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GENERAL PHYSICS

THREE-DIMENSIONAL GINZBURG-LANDAU SOLITONS: COLLISION SCENARIOS

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Abstract. We give an overview of recent results obtained in the study of collisions between coaxial stable three-dimensional Ginzburg-Landau dissipative solitons. We present the generic outcomes of collisions between coaxial nonspinning (vorticityless) and spinning (vortex) solitons in the three-dimensional cubic-quintic complex Ginzburg-Landau equation. We identify the generic scenarios of collisions between both corotating and counter-rotating vortex solitons.

Key words: spatiotemporal optical solitons, vortex solitons, Ginzburg-Landau solitons, dissipative solitons.

1. INTRODUCTION

In the past years there has been an increasing interest in the theoretical and experimental study of multidimensional localized structures arising both in optics and in Bose-Einstein condensates (BEC) [1–11]. These localized (soliton-like) structures either nondissipative or dissipative [1] ones are quite complex physical objects, which are spatially confined on the order of wavelength (or even in the sub-wavelength scale). They represent the "particle-like" counterpart of the more common extended light structures.

However, the optical media that might sustain such confined self-guiding structures should be nonlinear, that is, their refractive index should be dependent on the light intensity (for typical Kerr-like nonlinear media the nonlinear shift of the refractive index is proportional to the light intensity). Different types of nonlinearities of optical materials such as absorptive, dispersive, second-order (or quadratic), third-order (or cubic, Kerr-like) can be used in practice to prevent either temporal dispersion or spatial diffraction of light beams or both of them. The field of both temporal and spatial optical solitons emerged from these fundamental studies of the interaction of intense laser beams with matter. However, there exist a third kind of optical solitons, which are spatially confined pulses of light, the so-called spatiotemporal optical solitons [4], alias “light bullets” [12]. These spatiotemporal optical solitons are nondiffracting and nondispersing wavepackets propagating in nonlinear optical media. The three-dimensional (3D) spatiotemporal optical solitons are localized (self-guided) in the two transverse (spatial) dimensions and in the direction of propagation due to the balance of anomalous group-velocity dispersion (GVD) of the medium in which they form and nonlinear self-phase modulation. Therefore, the "light bullet" is a fully three-dimensional localized object in both space and time. It is believed that the spatiotemporal optical solitons could be used as information carriers in future all-optical processing information systems [4].

Both two-dimensional (2D) and three-dimensional solitons in self-focusing cubic (Kerr-like) media are unstable because of the occurrence of collapse in the governing nonlinear Schrödinger model [13]. However, several possibilities to arrest this intrinsic collapse were considered, such as periodic alternation of self-focusing and defocusing layers [14] and various generalizations of this setting [15], and the use of weaker instabilities, *viz*}, saturable [16] or quadratic $\left(\chi^{(2)}\right)$ ones [17–20]. Tandem layered structures, composed of alternating linear and quadratic layers, were also proposed and investigated [21]. Other theoretically developed approaches use off-resonance two-level systems [22] and self-induced-transparency media [23].

The undesired collapse effect does not occur in cubic nonlinear media whose optical nonlinearity is nonlocal [24], therefore these media may also give rise to stable multidimensional optical solitons, see [25, 26]. Two-dimensional spatial optical solitons stabilized by the nonlocality were observed in vapors [27] and lead glasses featuring strong thermal nonlinearity [28]; in the latter case, elliptic and vortex-ring solitons were reported. One-dimensional optical solitons supported by a nonlocal $\left(\chi^{(3)}\right)$ nonlinearity were also created in liquid crystals[29]. Moreover, photonic lattices [30], vortices [31], spatial solitons in soft matter [33], multipole vector solitons in nonlocal nonlinear media [33], and one-dimensional solitons of even and odd parities supported by competing nonlocal nonlinearities [34] were considered in the context of nonlocality of the optical nonlinearity. The long-range cubic nonlinearity induced by long-range interactions between atoms carrying polarized magnetic momenta in an effectively 2D BEC also leads to the prediction of stable 2D solitons [35]. Moreover, 2D vortex solitons [31] and 3D fundamental (vorticityless) and spinning solitons [36] were considered in the context of nonlocality in various models in optics and BEC.

Localized optical vortices (alias vortex solitons), have drawn much attention as objects of fundamental interest, and also due to their potential applications to all optical information processing, as well as to the guiding and trapping of atoms

[37]. Unique properties are also featured by vortex clusters, such as rotation similar to the vortex motion in superfluids. The complex dynamics of vortex clusters in optical media with competing nonlinearities has been studied too [38]. Various complex patterns based on vortices were theoretically investigated in the usual BEC models, based on the Gross-Pitaevskii equation with the local nonlinearity [39].

Soliton necklaces [40–41] and rotating soliton clusters [42] were studied, too. Moreover, in nondissipative optical media with competing nonlinearities, robust soliton complexes (in the form of "clusters" or soliton "molecules") composed by several fundamental (nonspinning) solitons were thoroughly investigated [42]. The quasi-stable propagation of such robust soliton clusters is a generic feature of media with competing nonlinearities (self-focusing cubic and self-defocusing quintic nonlinearities or quadratic nonlinearities in competition with self-defocusing cubic nonlinearities) [42].

A class of both 2D and 3D spatially modulated vortex solitons, the so-called azimuthons, was introduced in Ref. [43]. Azimuthons represent intermediate states between the radially symmetric vortices and nonrotating multipole solitons.

Experimentally, only two-dimensional spatiotemporal optical solitons that overcome diffraction in one transverse spatial dimension have been created in quadratic nonlinear media [44]. With regard to theory, both fundamental (nontopological) and topological (vorticity-carrying) stable three-dimensional spatiotemporal optical solitons have been predicted, in media with competing optical nonlinearities (quadratic in competition with self-defocusing cubic or self-focusing cubic in competition with self-defocusing quintic) [45].

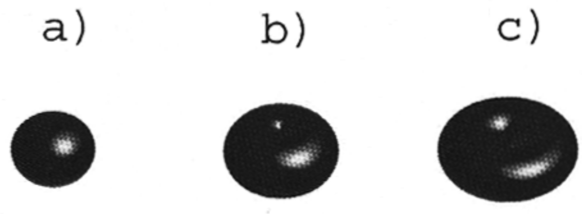

Fig. 1 – Isosurface plots of soliton local intensity,showing the stable input 3D Ginzburg-Landau solitons: a) S=0; b) S=1; c) S=2.

Recently, it was shown the existence of stable three-dimensional spatiotemporal optical solitons confined by either harmonic two-dimensional optical lattices [46] or radially symmetric Bessel lattices [47]. Thus it was predicted the existence of stable three-dimensional spatiotemporal solitons in a two-dimensional photonic lattice and it was found that the Hamiltonian (H)-versus-soliton norm (N) diagram exhibits a generic two-cusp structure. Correspondingly, a "swallowtail" shape of the H-N diagram emerged, which is a quite rare physical phenomenon [46]. This unique feature is a generic one for both nontopological (nonspinning) [46–49] and topological (spinning) [50] 3D solitons. This unique property has been also found both in the case of radially symmetric Bessel lattices [47], which is a result suggesting a promising approach to generate stable "light bullets" in optics and stable three-dimensional solitons in attractive Bose-Einstein condensates [48], and in the search for stable light bullets in media with quadratic nonlinearities in competition with self-focusing cubic nonlinearities [49].

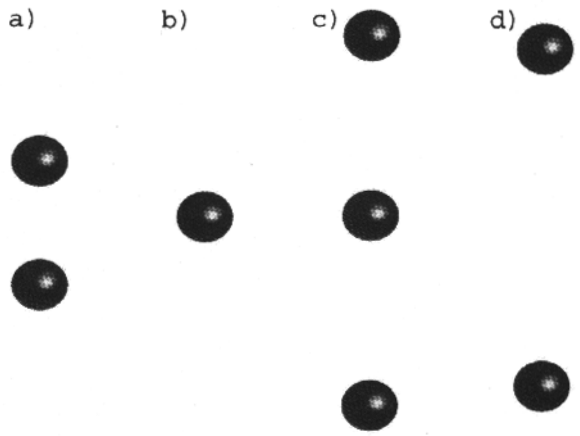

Fig. 2 – Isosurface plots of soliton local intensity, showing the initial and final sets of zero-vorticity (S=0) solitons involved in the collision, at different values of kick χ : a) input (at z = 0); b) merger into a single soliton, for $\chi = 1$ (at z = 120); c) creation of an extra soliton, for $\chi = 2$ (at z = 30); d) quasielastic collision for $\chi = 4$ (at z = 15). The simulations were run on the grid of size 160 × 160 × 601.

In the past years there was an increasing interest in the study of multidimensional dissipative localized structures (dissipative solitons). These unique physical objects are modeled by nonlinear partial differential equations involving gain and loss terms in addition to the common nonlinear and dispersive/diffractive terms. These nonlinear dynamical systems allow for the formation under certain conditions of *stable dissipative solitons* [1]. One of the prototype dissipative dynamical system is that governed by the complex Ginzburg-Landau equation, which is one of the most studied nonlinear partial differential equation in nonlinear science [51]. Recently stable fundamental (vorticityless) and spinning (with nonzero intrinsic vorticity) spatiotemporal dissipative optical solitons described by the complex cubic-quintic Ginzburg-Landau equation were found [52–60] and both elastic and inelastic collision scenarios were identified 61–64].

In this work we briefly overview some recent theoretical studies of ollisions between coaxial stable three-dimensional Ginzburg-Landau dissipative solitons. We present the generic outcomes of collisions between both nonspinning and spinning solitons in the three-dimensional cubic-quintic complex Ginzburg-Landau equation. Moreover, the collisions between both corotating and counter-rotating coaxial vortex solitons are briefly described.

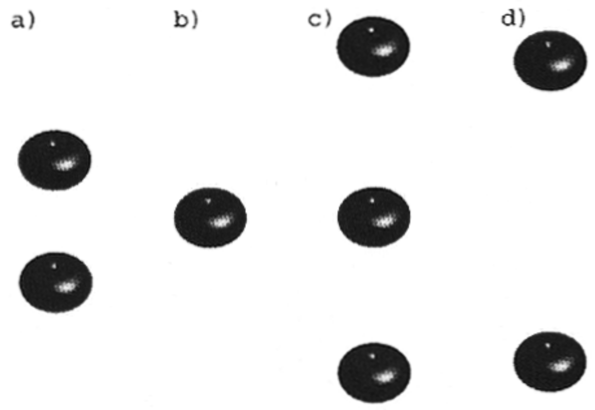

Fig. 3 – The same as in Fig. 2, but for the vortex solitons with S=1: a) input at z = 0; b) $\chi = 1$ (at z = 100); c) $\chi = 2$ (at z = 30); d) $\chi = 4$ (at z = 15). The grid size is 193 × 193 × 601.

COAXIAL GINZBURG-LANDAU SOLITON COLLISIONS

We investigate soliton collisions in the framework of the following cubic-quintic complex Ginzburg-Landau equation in three dimensions:

$$\begin{aligned} &\mathrm{i}U_z + \left(\frac{1}{2} - \mathrm{i}\beta\right)\left(U_{xx} + U_{yy}\right) + \left(\frac{D}{2} - \mathrm{i}\gamma\right)U_{tt} + \\ &+\left[\mathrm{i}\delta + (1-\mathrm{i}\varepsilon)|U|^2 - (\nu - \mathrm{i}\mu)|U|^4\right]U = 0. \end{aligned} \tag{1}$$

In terms of nonlinear optics, U is the local amplitude of the electromagnetic wave in the bulk medium which propagates along axis z, the transverse coordinates are x and y, while the temporal variable is $t = T - z/V_0$, where T is time and V_0 the group velocity of the carrier wave. The coefficients which are scaled to be 1/2 and 1 account, respectively, for diffraction in the transverse plane and the self-focusing Kerr nonlinearity. Here $\beta \geq 0$ is the effective diffusivity in the transverse plane, real constants δ, ε and μ represent, respectively, the linear loss, cubic gain, and quintic loss. The parameter $\nu \geq 0$ accounts for the self-defocusing quintic nonlinearity, that may compete with the cubic term, D is the GVD coefficient [D>0 (D<0) corresponds to the anomalous/normal GVD], and $\gamma \geq 0$ accounts for the dispersion of the linear loss.

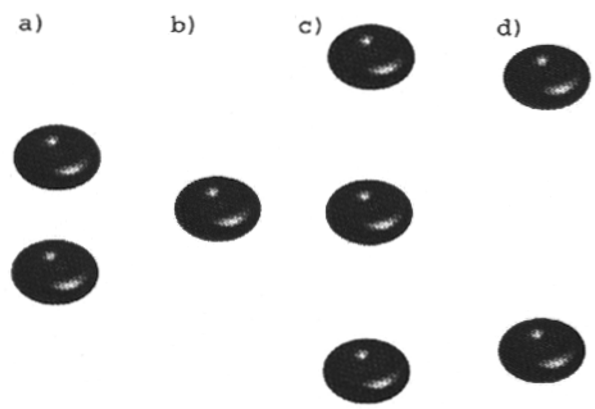

Fig. 4 – The same as in Fig. 2, but for the vortex solitons with S=2: a) input at $z = 0$; (b) $\chi = 0{,}5$ (at $z = 120$); c) $\chi = 2$ (at $z = 32$); d) $\chi = 4$ (at z=16). In this case, the largest grid was used, of size $201 \times 201 \times 601$.

The Ginzburg-Landau equation is a ubiquitous model in many physical problems [51], and in different forms it appears as the simplest model for describing *dissipative solitons* [1], clusters of localized states rotating around a central vortex core [65], and laser patterns in cavities [66].

Next we will consider the cubic-quintic Ginzburg-Landau model governed by Eq. (1) with *zero spectral filtering parameter* $(\gamma = 0)$. This model admits free motion of solitons along axis z, and thus makes collisions between them possible [62]. Notice that the free motion in plane (x, y) is impeded by the diffusivity term in Eq. (1), which contains the parameter $\beta > 0$. As shown in Refs. [53, 54], this term is necessary for the stability of dissipative vortex solitons, while fundamental solitons, with S=0, may be stable at $\beta = 0$.

The stationary soliton solutions to Eq. (1) are given by

$$U(z, x, y, t) = \psi(r, t)\exp(\mathrm{i}kz + \mathrm{i}S\theta), \tag{2}$$

where r and θ are the polar coordinates in plane (x, y, the integer number $S \geq 0$ is the soliton vorticity (the fundamental solitons correspond to $S = 0$), k is the soliton nonlinear wave number, and the complex function $\psi(r, t)$ obeys the stationary equation,

$$\begin{aligned}&\left(\frac{1}{2} - \mathrm{i}\beta\right)\left(\psi_{rr} + \frac{1}{r}\psi_r - \frac{S^2}{r^2}\psi\right) + \frac{1}{2}D\psi_{tt} + \\ &+\left[\mathrm{i}\delta + (1 - \mathrm{i}\varepsilon)|\psi|^2 - (\nu - \mathrm{i}\mu)|\psi|^4\right]\psi = k\psi.\end{aligned} \tag{3}$$

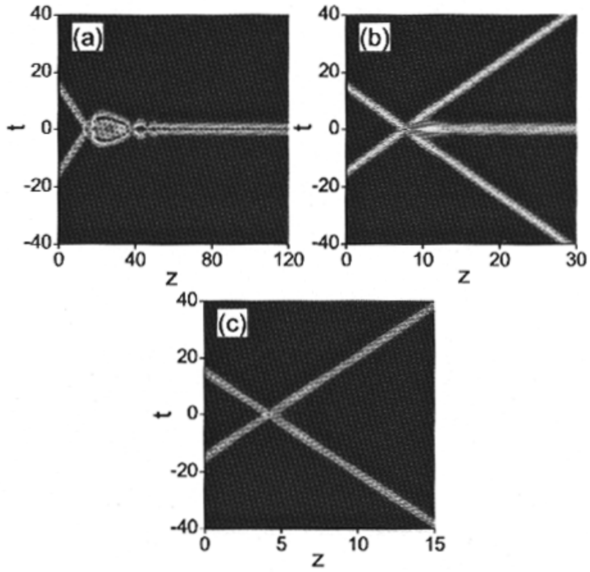

Fig. 5 – Contour plots display the evolution of field $|U|$ in plane (t,z), for three collision scenarios at different values of kick parameter χ for the solitons with S=0: a) merger into a single soliton, at $\chi = 1$; b) creation of an extra soliton, at $\chi = 2$; c) quasielastic collision, at $\chi = 4$.

In the above equations we set the filtering parameter $\gamma = 0$, as said above. Localized solutions to this equation must decay exponentially at r, $|t| \to \infty$, and must behave as r^S at $r \to 0$. We will consider the following set of parameters: $D = 1$ (anomalous GVD), $\mu = 1, \nu = 0.1, \delta = 0.4$, and $\beta = 0.5$. We will consider collisions between stable stationary dissipative solitons with vorticities $S = 0$, $S = 1$, and $S = 2$ [62].

In order to study generic outcomes of collisions between Ginzburg-Landau solitons, one should take a pair of stable three-dimensional solitons, separated by a large temporal distance T. The solitons are set in motion by "kicking" them in the axial direction, i.e., multiplying each soliton by $\exp(\pm i\chi t)$, where χ is the corresponding "kick" parameter. Next we will fix the cubic gain $\varepsilon = 2.3$ and we vary the kick parameter χ . For the above set of parameters the three-dimensional solitons with $S = 0, 1, 2$ are all stable [53, 54], being characterized by the following values of the soliton energy (soliton norm),

$$E \equiv 2\pi \int_0^\infty r\mathrm{d}r \int_{-\infty}^{+\infty} \mathrm{d}t \left|\psi(r,t)\right|^2 , \tag{4}$$

$$E(S=0) \approx 52, \quad E(S=1) \approx 171, \quad E(S=2) \approx 310. \tag{5}$$

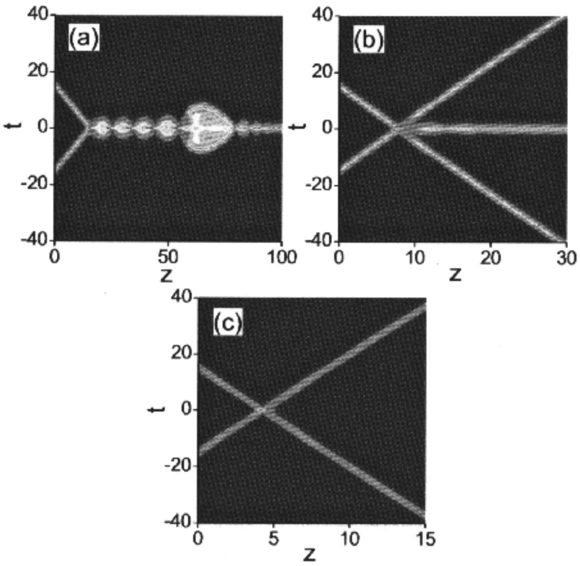

Fig. 6 – The same as in Fig. 5 but for vortex solitons with S=1: a) $\chi = 1$; b) $\chi = 2$; c) $\chi = 4$.

In Figs. 2–4 we show the isosurface plots of soliton local intensity, for both the initial and final sets of solitons involved in collision, at different values of the kick parameter χ. Gradually increasing initial transverse collision momentum χ, we have observed the following outcomes (the initial separation was typically T = 30, but variation of T did not affect the numerical results):

(a) *Merger* of the two solitons into one, at small values of χ, namely, in intervals $\chi \le 1.1$ for $S = 0$, $\chi \le 1.2$ for $S = 1$, and $\chi \le 1.4$ for $S = 2$.

(b) Generation of an *extra soliton*, at intermediate values of χ, namely, in intervals $1.1 < \chi \le 2.2$ for $S = 0$, $1.2 < \chi \le 2.4$ for $S = 1$, and $1.4 < \chi \le 2.4$ for $S = 2$.

(c) *Quasielastic* interactions at larger χ, i.e., $\chi > 2.2$ for $S = 0$, $\chi > 2.4$ for $S = 1$, and $\chi > 2.4$ for $S = 2$. In this case, the solitons pass through each other, and after the collision they feature velocities slightly smaller than they had originally.

These three collision scenarios are further illustrated in Figs. 5–7 by pictures of the evolution of the field in the plane of (t, z). Note that the merger of the two colliding solitons into a single one [see also Ref. [61] for examples of the interaction of two nonspinning light bullets with zero transverse velocities, when they fuse if the initial separation (in time) is small], a promising effect for potential applications has been reported before in other physical settings: (*i*) the study of solitons in saturable materials with a linear and quadratic intensity depending refraction index change [67], and (*ii*) the study of dynamics and collisions of moving solitons in Bragg gratings with dispersive reflectivity [68].

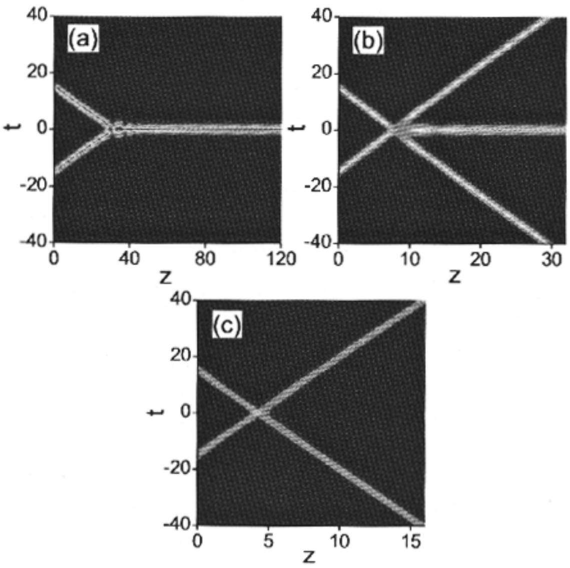

Fig. 7 – The same as in Fig. 5 but for vortex solitons with S=2: a) $\chi = 0,5$; b) $\chi = 2$; c) $\chi = 4$.

The transformation of two colliding solitons into three, one quiescent and two moving has also been reported in collisions of solitons in media with saturable nonlinearity [67], collisions of two one-dimensional dissipative spatial solitons in periodically patterned semiconductor amplifiers [69], and collisions of moving solitons in Bragg gratings with dispersive reflectivity [68]. Thus our results show that the soliton creation is a generic feature of collisions of both fundamental ($S = 0$) and spinning $(S \neq 0)$ three-dimensional solitons described by the complex Ginzburg-Landau equation.

3. COUNTER/ROTATING COAXIAL GINZBURG-LANDAU SOLITON COLLISIONS

Next we present a natural extension of the analysis performed in the preceding section to the case of collisions between "counter-rotating" vortex solitons, i.e., ones with opposite vorticities, $S_1 = -S_2 = 1$ and 2.

Thus, to study the collisions between counter-rotating three-dimensional Ginzburg-Landau solitons, we started, at $z = 0$, with a pair of stable vortex solitons in the form of $\psi(r,\ t + T/2)\exp(\mathrm{i}S\theta)$ and $\psi(r,\ t + T/2)\exp(-\mathrm{i}S\theta)$, with $S = 1$ or 2, which are separated by a large initial temporal distance, $\Delta t = T$. In most cases, we took $T = 30$, but varying the initial separation did not affect outcomes of the collisions.

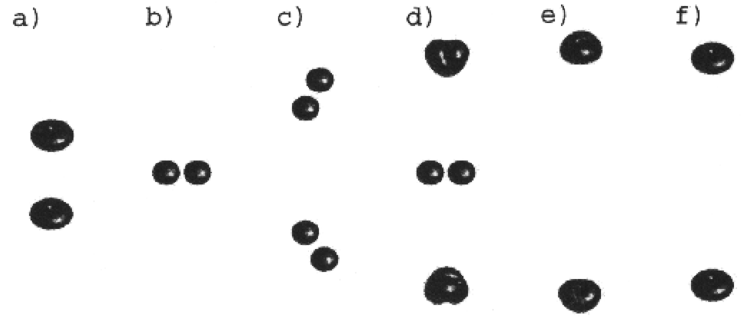

Fig. 8 – Generic outcomes of collisions between 3D solitary vortices with $(S_1, S_2) = (+1, -1)$ are shown by means of isosurface plots of local intensity $|U(x,y,t)|^2$, for different values of kick χ. a) The input configuration (at z=0); the localized vortices move, towards their collision, along their common axis, i.e., in the positive and negative vertical directions. b) A single nonrotating dipole cluster composed of two fundamental solitons, which is the outcome for $\chi = 1$ (shown at z=100). c) Two counter-rotating dipole clusters, for $\chi = 1.5$ (at z=170); both dipoles lie in planes $t = \pm 30$ oriented perpendicular to the original axis. d) Two counter-rotating double-humped "unfinished vortices", plus a single dipole cluster, for $\chi = 2$ (at z=34). e) Two counter-rotating "unfinished vortices", without the additional cluster, for $\chi = 2.4$ (at z=27). f) A quasielastic collision, for $\chi = 4$ (shown at z=15).

The vortex solitons are set in motion by kicking them in the opposite directions along the common axis, i.e., multiplying each one by $(\pm \mathrm{i}\chi t)$. Thus, the full initial configuration $U(0, x, y, t)$ was

$$\begin{aligned} U(0,x,y,t) &= \psi(r,t+T/2)\exp(\mathrm{i}S\theta + \mathrm{i}\chi t) + \\ &+ \psi(r,t-T/2)\exp(-\mathrm{i}S\theta - \mathrm{i}\chi t). \end{aligned} \tag{6}$$

Because, at $\gamma = 0$, the governing three-dimensional Ginzburg-Landau equation (1) is Galilean-invariant in the longitudinal (i.e., axial) direction, the application of the kick to an isolated quiescent soliton, $U_0(z,t,x,y)$, generates an exact solution in the form of a "walking soliton", see, e.g. [9]:

$$U_\chi(z,x,y,t) = U_0(z,x,y,t \mp a\chi)\exp\left(\pm \mathrm{i}\chi t - \mathrm{i}a\chi^2 z/2\right). \tag{7}$$

At small values of the transverse collision momentum χ, slow collisions are *inelastic*, leading to merger of the vortices into one or two clusters of fundamental solitons (dipoles or quadrupoles in the cases of $S = \pm 1$ and $S = \pm 2$, respectively). In the case when two clusters are generated by the collision, they feature decelerating rotation in opposite directions. With the increase of the kick parameter

χ, two "unfinished vortices" (counter-rotating multi-humped objects without a through hole in the center) emerge from the collision; at intermediate values of χ, they appear along with a cluster of fundamental solitons. As a matter of fact, the "unfinished vortices" replace the original solitary vortices. Finally, the collision becomes *elastic* at large values of transverse collision momentum χ. It is worthy to notice that the dipolar and quadrupolar clusters feature very slow expansion, being robust against strong perturbations. On the contrary, the "unfinished vortices" eventually split into dipolar pairs of fundamental solitons (see below).

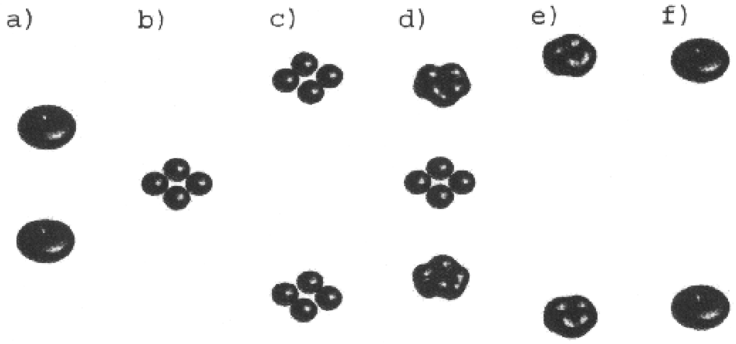

Fig. 9 – The same as in Fig. 8, but for the collision of solitary vortices with (S_1, S_2) = = (+2, –2). a) The input at $z = 0$ (as well as in Fig. 8, the localized vortices move, towards their collision, in the vertical directions). b) A single nonrotating quadrupole cluster composed of four fundamental solitons, which is the outcome of the collisions for $\chi = 0.2$ (shown at $z = 120$). c) Two counter-rotating quadrupole clusters, for $\chi = 0.5$ (at $z = 120$); the clusters lie in planes $t = \pm 33.4$ oriented perpendicular to the original axis. d) Two counter-rotating four-humped "unfinished vortices" plus a single quadrupole cluster, for $\chi = 1$ (at $z = 120$). e) Two counter-rotating "unfinished vortices", without the additional cluster, for $\chi = 2.5$ (at $z = 25$). (f) A quasielastic collision, for $\chi = 4$ (shown at z=16).

The five collision scenarios identified above (see Ref. [63] are illustrated by means of isosurface plots of local intensity $\left|U\left(x,y,t\right)\right|^2$, for different values of kick χ (Figs. 8 and 9). The typical splitting of "unfinished vortices" into dipolar pairs of fundamental solitons is shown in Fig. 10 by plotting the evolution of the total soliton energy; it converges to the value $E \approx 104$, which is the total energy of the two identical fundamental (spinless) solitons.

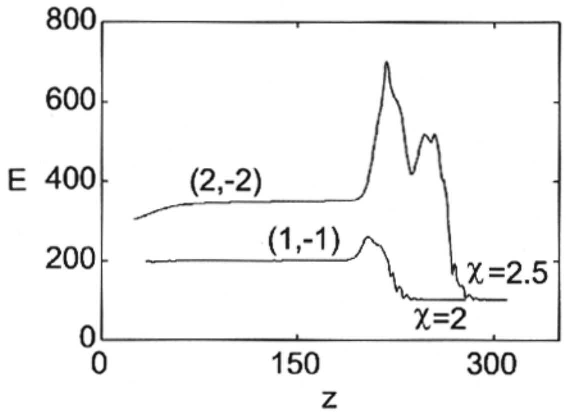

Fig. 10 – Evolution of the total soliton energy in the case of the splitting process of "unfinished vortices" into pairs of fundamental solitons at $\chi = 2$ (for (1, –1) solitons) and at $\chi = 2.5$ (for (2, –2) solitons).

4. CONCLUSIONS

In this work we overviewed some recent results concerning collisions between coaxial stable three-dimensional Ginzburg-Landau dissipative solitons. We presented the generic outcomes of collisions between coaxial fundamental (vorticityless) and higher-order (vortex) solitons in the three-dimensional cubic-quintic complex Ginzburg-Landau equation. We identified several generic scenarios of collisions between both corotating and counter-rotating vortex solitons. However, in the present paper we briefly overviewed recent results obtained for relatively simple coaxial configurations. A challenging problem is the analysis of three-dimensional soliton collisions in a more general geometrical setting.

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