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GAUGE THEORIES OF GRAVITATION: COMMUTATIVE AND NONCOMMUTATIVE

G. ZET

Department of Physics, "Gh. Asachi" Technical University, Iasi 700050, Romania
e-mail: gzet@phys.tuiasi.ro

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Abstract. Gauge theories of gravitation both commutative and non-commutative are presented and a comparison between their results is given. The commutative gauge theory of gravitation is developed in a similar way with internal gauge theory by using the formalism of differential forms. In the noncommutative case we use a covariant star product between differential forms in order to construct a gauge invariant action for the gravitational field. Because the Poincaré algebra do not closes with respect to this star product we use *GL*(2, *C*) as gauge group.

Key word: noncommutative gauge theory, covariant star product, *GL*(2, *C*) group.

1. INTRODUCTION

Gauge symmetry is a basic principle in classical and quantum field theory and in particular in elementary particle physics. It allows a deductive description of the fundamental interactions between physical fields.

A very important characteristic of the gauge theory is that we can consider the gravitational field also as a gauge field in a very similar way with other interactions.

Many research efforts have been oriented in the last time towards a quantum theory of the gravitation. One possible way for this aim is to develop a non-commutative gauge theory of gravitation. Generally speaking, such a theory is defined over a noncommutative space-time whose coordinates do not commute

$$\left[x^{\mu}, x^{\nu}\right] = \mathrm{i}\theta^{\mu\nu}, \tag{1.1}$$

where $\theta^{\mu\nu}$ is supposed to be an anti-symmetric constant matrix. Then, it is possible to use the so called Moyal star product between functions and/or fields to build non-commutative generalizations of commuting theories. Because $\theta^{\mu\nu}$ is constant, the Lorentz invariance of (1.1) breaks down and as a consequence, it is not possible to define gauge covariant quantities with this star product. It was proposed [3] that the non-commutative field theories in general, and gauge theories in particular formulated with Moyal-product, are invariant under the twisted Poincaré symmetry. However, it was proven [6] that the concept of twist symmetry, originally obtained for the non-commutative space-time (1.1), cannot be extended to include internal gauge symmetry. In other words, it is not possible to obtain a gauge covariant twist if the property (1.1) is adopted with $\theta^{\mu\nu}$ constant.

The same result appears to be valid also in the case of non-commutative gauge theory of gravitation. In Ref. [7] it has been shown that the twisted Poincaré symmetry cannot be gauged by generalizing the Abelian twist to a covariant non-Abelian twist, nor by introducing a more general covariant twist element defined with $\theta^{\mu\nu}$ constant.

One possible way to define a covariant star product is to consider models of non-commutative theories with $\theta^{\mu\nu}$ depending on the space-time coordinates: $\theta^{\mu\nu} = \theta^{\mu\nu}(x)$. In Ref. [1] a covariant star product between differential forms has been defined. For ordinary functions this product reduces to results well known in literature.

In this paper we present first the gauge theory of gravity based on the Poincaré symmetry group and then generalize the results to the noncommutative case using the covariant star product. Because the Poincaré algebra do not closes with respect to this star product we use $GL(2,C)$ as gauge group.

In Sect. 2 we present the gauge theory of gravitation in a similar way with that corresponding to internal symmetries. The gravitational gauge fields are introduced and their transformation laws are written. Then, the Lorentz curvature and torsion are obtained by calculating the commutator of the gauge covariant derivatives. Section 3 is devoted to the noncommutative gauge theories of the gravitation. We use a covariant star product [1, 2] between differential forms in order to define a gauge invariant action. The gauge group is $GL(2,C)$ and the invariance of the action is verified up to the second order in $\theta^{\mu\nu}(x)$.

2. COMMUTATIVE GAUGE THEORY

The gauge theory of gravity is formulated usually using the Poincaré, de-Sitter or affine groups as local symmetry of the Lagrangian $L = L(\varphi_i, \partial_\mu \rho_i)$ associated to a system of fields φ_i. We will choose the Poincaré group as gauge group for gravitation. This group has the generators P_a (of the space-time translations) and $M_{ab} = -M_{ba}$ (of the homogeneous Lorentz transformations), with $a, b = 0, 1, 2, 3$. The equations of structure of this group are

$$\left[M_{ab},M_{cd}\right]=\eta_{bc}M_{ad}-\eta_{ac}M_{bd}-\eta_{bd}M_{ac}+\eta_{ad}M_{bc},$$
$$\left[M_{ab},P_c\right]=\eta_{bc}P_a-\eta_{ac}P_b, \tag{2.1}$$
$$\left[P_a,P_b\right]=0.$$

The gravitational gauge fields can be introduced as in the case of the Yang-Mills theory, and they are denoted by $\omega_\mu^{ab}\left(x\right)$ (the spin connection) and $e_\mu^a\left(x\right)$ (the tetrad fields) which define the gauge potential 1-form $A=A_\mu \mathrm{d}x^\mu$, where

$$A_\mu=e_\mu^a P_a+\frac{1}{2}\omega_\mu^{ab}M_{ab}. \tag{2.2}$$

The Lorentz indices *a*, *b*, *c*, ... = 0, 1, 2, 3 are raised and lowered using the flat metric tensor $\eta_{ab}=\mathrm{diag}\left(-1,1,1,1\right)$. The transformations of the gauge fields with the parameter

$$\lambda=\lambda^a P_a+\frac{1}{2}\lambda^{ab}M_{ab}, \tag{2.3}$$

where λ^a and $\lambda^{ab}=-\lambda^{ba}$ depend of space-time coordinates, have the form:

– under *Lorentz rotations*

$$\delta e_\mu^a=\lambda_b^a e_\mu^b,\quad \delta\omega_\mu^{ab}=\partial_\mu\lambda^{ab}+\omega_\mu^{ac}\lambda_c^{\ b}-\omega_\mu^{bc}\lambda_c^{\ a}, \tag{2.4a}$$

– under *space-time translations*

$$\delta e_\mu^a=\partial_\mu\lambda^a+\omega_\mu^{ab}\lambda_b,\quad \delta\omega_\mu^{ab}=0. \tag{2.4b}$$

We can define as usually the gauge covariant derivative

$$\nabla_\mu=\partial_\mu+A_\mu=\partial_\mu+e_\mu^a P_a+\frac{1}{2}\omega_\mu^{ab}M_{ab}, \tag{2.5}$$

where we have not introduced a gravitational gauge coupling constant g but it will be considered later.

The Lorentz curvature and torsion are defined by

$$\left[\nabla_\mu,\nabla_\nu\right]+\frac{1}{2}R_{\mu\nu}^{ab}M_{ab}+T_{\mu\nu}^a P_a, \tag{2.6}$$

where

$$R_{\mu\nu}^{ab}=\partial_{\left[\mu\omega_\nu^{ab}\right]}+\omega_{\left[\mu\omega_\nu^{b}\right]c}^{ac} \tag{2.7}$$

$$T^a_{\mu\nu} = \partial_{\left[\mu e^a_\nu\right]} + \omega^{ab}_{[\mu e_\nu]b} \tag{2.8}$$

respectively, and they satisfy the Bianchi identity

$$\partial_{\left[\mu R^{ab}_\nu\right]\rho} + \omega^a_{c\left[\mu R^{cb}_\nu\right]\rho} - R^a_{c[\mu\nu]}\omega^{cb}_\rho , \tag{2.9}$$

$$\partial_{\left[\mu T^a_{\nu\rho}\right]} + \omega^a_{b\left[\mu T^b_{\nu\rho}\right]} - R^a_{b\left[\mu\nu e^b_\rho\right]} . \tag{2.10}$$

The action of the gauge gravitational field is chosen as

$$S_g = -\frac{1}{16\pi G}\int \mathrm{d}^4 x \varepsilon_{abcd}\varepsilon^{\mu\nu\rho\sigma} R^{ab}_{\mu\nu} e^c_\rho e^d_\sigma , \tag{2.11}$$

where G is the gravitational constant and and it is connected with the gravitational gauge coupling constant g by the relation $4\pi G = g^2$ [11]. It is invariant under the gauge transformations (2.4) and can be used to determine the field equations for the gravitational gauge fields $e^a_\mu(x)$ and $\omega^{ab}_\mu(x)$. Indeed, if we impose the variational principle $\delta S_g = 0$ with respect to $e^a_\mu(x)$ and $\omega^{ab}_\mu(x)$, we obtain respectively

$$R^a_\mu - \frac{1}{2} R e^a_\mu = 0, \tag{2.12}$$

$$T^a_{\mu\nu} = 0. \tag{2.13}$$

Here

$$R^a_\mu = R^{ab}_{\mu\nu}\bar{e}^\nu_b , \quad R = R^{ab}_{\mu\nu}\bar{e}^\mu_a \bar{e}^\nu_b , \tag{2.14}$$

and $\bar{e}^\mu_a$ is the inverse of e^a_μ satisfying the usual properties:

$$e^a_\mu \bar{e}^\mu_b = \delta^a_b , \quad e^a_\mu \bar{e}^\nu_a = \delta^\nu_\mu . \tag{2.15}$$

Therefore, the action (2.11) corresponds to the Einstein theory of General Relativity where the gravitational field is described by a curved space-time with vanishing torsion. We can conclude that the action (2.11) is invariant under the space-time translations provided the torsion is set to zero. In other words, enforcing the zero torsion constraint allows us to consider an invariant action, without the need to define a metric on the four-dimensional manifold, by making use of the differential form representation.

3. NONCOMMUTATIVE GAUGE THEORY

Open string theories as well as D-branes in the presence of a background anti-symmetric $B_{\mu\nu}$-field give rise to non-commutative effective field theories. This is equivalent to field theories deformed with the star product. The starting point is the assumption that space-time coordinates x^{μ} do not commute

$$\left[x^{\mu}, x^{\nu}\right] = \mathrm{i}\theta^{\mu\nu}\left(x\right). \tag{3.1}$$

To obtain a covariant star product between differential forms, the parameter functions $\theta^{\mu\nu}\left(x\right)$ have to satisfy some constraint conditions. These constraints have been given in Ref. [1] where a covariant star product was also constructed. Here we just recall the definitions and properties to fix the ideas. However, we generalize these results to the case of Lie valued forms [13]. This means that the differential forms determine a graded Lie valued Poisson algebra. Therefore, the star-commutator of differential forms can be a commutator or an anti-commutator, depending on their degrees.

Assuming that $\theta^{\mu\nu}\left(x\right)$ is invertible, we can always write the Poisson bracket $\left\{x^{\mu}, \mathrm{d}x^{\nu}\right\}$ in the form [1, 2]

$$\left\{x^{\mu}, x^{\nu}\right\} = -\theta^{\mu\sigma}\Gamma^{\nu}_{\sigma\rho}\mathrm{d}x^{\rho}, \tag{3.2}$$

where $\Gamma^{\nu}_{\sigma\rho}$ are some functions transforming like a connection under general coordinate transformations. As $\Gamma^{\nu}_{\sigma\rho}$ is generally not symmetric, on can use the 1-forms connection

$$\tilde{\Gamma}^{\mu}_{\nu} = \Gamma^{\mu}_{\nu\rho}\mathrm{d}x^{\rho}, \quad \Gamma^{\mu}_{\nu} = \mathrm{d}x^{\rho}\Gamma^{\mu}_{\rho\nu}, \tag{3.3}$$

to define two kinds of covariant derivatives $\tilde{\nabla}$ and ∇, respectively. For example, if $\alpha = \alpha_{\nu}\mathrm{d}x^{\nu}$ is an 1-form, then

$$\tilde{\nabla}_{\mu}\alpha = \left(\partial_{\mu}\alpha_{\nu} - \Gamma^{\rho}_{\mu\nu}\alpha_{\rho}\right)\mathrm{d}x^{\nu}, \tag{3.4}$$

and analogous formula for $\nabla_{\mu}\alpha$. Given θ and Γ, all Poisson brackets are determined [2].

Now, if α and β are two arbitrary differential forms, then their Poisson bracket is given by [1]

$$\{\alpha,\beta\} = \theta^{\mu\nu}\nabla_\mu\alpha\nabla_\mu\beta + (-1)^{[\alpha]}\,\tilde{R}^{\mu\nu}\left(i_\mu\alpha\right)\left(i_\nu\beta\right), \tag{3.5}$$

where

$$\tilde{R}^{\mu\nu} = \tilde{R}^{\mu\nu}_{\rho\sigma}\mathrm{d}x^\rho\mathrm{d}x^\sigma, \quad \tilde{R}^{\mu\nu}_{\rho\sigma} = \theta^{\mu\lambda}\tilde{R}^{\nu}_{\lambda\rho\sigma}, \tag{3.6}$$

and

$$\tilde{R}^{\nu}_{\lambda\rho\sigma} = \partial_\rho\Gamma^{\nu}_{\lambda\sigma} - \partial_\sigma\Gamma^{\nu}_{\lambda\rho} + \Gamma^{\nu}_{\tau\rho}\Gamma^{\tau}_{\lambda\sigma} - \Gamma^{\nu}_{\tau\sigma}\Gamma^{\tau}_{\lambda\rho}. \tag{3.7}$$

Also, i_μ denotes the interior product which maps k-forms into (k – 1)-forms [12]. The definition (3.5) has been used in Ref. [1] to construct a covariant star product between two arbitrary differential forms. Then, the results have been extended to the case of Lie algebra valued differential forms [13], with the ultimate aim of constructing noncommutative gauge field theories. Thus, if $\alpha = \alpha^a T_a$ and $\beta = \beta^b T_b$ are two arbitrary such forms, where α^a and β^b are ordinary differential forms of degrees $|\alpha|$ and respectively $|\beta|$, then their star product has the expression

$$\alpha * \beta = \alpha\beta + \sum_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^2 C_n\left(\alpha,\beta\right) = \sum_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^2 C_n\left(\alpha^a,\beta^b\right)T_aT_b, \tag{3.8}$$

where $C_n\left(\alpha^a,\beta^b\right)$ are bilinear differential operators. Here $\alpha\beta$ denotes the exterior product $\alpha\wedge\beta$ and we use this convention everywhere in our paper. We must impose also the condition that the star product (3.8) satisfies the property of associativity

$$\left(\alpha*\beta\right)*\gamma = \alpha*\left(\beta*\gamma\right), \tag{3.9}$$

which determines some constraints on the operators $C_n\left(\alpha^a,\beta^b\right)$. In Ref. [1] their expressions were obtained up to the second order in $\theta^{\mu\sigma}$. We admit that these results are also valid in our case of Lie valued differential forms with adequate definitions. They are

$$\begin{aligned} C_1\left(\alpha^a,\beta^b\right) &= \left\{\alpha^a,\beta^b\right\} = \\ &= \theta^{\mu\nu}\left[\nabla_\mu\alpha^a\nabla_\mu\beta^b + (-1)^{|\alpha|}\,\tilde{R}^{\mu\nu}\left(i_\mu\alpha^a\right)\left(i_\nu\beta^b\right)\right], \end{aligned} \tag{3.10}$$

$$\begin{aligned} C_2\left(\alpha^a,\beta^b\right) &= \frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\nabla_\rho\alpha^a\nabla_\nu\nabla_\sigma\beta^b + \frac{1}{3}\theta^{\mu\rho}\partial_\rho\theta^{\nu\sigma}\left(\nabla_\mu\nabla_\nu\alpha^a\nabla_\sigma\beta^b - \right.\\ &\left.-\nabla_\nu\alpha^a\nabla_\mu\nabla_\sigma\beta^b\right) - \frac{1}{2}\tilde{R}^{\mu\nu}\tilde{R}^{\rho\sigma}\left(i_\mu i_\rho\alpha^a\right)\left(i_\nu i_\sigma\beta^b\right) - \\ &-\frac{1}{3}\tilde{R}^{\mu\nu}\left(i_\nu\tilde{R}^{\rho\sigma}\right)\left[(-1)^{|\alpha|}\left(i_\mu i_\rho\alpha^a\right)\left(i_\sigma\beta^b\right) + \right.\\ &\left.+\left(i_\rho\alpha^a\right)\left(i_\mu i_\sigma\beta^b\right)\right] + (-1)^{|\alpha|}\theta^{\mu\nu}\tilde{R}^{\rho\sigma}\left(i_\rho i_\mu\alpha^a\right)\left(i_\sigma i_\nu\beta^b\right). \end{aligned} \tag{3.11}$$

It is important to observe that the operators $C_n\left(\alpha^a,\beta^b\right)$ have the generalized Moyal symmetry [1]

$$C_n\left(\alpha^a,\beta^b\right) = (-1)^{|\alpha||\beta|+n} C_n\left(\beta^b,\alpha^a\right). \tag{3.12}$$

Taking into account the graded structure of our Poisson algebra, we define star commutator of two Lie valued differential forms $\alpha = \alpha^a T_a$ and $\beta = \beta^b T_b$ by

$$[\alpha,\beta]_* = \alpha * \beta + (-1)^{|\alpha||\beta|}\beta * \alpha. \tag{3.13}$$

For example, if α and β are Lie valued 1-forms, we have

$$\begin{aligned} [\alpha,\beta]_* &= \alpha^a\beta^b[T_a,T_b] + \frac{i\hbar}{2}C_1\left(\alpha^a,\beta^b\right)\{T_a,T_b\} + \\ &+\left(\frac{i\hbar}{2}\right)^2 C_2\left(\alpha^a,\beta^b\right)[T_a,T_b] + \dots. \end{aligned} \tag{3.14}$$

This result shows that the star commutator of Lie valued differential forms do not close in general on Lie algebra but in its universal enveloping algebra. Exceptions are the unitary groups $U(n)$ whose Lie algebras coincide with their universal enveloping algebras. However, we can extend the above results to any symmetry algebra considering the associated Hopf algebra. The universal enveloping of any Lie algebra can be always organized as a Hopf algebra.

As an example, we suppose that the gauge group for gravitation is $GL(2,C)$ [4, 5] whose generators are $\left(\sigma_{ab},\gamma_5,1\right)$, where

$$\sigma_{ab} = -\frac{i}{4}\left[\gamma_a,\gamma_b\right], \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \tag{3.15}$$

with γ_a, a = 0, 1, 2, 3 denoting the Dirac matrices and 1 – the unit matrix. We must observe that it is not possible to choose the Lorentz or equivalent $SL(2,C)$

algebra as gauge symmetry because it no longer closes upon going to the non-commutative version of the standard gauge theory formulation. It is convenient to utilize Dirac spinor notation

$$\hat{\omega}_\mu = \frac{1}{2}\hat{\omega}_\mu^{ab}\sigma_{ab}, \quad \hat{e}_\mu = \hat{e}_\mu^a\gamma_a. \tag{3.16}$$

Then, the 1-form of the gauge gravitational field and gauge parameters can be written as

$$\hat{A}_\mu = \hat{\omega}_\mu + \hat{a}_\mu 1 + \mathrm{i}\hat{b}_\mu\gamma_5, \quad \hat{\Lambda} = \lambda + \alpha 1 + \mathrm{i}\beta\gamma_5, \tag{3.17}$$

where $\hat{\omega}_\mu$ denotes the gauge gravitational field, $\hat{a}_\mu$ and $\hat{b}_\mu$ are two $U(1)$ gauge potentials, and α and β are two infinitesimal parameters associated to the generators 1 and γ_5. In addition, following the Ref. [5], we introduce a complex vierbein matrix

$$\hat{E}_\mu = \hat{e}_\mu + \mathrm{i}\hat{f}_\mu, \quad \hat{f}_\mu = \hat{f}_\mu^a\gamma_5\gamma_a. \tag{3.18}$$

Then, we can write down a consistent set of noncommutative gauge transformations

$$\hat{\delta}\hat{A} = \mathrm{d}\hat{\Lambda} + \mathrm{i}\left[\hat{A}, \hat{\Lambda}\right]_*, \quad \hat{A} = \hat{A}_\mu \mathrm{d}x^\mu, \tag{3.19}$$

$$\hat{\delta}\hat{E} = \mathrm{i}\left[\hat{E}, \hat{\Lambda}\right]_*, \quad \hat{E} = \hat{E}_\mu \mathrm{d}x^\mu, \tag{3.20}$$

where we must use the definition (4.19) for the star commutator $[,]_*$. The transformations (3.19) – (3.20) lead to rather involved variations for the component fields $\hat{\omega}_\mu^{ab}, \hat{a}_\mu, \hat{b}_\mu, \hat{e}_\mu^a$ and $\hat{f}_\mu^a$, so we will not write here theirs explicit form.

We can construct then the noncommutative strength field using the usual definition

$$\hat{F} = \mathrm{d}\hat{A} - \frac{i}{2}\left[\hat{A}, \hat{A}\right]_*, \tag{3.21}$$

which is gauge covariant, i.e.

$$\hat{\delta}\hat{F} = \mathrm{i}\left[\hat{F}, \hat{\Lambda}\right]_*. \tag{3.22}$$

A noncommutative gauge invariant action under $GL(2,C)$ can be constructed using only the fields $\hat{A}$ and $\hat{E}$ [5]

$$\hat{S}_g = -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big(\left(\alpha_1 + \beta_1\gamma_5\right)\left(\hat{F}_{\mu\nu} * \hat{E}_\rho * \hat{E}_\sigma\right)\Big), \tag{3.23}$$

where α_1 and β_1 are two constants that have to be chosen such that in the commutative limit we obtain the results from gravity. The property of gauge invariance of the action $\hat{S}_g$ results if we consider the transformations (3.20) and (3.21) and take into account that the integral on the right hand side of (3.23) is cyclic in the Poisson limit [2]. Imposing then the variational principle $\hat{\delta}\hat{S}_g = 0$ with respect to the gauge fields $\hat{\omega}_\mu^{ab}$ and $\hat{e}_\mu^a$, we can obtain the noncommutative field equations of the gravitation.

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