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PARTICLE IN A BOX BY MEANS OF A FRACTAL HYDRODYNAMIC MODEL

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Abstract. In the frame of a fractal hydrodynamic model (the hydrodynamic model of the scale relativity theory in fractal dimension $D_F = 2$), the static system (particle in a box) is analyzed. The static system of fractal type is coherent (the speed is null at differentiable scale) and the observable in the form of the energy is generated by the fractal speed, through the fractal potential.

Key words: scale relativity theory, fractal model, static system.

1. INTRODUCTION

The scale relativity (SR) model provides a new insight into the origin of the fundamental laws in physics [1-8]. This model is based both on the fractal space-time concept (also Ord [9] and El Naschie [10] introduced and "operated" with the fractal space-time) and on a generalization of Einstein's principle of relativity to scale transformations. Three scales of interaction of SR were developed: (i) A 'Galileian' version [1-8, 12-16]; (ii) A special scale-relativistic version [1, 2, 17]; (iii) A 'general scale-relativistic' version [1, 2, 18].

Recently, using the SR model, for $D_F = 2$ and the dissipative approximation of motion, the fractal model of atom was build, while for $D_F = 3$ and the dispersive approximation of motion, some properties of the matter were explained (the anomaly of nano-fluids thermal conductivity, the superconductivity etc.). For any of the analyzed fractal dimensions, both the differential (classical and deterministic) component and the fractal (stochastic) component of the complex speed field have been proven to be essential. For example, in the fractal model of atom ($D_F = 2$), the real part (classical and deterministic) of the complex speed field describes the electron motion on stationary orbits, while the imaginary stochastic one, the electron energy quantification [16].

In the present paper, more implications of the fractal hydrodynamic model given in [16] for fractal dimension $D_F = 2$ are analyzed. Thus, the physical particularities of the fractal hydrodynamic model and the mathematical procedure for solving the fractal hydrodynamic differential equations are discussed in Paragraph 2. In this connection, static system is considered: the particle in a box (Paragraph 3).

2. FRACTAL HYDRODYNAMIC MODEL IN FRACTAL DIMENSION

$D_F = 2$

You Let us suppose that the motion of the particles takes place on continuous but non-differentiable curves, *i.e.* on fractals [11]. The "non-differentiability" in the topological dimension $D_F = 2$ implies the replacement of the standard time derivative $\mathrm{d}/\mathrm{d}t$ by a new complex operator $\hat{\partial}/\partial t$ [1-8, 12-14, 16]

$$\frac{\hat{\partial}}{\partial t} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla - \mathrm{i} D \Delta , \tag{1}$$

where $\boldsymbol{V}$ is the complex speed field, and D is the Nottale's fractal – non-fractal coefficient. Consequently, the covariant form of the first Newton's principle in the fractal space-time is reduced to equation

$$\frac{\hat{\partial} \mathrm{V}}{\partial t} = \frac{\partial \mathrm{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} - \mathrm{i} D \Delta \boldsymbol{V} = 0 . \tag{2}$$

If the fractal fluid is irrotational, $\boldsymbol{\Omega} = \nabla \times \mathrm{V} = 0$, we can choose $\boldsymbol{V}$ of the form [1-8, 12-14],

$$\boldsymbol{V} = -2\mathrm{i} D \nabla \ln \psi , \tag{3}$$

with ψ a scalar complex function. Then, equation (1) becomes a Navier-Stokes type equation,

$$\frac{\hat{\partial} \boldsymbol{V}}{\partial t} = \frac{\partial \boldsymbol{V}}{\partial t} + \nabla \left(\frac{\mathrm{V}^2}{2} \right) - \mathrm{i} D \Delta \boldsymbol{V} = 0 , \tag{4}$$

with an imaginary viscosity coefficient, ν

$$\nu = \mathrm{i} D , \tag{5}$$

while in terms of the ψ function, a “Schrödinger” type equation results

$$D^2 \Delta\psi + \mathrm{i}D \frac{\partial \psi}{\partial t} = 0 . \tag{6}$$

Let us choose the scalar function ψ in the form $\psi = \sqrt{\rho}\mathrm{e}^{\mathrm{i}S}$, with $\sqrt{\rho}$ an amplitude and S a phase. Thus, the complex speed field (3),

$$\boldsymbol{V} = \boldsymbol{v} - \mathrm{i}\boldsymbol{u} , \tag{7}$$

has the components

$$v = 2D\nabla S, \qquad u = D\nabla \ln \rho . \tag{8 a, b}$$

Introducing (7) and (8a,b) in (4) and separating the real and imaginary parts, we obtain

$$\begin{gathered} \frac{\partial \boldsymbol{v}}{\partial t} + \nabla\left(\frac{\boldsymbol{v}^2}{2} - \frac{\boldsymbol{u}^2}{2} - D\nabla \cdot \boldsymbol{u}\right) = 0, \\ \frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\boldsymbol{v} \cdot \boldsymbol{u} + D\nabla \cdot \boldsymbol{v}\right) = 0 \end{gathered} \tag{9 a, b}$$

or, up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of ψ ,

$$\begin{gathered} m_0\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\right) = -\nabla\left(Q\right) \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \end{gathered} , \tag{10a, b}$$

with Q the fractal potential,

$$Q = -2m_0 D^2 \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = -\frac{m_0 \boldsymbol{u}^2}{2} - m_0 D\nabla \cdot \boldsymbol{u} . \tag{11}$$

The fractal potential depends only on the imaginary part $\boldsymbol{u}$ of the complex speed field, $\boldsymbol{V}$, and it comes from the non-differentiability of the fractal space-time. Equation (10a), *i.e.* the momentum conservation law, and equation (10b), *i.e.* the probability density conservation law, form the fractal hydrodynamic model.

Two types of stationary states are to be distinguished:

i) Dynamic states. For $\partial / \partial t = 0$ and $\boldsymbol{v} \neq 0$, equations (10a,b) give

$$\nabla\left(\frac{m_0\boldsymbol{v}^2}{2}-\frac{m_0\boldsymbol{u}^2}{2}-m_0D\nabla\cdot\boldsymbol{u}\right)=0$$
$$\nabla\cdot(\rho\boldsymbol{v})=0 \tag{12a, b}$$

namely,

$$\frac{m_0\boldsymbol{v}^2}{2}-\frac{m_0\boldsymbol{u}^2}{2}-m_0D\nabla\cdot\boldsymbol{u}=E$$
$$\rho\boldsymbol{v}=\nabla\times\boldsymbol{F}. \tag{13a, b}$$

Consequently, the non-fractal inertia, $m_0\boldsymbol{v}\cdot\nabla\boldsymbol{v}$, and the fractal force, $-\nabla Q$, are in balance at every field point, equation (12a). The sum of the non-fractal kinetic energy, $m\boldsymbol{v}^2/2$, and fractal potential, Q, is invariant, *i.e.*, equal to the integration constant $E\neq E(\mathrm{r})$ – equation (13a). $E\equiv\ <E>$ represents the total energy of the dynamic system. The probability flow density ρv has no sources - equation (12b), *i.e.*, its streamlines are closed – equation (16b).

ii) Static states. For $\partial/\partial t = 0$ and $\boldsymbol{v}=0$, equations (10a,b) give

$$\nabla\left(-\frac{m_0\boldsymbol{u}^2}{2}-m_0D\nabla\cdot\boldsymbol{u}\right)=0\,, \tag{14a}$$

i.e.

$$-\frac{m_0\boldsymbol{u}^2}{2}-m_0D\nabla\cdot\boldsymbol{u}=E\,. \tag{14b}$$

Thus, the fractal force, $-\nabla Q$ has the zero value – equation (14a). The fractal potential, Q, is invariant, *i.e.* equal to the integration constant $E\neq E(\boldsymbol{r})$ – equation (14b). $E\equiv\ <E>$ represents the total energy of the static system. Equation (10b) is identically satisfied. As an illustration of the fractal hydrodynamic formalism, static fractal system is further analyzed.

3. PARTICLE IN A BOX IN A FRACTAL SPACE-TIME

One of the simplest fractal systems is a particle enclosed in a one-dimensional box of infinite potential walls at $x=0$ and $x=a$

$$U(x)=0,\ \ 0<x<a\ ;\quad U(x\leq 0)=\infty;\quad U(x\geq a)=\infty\,. \tag{15a-d}$$

In the stationary state

$$\rho(x)\boldsymbol{v}(x)=\mathrm{c}=\mathrm{const.}, \tag{16}$$

by equation (12b), where either of the boundary conditions

$$\rho(0)=0 \text{ or } \rho(a)=0, \tag{17a,b}$$

specifies that $c=0$. Hence

$$\boldsymbol{v}(x)=0. \tag{18}$$

Accordingly, equation (14b) applies, which leads (under consideration of equations (15a-d)) to the linear boundary value problem,

$$\frac{\mathrm{d}^2\sqrt{\rho}}{\mathrm{d}x^2}+\frac{E}{2m_oD^2}\sqrt{\rho}=0,\ 0<x<a \quad , \quad \rho(0)=\rho(a)=0. \tag{19a-c}$$

Usually [19, 20, 21], a solution $\rho=\rho_n$ exists if the total energy of the system assumes certain eigenvalues $E=E_n$. One finds

$$\rho_n=\frac{2}{a}\sin^2\left(\frac{n\pi}{a}x\right) \tag{20}$$

and

$$E_n=2m_0D^2\left(\frac{n\pi}{a}\right)^2, \quad n=1,2.... \tag{21}$$

According with the fractal space-time theory [1-8, 16], equations (18) and (20) specify the followings:

i) at the differentiable scale the speed field, $\boldsymbol{v}$, is zero – see equation (18). According to equation (8a), the fractal fluid is coherent, *i.e.* the fluid particles have the same phase;

ii) at the non-differentiable scale, the speed field $\boldsymbol{u}$ is nonzero, and has the expression (according with equation (8b)),

$$u_x=D\frac{d\ln\rho}{dx}=2D\frac{n\pi}{a}\mathrm{ctg}\left(\frac{n\pi}{a}x\right). \tag{22}$$

Then, the fractal one-dimensional potential (11) takes the form

$$\begin{aligned}Q_n&=-\frac{m_0u_x^2}{2}-m_0D\frac{\mathrm{d}u_x}{\mathrm{d}x}=\\&=-2m_0D^2\left(\frac{n\pi}{a}\right)^2\mathrm{ctg}^2\left(\frac{n\pi}{a}x\right)+2m_0D^2\left(\frac{n\pi}{a}\right)^2\frac{1}{\sin^2\left(\frac{n\pi}{a}x\right)}=\\&=2m_0D^2\left(\frac{n\pi}{a}\right)^2.\end{aligned} \tag{23}$$

Consequently, the fractal system is coherent, *i.e.* the speed is null at differentiable scale, while at the non-differentiable scale, the observable in the form of energy is generated by the fractal speed, through the fractal potential.

4. CONCLUSIONS

Using a fractal hydrodynamic model in the fractal dimension $D_F = 2$, the static systems of potential box type is given. Thus:

i) The momentum transfer through the differential component is null. Since the fractal fluid particles have the same phase, the fluid is coherent;

ii) The momentum transfer is achieved only through the fractal component. Moreover, the same component generates also the observable quantity in the energy form.

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