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Dedicated to Professor Ioan Gottlieb's 80^{th} anniversary

THERMODYNAMICS OF BOSONS IN AN UNIVERSE WITH GLOBAL PATHOLOGY

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Abstract. The aim of the present paper is to investigate an exact class of solutions belonging to the Plebanski-Petrov $D-[2S-2T]_{(11)}$ – type with a $G_6 = \mathrm{VII}_0 \times \mathrm{VIII}$ group of motion, supported by a suitable matter-source. In the spacetime endowed with $g_{44} = -\cosh^2(\alpha z)$, we are pointing out some unusual (pathological) features and solve the Klein–Gordon equation for the massless bosons. Finally, within a thermodynamic analysis, we derive the characteristic function and the main thermodynamic quantities.

Key words: globally pathological manifolds, Gordon-type equation, thermodynamic properties.

1. THE GEOMETRY

Studies of supernova explosions suggesting an accelerating rate of the expanding Universe as well as the largest computer simulation of the evolution of the Universe, performed by the Virgo consortium two years ago, have revived talk of Einstein's cosmological constant. Although the existence of a non-zero value for the cosmological constant has come into play previously, now it has gone hand in hand with the increased accuracy of observational data on the distribution of clustering that, for the first time, perfectly matched the theoretical results [1]. On the other hand, in the past decades, a wide interest has been focused on the globally pathologic manifolds, and radical changes have occured in understanding gravity, matter fields and spacetime [2–6]. In this respect, not only intensive studies on the cosmic strings, naked singularities, Bianchi spacetimes, dynamical isotropization or topological domain walls have been the main targets [7–9], but also black holes in less than four dimnensions [10], revealed new intriguing features due to their causal structure singularities, such as the closed timelike curves and/or additional Taub-Nut pathologies at the metric *singular point*.

A brief overview of Universes with $VII_0 \times VIII$ isometries pointed out how the extreme pathology induced by the cosmic temporal trap led to unexpected behaviours of electric or magnetic static modes, such as the appearance of a gravitoelectromagnetic resonance, in some spatially finite regions [11]. Moreover, our analysis has led to similar results as for the *BTZ* black hole, where the reduced one-dimensional motion of the test particle evolves in a parabolic well periodically crossing the two $r = \pm 1$ horizons. As the problem of time remains very controversal, it turns out that it is possible to get cosmic-time traps and temporally imprisoned geodesics even when the metric contains no singular points.

Our interest being motivated by the recent investigations on the eigenvalue problem of scalar fields in BTZ black holes [12], in the present work are are writing down the univoc-regular solutions to the Gordon equation, using the same geometry as in the Ref. [11]. Since one of the most attractive aspects of black hole physics is its thermodynamical properties, in the final section, we are deriving the characteristic function and the main thermodynamic quantities, for the massless bosons evolving in a spacetime described by a metric with $g_{44} = -\cosh^2(\alpha z)$.

Let us start with the metric

$$\mathrm{d}s^2 = \delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - \mathrm{e}^{2f(z)}\left(\mathrm{d}t\right)^2; \quad \mu,\nu = \overline{1,3} \tag{1}$$

and, using the dually-related pseudo-orthonormal tetrads

$$\vec{e}_\mu = \partial_\mu, \quad e_4 = \mathrm{e}^{-f(z)}\partial t; \quad \omega^\mu = \mathrm{d}x^\mu, \quad \omega^4 = \mathrm{e}^{f(z)}\mathrm{d}t,$$

we arrive at the following essential components of the Riemann, Ricci and Einstein tensors

$$R_{3434} = -R_{33} = R_{44} = -\frac{1}{2}R = f_{|33} + \left(f_{|3}\right)^2 \tag{2}$$

$$G_{AB} = \left[f_{|33} + \left(f_{|3}\right)^2\right]\delta_{AB}, \quad A,B = 1,2. \tag{3}$$

As the total energy density, T_{44}, should be zero and any conventional source possesses a positive energy density $T_{44}^{(cs)}$, one must necessarily use a *false vacuum* state described by $T_{ab}^{(\upsilon)} = \lambda\eta_{ab}$, with $\lambda > 0$, in order to get $T_{44}^{(cs)} - \lambda = 0$. Considering, for simplicity, the conventional matter as being characterized by a pressureless ideal fluid at rest, i.e. $T_{ab}^{(d)} = \rho u_a u_b$ and $u_a = \eta_{a4}$, it obviously results $\rho = \lambda$, such that $T_{ab}^{(d)} = \lambda\eta_{a4}\eta_{b4}$. To also get $G_{33} = 0$ without violating the rest of the G_{ab} values, we need an extra source that can be thought of as a global cosmic string of unitary elongation effort $\mu = \lambda$ along the $X_a = \eta_{a3}$ direction. Hence, the total energy-momentum tensor

$$T_{ab} = \lambda\left[\eta_{a4}\eta_{b4} - \eta_{a3}\eta_{b3} + \eta_{ab}\right], \text{ with } \lambda > 0, \tag{4}$$

describes a combined matter-source made of stuck universal dust, with $\rho = \lambda$, on a z-directed global string immersed in a medium of negative energy density and *equal positive* pressure that floods everything all around. Finally, the conservation law $T^{ab}_{;b} = 0$ requires a constant λ and expresses the (false) vacuum-type contribution as a true Λ-**term**.

Under these assumptions, the Einstein equations turn to the essential one,

$$f_{|33} + \left(f_{|3}\right)^2 = \pm\alpha^2, \tag{5}$$

where $\alpha = \left(\kappa_0\lambda\right)^{1/2}$, which is satisfied by the general solution which brings the metric (1) to the *hyperbolic* form

$$\mathrm{d}s^2 = \delta_{AB}\mathrm{d}x^A\mathrm{d}x^B + \left(\mathrm{d}z\right)^2 - \cosh^2\left(\alpha z\right)\left(\mathrm{d}t\right)^2, \quad A, B = \overline{1,2}\,. \tag{6}$$

As it can be noticed, the above metric is defined on $M_4 = R^2 \times M_2$ and is free of finite singular points. According to Estabrook–Ellis–MacCallum method [13], by analyzing the invariant properties characterized by $A_\mu = C^{\sigma}_{\cdot\mu\sigma} = 0$ and $N^{\mu\nu} = \frac{1}{2}C^{\mu}_{\cdot\alpha\beta}\varepsilon^{\alpha\beta\nu} - \varepsilon^{\mu\nu\vartheta}A_\vartheta$, we conclude that we deal with a $G_6 = E_3 \times G_3' = \mathrm{VII}_0 \times \mathrm{VIII}$ group of motion, acting on the manifold described by (6).

By computing the timelike geodesics,

$$z\left(\sigma\right) = \frac{1}{\alpha}\ln\left[\sinh\left(\alpha z_0\right)\cos\left(\alpha\sigma\right) + \sqrt{\sinh^2\left(\alpha z_0\right)\cos^2\left(\alpha\sigma\right) + 1}\right],$$

$$t\left(\sigma\right) = \frac{1}{\alpha}\arctan\left[\frac{\tan\left(\alpha\sigma\right)}{\cosh\left(\alpha z_0\right)}\right], \quad \text{with} \quad t_1 = -\frac{\pi}{2\alpha} < \sigma < t_2 = \frac{\pi}{2\alpha}, \tag{7}$$

represented in Fig. 1, one points out a genuine temporal imprisonment, since they cannot be emitted earlier than $t_1 = -\frac{\pi}{2\alpha}$, or extended to the future beyond $t_2 = \frac{\pi}{2\alpha}$, getting trapped inbetween these two universal moments.

As a matter of fact, in 5D-dimensional models, such hyperbolic metrics have been used for the analysis of test particles trajectories and the so-called *trapping solution* of multidimensional Einstein equations, $\cosh(\alpha x^5)$, has been considered for explaining the matter-confinement [14], in agreement with the anzats of Rubakov–Shaposhnikov [15].

2. SCALAR FIELD QUANTIZATION

In the spacetime endowed with the metric (6), the Gordon equation,

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left[\sqrt{-g}\,g^{ik}\frac{\partial\phi}{\partial x^k}\right]-m_0^2\phi=0\,, \tag{8}$$

does actually read

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}+\alpha\tanh\left(\alpha z\right)\frac{\partial\phi}{\partial z}-\frac{1}{\cosh^2\left(\alpha z\right)}\frac{\partial^2\phi}{\partial t^2}-m_0^2\phi=0\,, \tag{9}$$

where $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$. Thence, it comes to be written as

$$\frac{\partial^2\phi}{\partial z^2}+\alpha\tanh\left(\alpha z\right)\frac{\partial\phi}{\partial z}-\frac{1}{\cosh^2\left(\alpha z\right)}\frac{\partial^2\phi}{\partial t^2}+\left(\Delta_\perp-m_0^2\right)\phi=0\,,$$

where $\Delta_\perp=\delta^{AB}\partial^2_{AB}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. Since the three operators, $\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}$, commute with each other and also with the main operator

$$\hat{D}=\frac{\partial^2}{\partial z^2}+\alpha\tanh\left(\alpha z\right)\frac{\partial}{\partial z}-\frac{1}{\cosh^2\left(\alpha z\right)}\frac{\partial^2}{\partial t^2}+\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-m_0^2\right],$$

they "share" a common set of eigenfunctions, respectively – for the positive frequency and planary out-going modes – $\mathrm{e}^{\mathrm{i}p_A x^A}\cdot\mathrm{e}^{-\mathrm{i}\omega t}=\mathrm{e}^{\mathrm{i}\left[p_A x^A-\omega t\right]}$, $A=\overline{1,2}$, so that the field ϕ can be expressed as

$$\phi_\lambda\left(x^A,z,t\right)=F_\lambda\left(z\right)\mathrm{e}^{\mathrm{i}\left(p_A x^A-\omega t\right)},\quad \text{where}\quad \lambda=\left(\omega,p_A\right), \tag{10}$$

the amplitude function $F_\lambda(z)$ being the regular solution, on $\{z\}=\mathbb{R}$, of the corresponding differential equation,

$$\frac{\mathrm{d}^2F_\lambda}{\mathrm{d}z^2}+\alpha\tanh\left(\alpha z\right)\frac{\mathrm{d}F_\lambda}{\mathrm{d}z}+\left[\frac{\omega^2}{\cosh^2\left(\alpha z\right)}-\left(p_\perp^2+m_0^2\right)\right]F_\lambda=0\,, \tag{11}$$

with $p_\perp^2=\delta^{AB}p_Ap_B$. Dividing by α^2 and defining

$$\zeta=\alpha z,\quad \Omega=\frac{\omega}{\alpha},\quad \varepsilon_\perp^2=\frac{1}{\alpha^2}\left(p_\perp^2+m_0^2\right),$$

it yields the form

$$\left(1+s^2\right)\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(1+s^2\right)\frac{\mathrm{d}F_\lambda}{\mathrm{d}s}\right]+\left[\Omega^2-\varepsilon_\perp^2\left(1+s^2\right)\right]F_\lambda=0,$$

with respect to the integration variable $s = \sinh\zeta$, where $s \in \mathbb{R}$, for $z \in (-\infty,\infty)$. This suggests the change of variable $\theta=\frac{\pi}{2}-\arctan\left(s\right)$, so that it becomes

$$\frac{\mathrm{d}^2F_\lambda}{\mathrm{d}\theta^2}+\left[\Omega^2-\frac{\varepsilon_\perp^2}{\sin^2\theta}\right]F_\lambda=0.$$

Finally, employing the function substitution $F_\lambda=\sqrt{\sin\theta}G_\lambda$, one gets for G_λ the well-known Legendre (generalized) equation [16], which basically sets the univoc-regular solution $G_\lambda=P_\ell^m\left(\cos\theta\right)$, with $m \in \mathbb{Z}$ and $|m| \leq \ell$, $\ell \in \mathbb{N}$, where

$$\Omega^2-\frac{1}{4}=\ell\left(\ell+1\right)\Rightarrow\Omega_\ell=\ell+\frac{1}{2},$$

$$\varepsilon_\perp^2+\frac{1}{4}=m^2\Rightarrow\varepsilon_\perp=\left[\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\right]^{1/2}.$$

As it can be noticed, even for the massless case, $\varepsilon_\perp=\alpha^{-1}\left|\vec{p}_\perp\right|$, m is restricted by the inequality $|m| \geq 1$, so that, because of $|m| \leq \ell$, the "second" quantum number ℓ must (compulsory) take only natural values, ℓ = 1, 2, 3, . . . Thence, with respect to the local coordinates $\left(x^A,\ \theta=\frac{\pi}{2}-\arctan\left[\sinh\left(\alpha x^3\right)\right],t\right)$, the positive-frequency modes of ϕ take on the form (inducing the normalization constant $\mathcal{N}$),

$$\begin{aligned}u_{\ell m\gamma}\left(x,y,\theta,t\right)&=\mathcal{N}\sqrt{\sin\theta}P_\ell^m\left(\cos\theta\right)\times\\&\times\exp\left\{\mathrm{i}\left[p_\perp\left(x\cos\gamma+y\sin\gamma\right)-\alpha\left(\ell+\frac{1}{2}\right)t\right]\right\},\end{aligned}\tag{12}$$

where

$$p_\perp=\left[\alpha^2\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)-m_0^2\right]^{1/2},$$

$$\gamma\in\left[0,2\pi\right),\quad\left|m\right|\geq\mathrm{Int}\left[\frac{m_0^2}{\alpha^2}+\frac{1}{4}\right]^{1/2},\quad\text{and}\quad\left|m\right|\leq\ell.\tag{13}$$

3. THERMODYNAMIC PROPERTIES OF MASSLESS BOSONS

Because of $\gamma \in (0, 2\pi)$ – in the momentum space – the one-particle states with $m = -|m|$, and $m \in \mathbb{Z}$, have been already considered. Therefore, in the massless case, $m_0 = 0$, it yields $p_\perp = \alpha\left[m^2 - 1/4\right]^{1/2} > 0$, meaning that $m \geq 1$ and $0 < m \leq \ell$, where $\ell = 1, 2, 3, \ldots \in \mathbb{N}$. Thus, the actual (proper) degeneration of each energy-level $\omega_\ell = \alpha\left(\ell + \frac{1}{2}\right)$ is $g_\ell = \ell$, coming from $g_\ell = \frac{1}{2}\left[(2\ell+1) - 1\right]$, because $m = 0$ is forbidden and we also have to account only for half of the eigenfunctions, as the azimuth γ in the momentum space runs from 0 to 2π. Thence, the one-particle partition function does effectively become

$$Z = \sum_{\ell=1}^{\infty} g_\ell \mathrm{e}^{-\beta\omega_\ell} = \sum_{\ell=1}^{\infty} \ell \mathrm{e}^{-\beta\alpha\left(\ell + \frac{1}{2}\right)}. \tag{14}$$

Inserting the physic dimensionless parameter $\chi = \beta\alpha$, it can be written as

$$Z = \mathrm{e}^{-\frac{\chi}{2}} \sum_{\ell=1}^{\infty} \ell \mathrm{e}^{-\chi^{\ell}}$$

yielding

$$Z = \frac{\mathrm{e}^{\frac{\chi}{2}}}{\left(\mathrm{e}^{\chi} - 1\right)^2}. \tag{15}$$

The corresponding free-energy becomes

$$\mathcal{F} \triangleq -\beta^{-1} \ln Z = \frac{2}{\beta} \ln\left[\mathrm{e}^{\beta\alpha} - 1\right] - \frac{\alpha}{2}, \tag{16}$$

and the one-particle entropy is going to read,

$$S = \beta^2 \frac{\partial \mathcal{F}}{\partial \beta} = \beta^2 \left[\frac{2\alpha}{\beta} \frac{\mathrm{e}^{\beta\alpha}}{\mathrm{e}^{\beta\alpha} - 1} - \frac{2}{\beta^2} \ln\left(e^{\beta\alpha} - 1\right)\right], \tag{17}$$

i.e. it comes to the (simple) expression (in terms of $\chi = \beta\alpha$)

$$S = 2\left[\chi + \frac{\chi}{\mathrm{e}^{\chi} - 1} - \ln\left(\mathrm{e}^{\chi} - 1\right)\right]. \tag{18}$$

As it can be noticed, at $T = 0_+$, where $\chi \rightarrow \infty$, it readily becomes

$$S(\chi \rightarrow \infty) = 2\left(\chi + \chi e^{-\chi} - \chi\right) = 2 \lim_{T \rightarrow 0_+} \left[\frac{\alpha}{T} \mathrm{e}^{-\frac{\alpha}{T}}\right] = 0,$$

and therefore it fulfills the Third Thermodynamical Principle. Concerning its sign, one can take the other limit, at large (positive) values of T, where $\chi \to 0_+$, and obtains

$$S(\chi \to 0_+) = 2 - 2\ln\chi,$$

which goes to $+\infty$ as χ runs into 0_+. Also, taking the derivative of S, it results

$$\frac{dS}{d\chi} = -2\frac{\chi e^x}{\left(e^\chi - 1\right)^2} < 0,$$

for all positive values of χ, meaning that the one-particle entropy is monotonically decreasing (with respect to χ running from 0_+ to infinity) from $+\infty$ to 0_+ standing *positive* all the way down.

The averaged thermal energy, estimated as

$$\varepsilon = -\frac{1}{Z}\frac{\partial Z}{\partial\beta},$$

is

$$\varepsilon = \frac{2\alpha}{e^{\beta\alpha} - 1} + \frac{3}{2}\alpha. \tag{19}$$

Finally, in order to be sure that the above expression is correctly related to the one of the entropy, we have to check that

$$\frac{dS}{d\varepsilon} \triangleq \beta.$$

Therefore, expressing $\chi = \beta\alpha$, from the averaged-energy formula, it yields

$$\chi = \ln\left[1 + \frac{2}{\varepsilon - \frac{3}{2}}\right], \quad \text{where} \quad \varepsilon = \alpha^{-1}\cdot\varepsilon,$$

so that, the (one-particle) entropy becomes

$$S(\varepsilon) = \left(\varepsilon + \frac{1}{2}\right)\ln\left[1 + \frac{2}{\varepsilon - \frac{3}{2}}\right] + 2\ln\left(\varepsilon - \frac{3}{2}\right) - 2\ln 2 \tag{20}$$

and subsequently, its derivative reads

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon} = \alpha^{-1}\frac{\mathrm{d}S}{\mathrm{d}\varepsilon} = \alpha^{-1}\ln\left(1 + \frac{2}{\varepsilon - \frac{3}{2}}\right),$$

which means, because the logarithm is nothing else than βα,

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon} = \beta. \tag{21}$$

In order to put these results into a more transparent form, we have to introduce the constants $\hbar$ and k, which have been set to one up to now. Thus, the free energy and the energy turn into the following functions of T,

$$F = -\frac{\hbar\omega_0}{2} + 2kT\ln\left[\mathrm{e}^{\frac{\hbar\omega_0}{kT}} - 1\right], \tag{22}$$

$$E = \frac{3}{2}\hbar\omega_0 + 2\frac{\hbar\omega_0}{\mathrm{e}^{\frac{\hbar\omega_0}{kT}} - 1}, \tag{23}$$

where we have introduced the notation $\omega_0 \equiv \alpha$. The energy expression appears to be physically reasonable, though not trivial, containing besides the averaged energy of an oscillator (multipled by 2), a zero-energy contribution, $E_0 = (3/2)\hbar\omega_0$. If we proceed to consider high temperatures or small α's so that $\chi \ll 1$, one may expend in series the above formulae and we get:

$$F = -\frac{\hbar\omega_0}{2} - 2kT\ln\frac{kT}{\hbar\omega_0}, \tag{24}$$

$$E = \frac{3}{2}\hbar\omega_0 + 2kT, \tag{25}$$

$$P = P_0 - 2\frac{kT}{\omega_0}. \tag{26}$$

On the other hand, the free energy allows us to compute the so-called *Casimir pressure*, defined as

$$P = \frac{\partial F}{\partial\alpha}, \tag{27}$$

by analogy to the definition for the usual configuration, namely two parallel metal plates separated by a gap. In our case, the Casimir pressure (in $\hbar$ units) is

$$P = -\frac{3}{2} - \frac{2}{e^{\frac{\hbar\omega_0}{kT}} - 1}, \tag{28}$$

pointing out the constant contribution $P_0 = -3\hbar/2$, which survives when $T \to 0$.

4. CONCLUSIONS

Recently, it has been analyzed if the classical singular spacetime with a naked singularity at the origin remains singular when tested with quantum particles [17]. It has turned out that, although that for massive scalar particles all the possible boundary conditions necessary to turn the spatial portion of the wave operator self-adjoint are found, for massless particles, the singularity is healed and no extra boundary condition are needed.

The present work is based on our previous papers [11], devoted to the pathologies of Universes with $VII_0 \times VIII$ isometries. We start with the general metric (1) and use a total energy-momentum tensor of a combined matter-source made of stuck universal dust on a *z*-directed global string immersed in a medium of negative energy density and equal *positive* pressure, expressing the (false) vacuum-type contribution as a true **Λ-term**. For the *hyperbolic* metric, (6), which is satisfying the Einstein equations (5), the null trajectories (7) gets trapped inbetween two universal moments, pointing out a temporal imprisonment.

In the spacetime endowed with the metric (6), we write down the Klein–Gordon equation, (9), and derive its solution, (12), with respect to the local coordinates $\left(x^A, \theta = \frac{\pi}{2} - \arctan\left[\sinh\left(\alpha x^3\right)\right], t\right)$.

Finally, within a thermodynamic analysis, we focus on the massless case and compute the one-particle partition function, the free-energy and the *one-particle* entropy which does not come into conflict with Nerst's theorem.

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