac{d^2q}{dt^2}\right)}, \tag{24}$$

with the higher-order Lagrangian

$$L_1\left(q, \frac{dq}{dt}, \frac{d^2q}{dt^2}\right) = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{\theta m^2}{2}\left(\dot{q}_1\ddot{q}_2 - \dot{q}_2\ddot{q}_1\right) - \frac{m\omega^2}{2}\left(q_1^2 + q_2^2\right), \tag{25}$$

whereas $\int DQ$ is easily shown to lead to

$$Z_{h.o.} = N_2 \int Dp_1 Dp_2 e^{iL_2\left(p, \frac{dp}{dt}\right)}. \tag{26}$$

Using the substitution $p_i \equiv m\omega Q_i$ (dimensionally correct),

$$L_2\left(Q, \frac{dQ}{dt}\right) = \frac{m}{2}\left(\dot{Q}_1^2 + \dot{Q}_2^2\right) + \frac{\theta m^2 \omega^2}{2}\left(Q_1\dot{Q}_2 - Q_2\dot{Q}_1\right) - \frac{m\omega^2}{2}\left(Q_1^2 - Q_2^2\right) \tag{27}$$

which is nothing but the ordinary Lagrangian for a 2D harmonic oscillator put in an effective magnetic field $B_{eff} = \theta m^2\omega^2$. Adding a term kq_1q_2 to the original potential will just lead to an extra term $-km^2\omega^2\dot{Q}_1\dot{Q}_2$ in L_2 above.

However, if one introduces also $Bq_1\dot{q}_2$ (i.e introduce B via the symplectic form), one gets a higher-order Lagrangian in the momenta, $L_3\left(p, \dot{p}, \ddot{p}\right)$, by performing $\int DQ$!

If one introduces the magnetic field via minimal substitution, $p^2 \to (p\text{–A})^2$, then one gets a common Lagrangian $L_3\left(p, \dot{p}\right)$, hence clearly NC is responsible for $\ddot{p}$ in L_{eff}, mainly because then $|p_1, p_2>$ states do not exist.

In the (p–A) case our previous column vector V gets components $\varepsilon\left(v_i - A_i\right)$, and the net result is the previous one plus

$A_1\dot{A}_2 - A_2\dot{A}_1 + v_1\dot{A}_2 - \dot{v}_1A_2 + v_2\dot{A}_1 - \dot{v}_2A_1$.

It is not clear how to match it with [2].

In the harmonic oscillator case, one can further explore the consequences of introducing θ via

$$q_1 \to q_1 - \theta p_2, \quad q_2 \to q_2 - \theta p_1. \tag{28}$$

This generates exactly the same equations of motion, but one gets the usual effective Lagrangian (since we do not have true NC, $|q_1, q_2>$ states do exist). The path integral has in fact to take note whether $|q_1, q_2>$ states do exist or not. In the first case (usual commutation relations, but same equations of motion as if NC) one obtains the usual L, in the second case one gets $L(\ddot{q})$.

5. PERTURBATION THEORY

5.1. PERTURBATION THEORY VIA PATH INTEGRALS

Given the universality of the correction term, one may hope to develop a (possibly singular – for the effective equations of motion) perturbation theory independently of the system considered. This would provide an efficient procedure to evaluate the NC path integral. More precisely, one calculates the commutative path integral, and then the NC one follows through a procedure similar to the one in field theory.

Define

$$Z\left[j_1, j_2, k_1, k_2\right] = \int Dq_1 Dq_2 e^{i\int_0^T dt\left[L + j_1\dot{q}_1 + j_2\dot{q}_2 + k_1\ddot{q}_1 + k_2\ddot{q}_2\right]}. \tag{29}$$

Then

$$Z_{NC} = e^{i\int_0^T dt\left[\frac{\delta}{\delta j_1}\frac{\delta}{\delta k_2} - \frac{\delta}{\delta j_2}\frac{\delta}{\delta k_1}\right]} Z_C. \tag{30}$$

Hence it is enough to calculate the relatively mild extension (29) of the commutative path integral, and then one may apply the above exponential – eventually perturbatively, if necessary. Consider now evaluation of (29). By integration by parts, the source terms in the exponent can be rewritten as

$$\left[j_i q_i\right]_0^T + \left[k_i \dot{q}_i\right]_0^T - \int_0^T dt\left(\dot{j}_i q_i + \dot{k}_i \dot{q}_i\right). \tag{31}$$

The first two terms are boundary terms; the first one is zero if the coordinates are specified at the end-points, whereas the second one vanishes in case the velocities are fixed. Both vanish if one has periodic boundary conditions. The last two terms amount to driving forces (for the last term, one more integration by parts is necessary) In general, it is quite difficult – with the exception of the harmonic oscilator – to perform explicitely the path integral with sources, even if the initial one is under good control. Nevertheless, standard perturbative (once more) or numerical approaches can be used. Our main point was to show that a perturbation theory in θ could be developed, at least formally, around the commutative case,

that it involves only standard path integrals (Lagrangians depending on coordinates and velocities only), and that the formalism is system independent. It is however true that, if in practice one has to resort to perturbation theory also for the source terms, one will need to use in the last stage matrix elements (operator averages via path integrals) which now depend on the specific potential under study – though everything is done of course at the level of the commutative theory.

5.2 REMARKS ON THE CLASSICAL EQUATIONS OF MOTION

Let us come back to the equations of motion engendered by the effective Lagrangian (obtained via path integration over momenta)

$$\varepsilon_{ij}\theta m^2 \frac{d^3 q_j}{dt^3} + m\ddot{q}_i + \partial_{q_i} V = 0. \tag{32}$$

A priori, they do not need to be related to the equations of motion (8), obtained via elimination of classical momenta. In the first place, they are not of the same order. Nevertheless, a comparison reveals some intriguing relationship. For Hamiltonians of the form (7) the completely classical coordinate equations of motion (8) read, for $\sigma = 0$,

$$\ddot{q}_i = \frac{\partial V}{\partial q_i} + \theta\varepsilon_{ij} \frac{d}{dt}\frac{\partial V}{\partial q_j}, \qquad i = 1, 2. \tag{33}$$

By taking one more time derivative, we obtain

$$\varepsilon_{ij}\theta m^2 \frac{d^3 q_j}{dt^3} + m\ddot{q}_i + \partial_{q_i} V = \varepsilon_{ij} m\theta^2 \frac{d^2}{dt^2}\frac{\partial V}{\partial q_j}. \tag{34}$$

In the limit of small θ, in which θ^2 terms can be neglected, (32) and (34) are identical! The effective classical solutions appear to be given by a small θ limit of the purely classical solutions!

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