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BIANCHI TYPE – I STRING COSMOLOGICAL MODEL IN THE PRESENCE OF A MAGNETIC FIELD: CLASSICAL AND QUANTUM LOOP APPROACH

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Abstract. Cosmic strings are expected generically formed in the early Universe as topological defects as a result of a series of phase transitions related to the spontaneous breaking of internal symmetry. A Bianchi type-I cosmological model in the presence of a magnetic field is investigated. A few plausibly assumptions are introduced and their outcomes are discussed using numerical and qualitative analyzes. Quantum effects of a cosmological string are examined in the framework of loop quantum cosmology.

Key words: Bianchi models, cosmological string, quantum loop cosmology, Einstein equation, Hubble rate.

1. INTRODUCTION

The Bianchi models which describe homogeneous anisotropic spacetimes are the most convenient to explain small but significant anisotropies in the Universe as seen, for example, in the cosmic microwave background (CMB). Among many possible alternatives, the simplest and most theoretically appealing of anisotropic models are Bianchi type-I (BI).

Cosmic strings are linear topological defects which could have formed during phase transitions in the early stages of the evolution of the Universe. They are included as a (sub-dominant) partner of inflation and can play a role in the framework of braneworld cosmologies.

On the other hand, the magnetic fields have an important role at the cosmological scale and are present in galactic and intergalactic spaces. Therefore it is natural to include the contribution of the magnetic fields in the energy momentum tensor of the early Universe. Any cosmological model which contains magnetic fields is necessarily anisotropic taking into account that the magnetic field vector implies a preferred spatial direction.

In what follows we shall investigate the evolution of BI cosmological models in presence of a cloud of strings and magnetic field. The paper has the following structure. We shall review the basic equations of an anisotropic BI model in the presence of a system of cosmic strings and magnetic field. The objective of this treatment is to generate solutions to the Einstein equations using a few plausible assumptions. In Section 3 we discuss the model in the framework of *loop quantum cosmology*. At the end we shall summarize the results and outline future prospects.

2. CLASSICAL EQUATIONS

The line element of a BI Universe is

$$\mathrm{d}s^2 = -\left(\mathrm{d}t\right)^2 + a_1\left(t\right)^2\left(\mathrm{d}x\right)^2 + a_2\left(t\right)^2\left(\mathrm{d}y\right)^2 + a_3\left(t\right)^2\left(\mathrm{d}z\right)^2 . \tag{1}$$

There are three scale factors a_i (i = 1, 2, 3) which are functions of time t only and consequently three expansion rates. In principle all these scale factors could be di_erent and it is useful to express the mean expansion rate in terms of the Hubble rate:

$$H = \frac{1}{3}\left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3}\right), \tag{2}$$

which is the average of the directional Hubble rates

$$H_i \equiv \frac{\dot{a}_i}{a_i}, \quad i = 1,2,3, \tag{3}$$

where over-dot means differentiation with respect to t.

In the absence of a cosmological constant, the Einstein's gravitational field equation has the form

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} = \kappa T_1^1 , \tag{4a}$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} = \kappa T_2^2 , \tag{4b}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} = \kappa T_3^3 , \tag{4c}$$

$$\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} = \kappa T_0^0 , \tag{4d}$$

where κ is the gravitational constant. We mention that a cosmological constant can be taken into account, but its inclusion does not alter the main results reported here.

The energy momentum tensor for a system of cosmic strings and magnetic field in a comoving coordinate is given by

$$T_{\mu}^{\nu} = \rho u_{\mu} u^{\nu} - \lambda x_{\mu} x^{\nu} + E_{\mu}^{\nu}, \tag{5}$$

where ρ is the rest energy density of strings with massive particles attached to them and can be expressed as $\rho = \rho_p + \lambda$. ρ_p is the rest energy density of the particles attached to the strings and λ is the tension density of the system of strings [1, 2, 3] which may be positive or negative. Here u_i is the four velocity and x_i is the direction of the string, obeying the relations

$$u_i u^i = -x_i x^i = 1, \quad u_i x^i = 0. \tag{6}$$

In (5) $E_{\mu\nu}$ is the electromagnetic field given by Lichnerowich [4]. In our case the electromagnetic field tensor $F^{\alpha\beta}$ has only one non-vanishing component, namely

$$F_{23} = h, \tag{7}$$

where h is assumed to be constant. For the electromagnetic field E_{μ}^{ν} one gets the following non-trivial components

$$E_0^0 = E_1^1 = -E_2^2 = -E_3^3 = \frac{h^2}{2\bar{\mu} a_2^2 a_3^2} \equiv \frac{1}{2} \frac{\beta^2}{\left(a_2 a_3\right)^2}, \tag{8}$$

where $\bar{\mu}$ is a constant characteristic of the medium and called the magnetic permeability. Typically $\bar{\mu}$ differs from unity only by a few parts in 10^5 ($\bar{\mu} > 1$ for paramagnetic substances and $\bar{\mu} < 1$ for diamagnetic).

Choosing the string along x1 direction and using comoving coordinates we have the following components of energy momentum tensor [5]:

$$T_0^0 = \rho + \frac{\beta^2}{2} \frac{a_1^2}{\tau^2}, \tag{9a}$$

$$T_1^1 = \lambda + \frac{\beta^2}{2} \frac{a_1^2}{\tau^2}, \tag{9b}$$

$$T_2^2 = -\frac{\beta^2}{2} \frac{a_1^2}{\tau^2}, \tag{9c}$$

$$T_3^3 = -\frac{\beta^2}{2}\frac{a_1^2}{\tau^2}, \tag{9d}$$

where we introduce the volume scale of the BI space-time

$$\tau = a_1 a_2 a_3, \tag{10}$$

namely, $\tau = \sqrt{-g}$ [6]. It is interesting to note that the evolution in time of τ is connected with the Hubble rate (2):

$$\frac{\dot{\tau}}{\tau} = 3H. \tag{11}$$

Taking into account the conservation of the energy-momentum tensor, i.e., $T^{\nu}_{\mu;\nu} = 0$, after a little manipulation of (9) one obtains [7, 8]:

$$\dot{\rho} + \frac{\dot{\tau}}{\tau}\sigma - \frac{\dot{a}_1}{a_1}\lambda = 0. \tag{12}$$

It is customary to assume a relation between ρ and λ in accordance with the state equations for strings. The simplest one is a proportionality relation [1]:

$$\rho = \alpha\lambda . \tag{13}$$

The most usual choices of the constant α are

$$\alpha = \begin{cases} 1 & \text{geometric string,} \\ 1+\omega & \omega \geq 0, \quad p \text{ string or Takabayasi string,} \\ -1 & \text{Reddy string} \end{cases} \tag{14}$$

From equation (12) with (13) we get

$$\rho_{string} = R a_1^{\frac{1-\alpha}{\alpha}} a_2^{-1} a_3^{-1}, \tag{15}$$

with R a constant of integration.

As a first example [7], we assumed that the average Hubble rate H (2) in the model is proportional to the eigenvalue σ_1^1

$$\sigma_1^1 = -\frac{1}{3}\left(4\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3}\right). \tag{16}$$

of the shear tensor σ_{μ}^{ν}.

Besides the generalized Hubble parameter (2) we considered two relative shear anisotropy parameters defined by:

$$R = \frac{1}{H}\left(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2}\right), \tag{17a}$$

$$S = \frac{1}{H}\left(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3}\right). \tag{17b}$$

when $R = S = 0$ the universe will be the isotropic flat Friedman Universe. In this second example we shall assume a deviation from the Friedman model.

We considered three different assumptions regarding the relative shear anisotropy parameters. Some numerical simulations and a more detailed discussion about these assumptions are given in [8].

3. EFFECTIVE LOOP QUANTUM DYNAMICS

In the loop quantum cosmology approach we shall use a Hamiltonian framework where the degrees of freedom of the Bianchi type-I model are encoded in triad components p_i and momentum components c_i as follows:

$$p_1 = a_2 a_3, \quad p_2 = a_1 a_3, \quad p_3 = a_1 a_2, \quad c_i = \gamma \dot{a}_i. \tag{18}$$

Here γ is the Barbero-Immirzi parameter and represent a quantum ambiguity of loop quantum gravity which is a non-negative real valued parameter.

In terms of these variables, the total Hamiltonian of the model is

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{grav} + \mathcal{H}_{matter} = \\ &= \frac{-1}{\kappa\gamma^2}\left(c_2 c_3 p_2 p_3 + c_1 c_3 p_1 p_3 + c_1 c_2 p_1 p_2\right) + p_1 p_2 p_3 \rho_M, \end{aligned} \tag{19}$$

where ρ_M is the matter energy density [9]. In our model ρ_M comprises the contribution of cosmological string density ρ_{string} (15) and the energy density of the magnetic field (8)

$$\rho_{mag} = \frac{1}{2}\left(\frac{\beta^2}{\left(a_2 a_3\right)^2}\right) = \frac{1}{2}\frac{\beta^2}{p_1^2}. \tag{20}$$

Without loss of generality, we shall restrict ourselves to the positive octant $p_i \geq 0$. The dynamics in any other seven octants can be recovered from the positive octant by the action of the discrete reflection symmetry $\Pi\left(p\right) = -p$ corresponding to the orientation reversal of triads [10].

Einstein's equations are derived from Hamilton's equations which for this system are:

$$\dot{p}_i = \{p_i, \mathcal{H}\} = -\kappa\gamma \frac{\partial \mathcal{H}}{\partial c_i}, \quad \dot{c}_i = \{c_i, \mathcal{H}\} = \kappa\gamma \frac{\partial \mathcal{H}}{\partial p_i}. \tag{21}$$

On the other hand, the total Hamiltonian $\mathcal{H}$ is of constrained type whereby it vanishes identically for solutions to Einstein's equations

$$\mathcal{H} = 0. \tag{22}$$

Using the explicit form of the Hamiltonian $\mathcal{H}$ we have for instance

$$\begin{aligned} \frac{\mathrm{d}p_1}{\mathrm{d}t} &= \frac{p_1}{\gamma}\left(c_2 p_2 + c_3 p_3\right), \\ \frac{\mathrm{dc}_1}{\mathrm{d}t} &= -\frac{c_1}{\gamma}\left(c_2 p_2 + c_3 p_3\right) + \kappa\gamma\, p_2 p_3 \left(\rho_M + p_1 \frac{\partial \rho_M}{\partial p_1}\right), \end{aligned} \tag{23}$$

and similar relations for the rest of p_i and c_i.

Let us observe that from (23) we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i c_i\right) = \kappa\gamma\, p_1 p_2 p_3 \left(\rho_M + p_i \frac{\partial \rho_M}{\partial p_i}\right). \tag{24}$$

From the vanishing of the Hamiltonian (22) we get the following relation between the directional Hubble rates:

$$H_1 H_2 + H_1 H_3 + H_2 H_3 = \kappa \rho_M . \tag{25}$$

In particular, taking into account the symmetry of the density ρ_M with respect to variables p_2 and p_3 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_2 c_2 - p_3 c_3\right) = 0. \tag{26}$$

This equation can be integrated to give

$$H_2 - H_3 = \frac{\alpha_{23}}{a_1 a_2 a_3} = \frac{\alpha_{23}}{\sqrt{p_1 p_2 p_3}}, \tag{27}$$

in terms of the directional parameter H_2, H_3 with α_{23} a constant.

In loop quantum cosmology the connection variables c_i do not have direct quantum analogues and are replaced by holonomies. The quantum effects are incorporated in the *effective* Hamiltonian constructed from the classical one by replacing the c_i terms with sine functions:

$$c_i \to \frac{\sin\left(\bar{\mu}_i c_i\right)}{\bar{\mu}_i}, \tag{28}$$

where $\bar{\mu}_i$ are real valued functions of the triad coeffcients p_i.

The effective Hamiltonian is given by:

$$\mathcal{H}_{eff} = -\frac{1}{\kappa\gamma^2}\left\{\frac{\sin\left(\bar{\mu}_2 c_2\right)\sin\left(\bar{\mu}_3 c_3\right)}{\bar{\mu}_2\bar{\mu}_3} p_2 p_3 + \text{cyclic terms}\right\} + p_1 p_2 p_3 \rho_M . \tag{29}$$

It is quite evident that in the limit $\bar{\mu}_i \to 0$, the classical Hamiltonian $\mathcal{H}$(19) is recovered. The expression of the parameters $\bar{\mu}_i$ as functions of the triad components pi represent an ambiguity of the quantization. Two most preferable constructions are discussed in [11, 12] and more recently in [10]. In what follows we shall use the so called $\bar{\mu}'$-scheme which presents some advantages as compared to other schemes [10].

3.1. EFFECTIVE DYNAMICS IN $\bar{\mu}'$-SCHEME

In $\bar{\mu}'$-scheme the parameters $\bar{\mu}_i'$ are chosen as follows:

$$\bar{\mu}_1' = \sqrt{\frac{p_1\Delta}{p_2 p_3}}, \quad \bar{\mu}_2' = \sqrt{\frac{p_2\Delta}{p_1 p_3}}, \quad \bar{\mu}_3' = \sqrt{\frac{p_3\Delta}{p_1 p_2}}, \tag{30}$$

where $\Delta = \frac{\sqrt{3}}{2}\left(4\pi\gamma\, l_{Pl}^2\right)$ is the area gap in the loop quantum cosmology with the Planck length $l_{Pl} := \sqrt{G\hbar}$.

In this scheme the Hamilton's equations (23) are:

$$\dot{p}_1 = \frac{1}{\gamma\sqrt{\Delta}} p_1^{\frac{3}{2}} p_2^{\frac{1}{2}} p_3^{\frac{1}{2}} \cos\left(\bar{\mu}_1' c_1\right)\left\{\sin\left(\bar{\mu}_2' c_2\right) + \sin\left(\bar{\mu}_3' c_3\right)\right\}, \tag{31a}$$

$$\dot{p}_2 = \frac{1}{\gamma\sqrt{\Delta}} p_1^{\frac{1}{2}} p_2^{\frac{3}{2}} p_3^{\frac{1}{2}} \cos\left(\bar{\mu}_2' c_2\right)\left\{\sin\left(\bar{\mu}_1' c_1\right) + \sin\left(\bar{\mu}_3' c_3\right)\right\}, \tag{31b}$$

$$
\begin{aligned}
\dot{c}_1 = & -\frac{1}{\gamma\sqrt{\Delta}} p_2 p_3 \times \\
& \times\left\{\sin\left(\bar{\mu}_2' c_2\right)\sin\left(\bar{\mu}_3' c_3\right) + \sin\left(\bar{\mu}_1' c_1\right)\sin\left(\bar{\mu}_3' c_3\right) + \sin\left(\bar{\mu}_1' c_1\right)\sin\left(\bar{\mu}_2' c_2\right)\right\} - \\
& -\frac{1}{2\gamma\sqrt{\Delta}} p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} p_3^{\frac{1}{2}} c_1 \cos\left(\bar{\mu}_1' c_1\right)\left\{\sin\left(\bar{\mu}_2' c_2\right) + \sin\left(\bar{\mu}_3' c_3\right)\right\} + \\
& +\frac{1}{2\gamma\sqrt{\Delta}} p_1^{-\frac{1}{2}} p_2^{\frac{3}{2}} p_3^{\frac{1}{2}} c_2 \cos\left(\bar{\mu}_2' c_2\right)\left\{\sin\left(\bar{\mu}_1' c_1\right) + \sin\left(\bar{\mu}_3' c_3\right)\right\} + \\
& +\frac{1}{2\gamma\sqrt{\Delta}} p_1^{-\frac{1}{2}} p_2^{\frac{1}{2}} p_3^{\frac{3}{2}} c_3 \cos\left(\bar{\mu}_3' c_3\right)\left\{\sin\left(\bar{\mu}_1' c_1\right) + \sin\left(\bar{\mu}_2' c_2\right)\right\} + \\
& +\kappa\gamma\, p_2 p_3 \left(-\frac{1}{2}\beta^2 p_1^{-2} + \frac{\alpha - 1}{2\alpha} R p_1^{-\frac{1+\alpha}{2\alpha}} p_2^{\frac{1-\alpha}{2\alpha}} p_3^{\frac{1-\alpha}{2\alpha}} \right),
\end{aligned}
\tag{32a}
$$

$$
\begin{aligned}
\dot{c}_2 = & -\frac{1}{\gamma\sqrt{\Delta}} p_1 p_3 \times \\
& \times\left\{\sin\left(\bar{\mu}_1' c_1\right)\sin\left(\bar{\mu}_3' c_3\right) + \sin\left(\bar{\mu}_2' c_2\right)\sin\left(\bar{\mu}_3' c_3\right) + \sin\left(\bar{\mu}_1' c_1\right)\sin\left(\bar{\mu}_2' c_2\right)\right\} - \\
& -\frac{1}{2\gamma\sqrt{\Delta}} p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} p_3^{\frac{1}{2}} c_2 \cos\left(\bar{\mu}_2' c_2\right)\left\{\sin\left(\bar{\mu}_1' c_1\right) + \sin\left(\bar{\mu}_3' c_3\right)\right\} + \\
& +\frac{1}{2\gamma\sqrt{\Delta}} p_1^{\frac{3}{2}} p_2^{-\frac{1}{2}} p_3^{\frac{1}{2}} c_1 \cos\left(\bar{\mu}_1' c_1\right)\left\{\sin\left(\bar{\mu}_1' c_1\right) + \sin\left(\bar{\mu}_3' c_3\right)\right\} + \\
& +\frac{1}{2\gamma\sqrt{\Delta}} p_1^{\frac{1}{2}} p_2^{-\frac{1}{2}} p_3^{\frac{3}{2}} c_3 \cos\left(\bar{\mu}_3' c_3\right)\left\{\sin\left(\bar{\mu}_1' c_1\right) + \sin\left(\bar{\mu}_2' c_2\right)\right\} + \\
& +\kappa\gamma\, p_1 p_3 \left(\frac{1}{2}\beta^2 p_1^{-2} + \frac{\alpha + 1}{2\alpha} R p_1^{-\frac{1+\alpha}{2\alpha}} p_2^{\frac{1-\alpha}{2\alpha}} p_3^{\frac{1-\alpha}{2\alpha}} \right).
\end{aligned}
\tag{32b}
$$

Similar equations for $\dot{p}_3$ and $\dot{c}_3$ can be obtained from equations (31b) and respectively (32b) *via* permutation of indices $2 \leftrightarrow 3$.

The complexity of these equations imposes numerical simulations which will be reported elsewhere [13]. Here we limit ourselves to remark that from the vanishing of the Hamiltonian (22) we have the bound:

$$
p_1 p_2 p_3 \rho_M \leq \frac{1}{\kappa\gamma^2}\left\{ \frac{p_2 p_3}{\bar{\mu}_2\bar{\mu}_3} + \frac{p_1 p_3}{\bar{\mu}_1\bar{\mu}_3} + \frac{p_1 p_2}{\bar{\mu}_1\bar{\mu}_2} \right\}. \tag{33}
$$

In particular, in the $\bar{\mu}'$ scheme the total density is bounded by

$$\rho_{M_{crit}} = 3\left(\kappa\gamma^2\Delta\right)^{-1}, \tag{34}$$

which is near the Planck density ρ_{Pl}.

The inequality (33) is remarkable showing that the matter energy density cannot increase unbounded as in the usual *big bang* scenario. Loop quantum cosmology offers a different picture of the early Universe, free of singularities.

4. CONCLUSIONS

In the first part of the paper we investigated a Bianchi type-I cosmological string model in the presence of a magnetic field. Einstein's equations were investigated using a few plausible assumptions usually accepted in the literature.

In the second part we consider the model in the frame of loop quantum cosmology. As an interesting result we note that the *big bang* singularity is avoided via a *big bounce*. Immediately following the bounce the anisotropies decay and the Universe isotropizes in the expanding phases.

Although Bianchi I models are the simplest among anisotropic cosmologies, they are suitable for investigations of the early Universe, dark matter, spacetimes singularities, etc. The extension of the effective loop quantum cosmology treatment of Bianchi I models is suffciently interesting to deserve further study.

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