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STATISTICAL APPROACH OF MODULATIONAL INSTABILITY IN CYLINDRICAL/SPHERICAL NLS EQUATION

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Abstract. The modulational (Benjamin-Feir) instability for the cylindrical and spherical nonlinear Schrödinger equation is studied using the statistical approach. A kinetic equation for a two-point correlation function is written and following the Wigner-Moyal approach an evolution equation for the Fourier transform of the correlation function is obtained. A linear stability analysis leads to an implicit integral dispersion relation. This is solved for several forms of the unperturbed correlation functions and the instability domains are determined. Unlike the ordinary NLS case these domains are time dependent and the total growth of the instability is calculated for a Lorentzian initial spectrum.

Key words: modulational instability, cylindrical and spherical NLS equations.

1. INTRODUCTION

The modulational instability – MI (also known as the Benjamin-Feir instability) is a phenomenon taking place when a (quasi)monochromatic wave is propagating through a weakly nonlinear medium. It was first discussed in the context of hydrodynamics by Benjamin and Feir [1, 2] and in plasma physics by Bespalov and Talanov [3]. Later on, this deterministic approach (DAMI) was developed by several authors, and now it is quite a common chapter in books on nonlinear science (see [4]-[9] and references therein). It represents the first stage of generation of robust coherent structures, solitary waves and solitons in completely integrable systems.

As an example let us consider the one-dimensional nonlinear Schrödinger equation (NLS eq.)

$$\mathrm{i}\frac{\partial\Psi}{\partial t}+\alpha\frac{\partial^2\Psi}{\partial x^2}+\beta\left|\Psi\right|^2\Psi=0. \tag{1}$$

It is a generic equation describing the time and space evolution of the amplitude of a narrow pulse propagating through a weakly nonlinear medium and it represents a well-known completely integrable system [7, 8, 10, 11]. It admits a plane wave solution $\Psi_S = a\exp\left[\mathrm{i}\left(kx-\omega t\right)\right]$ with an amplitude dependent dispersion relation $\omega\left(k,a\right)=\alpha k^2-\beta\left|a^2\right|$ (Stokes solution). Considering a slowly modulated solution

$$\Psi = a\left(1+\varphi\left(x,t\right)\right)\exp\left[\mathrm{i}\left(kx-\omega t\right)\right], \tag{2}$$

the slow modulation $\varphi\left(x,t\right)$ satisfies the linear evolution equation

$$\mathrm{i}\frac{\partial\varphi}{\partial t}+\alpha\frac{\partial^2\varphi}{\partial x^2}+2\mathrm{i}k\alpha\frac{\partial\varphi}{\partial x}+\beta\left|a\right|^2\left(\varphi+\varphi*\right)=0. \tag{3}$$

Looking for plane wave solutions

$$\varphi\left(x,t\right)=A\mathrm{e}^{\mathrm{i}\left(Qx-\Omega t\right)}+B^*\mathrm{e}^{\mathrm{i}\left(Qx-\Omega^* t\right)}, \tag{4}$$

where $*$ indicates complex conjugation and is a complex quantity (the instability is related to $\mathrm{Im}\,\Omega>0$) one obtains

$$\Omega = 2\alpha kQ+\mathrm{i}\left|\alpha\right|Q\sqrt{2\frac{\beta}{\alpha}\left|a\right|^2-Q^2} \tag{5}$$

and an instability develops if α,β have the same sign (the focusing case for NLS equation) and if

$$Q^2<2\frac{\beta}{\alpha}\left|a\right|^2 \tag{6}$$

i.e. in the long wave length limit.

Beside this deterministic treatment a complementary statistical approach (SAMI) was developed, emphasizing the wave-to-wave energy transfer due to weak nonlinear coupling [12]-[15]. In a seminal paper of Alber [16] a two-point correlation function was written for the complex amplitude of Davey and Stewartson equation and based on the Wigner transform method [17, 18] its linear stability was investigated. A similar approach was successfully applied in the theory or surface gravity waves [19, 20], wave propagation in nonstationary inhomogeneous plasma [21, 22], dynamics of charged particle beams in accelerators [23, 24], dynamics in Bosé-Einstein condensates [25], incoherent light propagation in nonlinear media [26]. Recently in a series of papers the SAMI was studied for several NLS type equations (discrete nonlinear lattices, derivative NLS equation, Manakov's system) [27-32].

Considering again the NLS equation (1) a kinetic equation for the two-point correlation function $\rho(x_1, x_2 t) = \langle \Psi(x_1, t)\Psi^*(x_2, t)\rangle$, ($\langle \ldots \rangle$ denotes an ensemble average) is found in the following way: write (1) for $\Psi(x_1, t)$ and multiply by $\Psi^*(x_2, t)$; write (1) for $\Psi^*(x_2, t)$ and multiply by $\Psi(x_1, t)$; add the two equations and take and ensemble average; decouple the four-point correlation function using a Gaussian approximation. Then, the following evolution equation for $\rho(x_1, x_2 t)$ is obtained

$$\mathrm{i}\frac{\partial \rho}{\partial t} + \alpha\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\rho + \beta\left[\overline{a}^2(x_1) - \overline{a}^2(x_2)\right]\rho = 0, \tag{7}$$

where

$$\overline{a}^2(x) = \langle \Psi(x)\Psi^*(x)\rangle$$

is the mean value of the squared amplitude. Using the Wigner-Moyal transform we introduce the center of mass coordinate $X = \frac{1}{2}(x_1 + x_2)$ and the relative coordinate $x = x_1 - x_2$ and make a Fourier transform with respect to the relative coordinate

$$F(k, X, t) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kx}\rho\left(X + \frac{x}{2}, X - \frac{x}{2}\right)\mathrm{d}x. \tag{8}$$

The Fourier transform F satisfies the equation

$$\frac{\partial F}{\partial t} + 2\alpha k\frac{\partial F}{\partial X} + 2\beta a^2(X)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)F = 0, \tag{9}$$

where the sin-operator is defined in terms of its Taylor expansion and the arrows give the direction of differentiation.

A linear stability analysis can be done assuming

$$F(k, X, t) = f_0(k) + \varepsilon F_1(k, X, t), \tag{10}$$

$$\overline{a}^2(X, t) = \overline{a}_0^2 + \varepsilon\overline{a}_1^2(X, t), \tag{11}$$

where

$$\overline{a}_0^2 = \int_{-\infty}^{+\infty} f_0(k)\,\mathrm{d}k, \quad \overline{a}_1^2(X, t) = \int_{-\infty}^{+\infty} F_1(k, X, t)\,\mathrm{d}k. \tag{12}$$

In (10) we considered that the two-point correlation function in the unperturbed state is time independent and depends only on the relative coordinate x.

The first order perturbation $F_1\left(k,X,t\right)$ satisfies the linearized equation

$$\frac{\partial F_1}{\partial t}+2\alpha k\frac{\partial F_1}{\partial X}+2\beta\overline{a}_1^2\left(X,t\right)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)f\left(k\right)_0=0. \tag{13}$$

Looking for plane wave solutions

$$\begin{gathered} F_1\left(k,X,t\right)=g\left(k\right)\exp\left[\mathrm{i}\left(QX-\Omega t\right)\right],\\ a_1\left(X,t\right)=G\exp\left[\mathrm{i}\left(QX-\Omega t\right)\right],\\ G=\int_{-\infty}^{+\infty}g\left(k\right)\mathrm{d}k, \end{gathered} \tag{14}$$

after few straightforward algebraic manipulations, the following integral form of the dispersion relation is obtained

$$1=\frac{\beta}{\alpha}\frac{1}{Q}\int_{-\infty}^{+\infty}\frac{f_0\left(k+\frac{Q}{2}\right)-f_0\left(k-\frac{Q}{2}\right)}{\frac{\Omega}{2\alpha Q}-k}\mathrm{d}k. \tag{15}$$

Different initial conditions for $f_0(k)$ may be considered, namely:

$$\begin{aligned} &\delta\text{ spectrum} && f_0\left(k\right)=\overline{a}_0^2\delta\left(k\right),\\ &\text{Lorentzian spectrum} && f_0\left(k\right)=\frac{\overline{a}_0^2}{\pi}\frac{p}{k^2+p^2},\\ &\text{Gaussian spectrum} && f_0\left(k\right)=\frac{\overline{a}_0^2}{\sqrt{2\pi\sigma}}\mathrm{e}^{-k^2/2\sigma^2}. \end{aligned} \tag{16}$$

We mention here only the final result in the case of the Lorentzian spectrum. Considering Ω purely imaginary $\Omega=\mathrm{i}\Omega_i$ and writing $\omega_i=\frac{\Omega_i}{2\alpha Q}$, we obtain

$$\omega_i=\sqrt{\frac{\beta}{2\alpha}\overline{a}_0^2-\frac{Q^2}{4}}-p \tag{17}$$

and the plane wave solution is unstable if $\left(\alpha,\beta\right)$ have the same sign and if

$$\frac{Q^2}{4}<\frac{\beta}{2\alpha}\overline{a}_0^2-p^2. \tag{18}$$

As $|a_0|$ is a small quantity, the instability region is restricted to the long wave length region and to small values of p, i.e. to long correlations in the unperturbed state.

In the next paragraphs both the deterministic and the statistical approach of modulational instability will be discussed for cylindrical and spherical NLS equations.

2. MODULATIONAL INSTABILITY FOR CYLINDRICAL AND SPHERICAL NLS EQUATION

Further on, the following NLS type equation will be considered

$$\mathrm{i}\frac{\partial \Phi}{\partial t}+\alpha\frac{\partial^2 \Phi}{\partial r^2}+\beta\left|\Phi\right|^2\phi+\mathrm{i}\frac{m}{2t}\Phi=0, \tag{19}$$

where $m = 1$ stands for the cylindrical case and $m = 2$ for the spherical one (here x and t are stretched variables introduced by the standard reductive perturbation technique). These equations model several processes in laboratory, space and astrophysical environments, especially in dust contaminated plasmas [33-36], and in fluid dynamics [37]. Instead of working with the quantity ϕ, it is convenient to make the transformation

$$\Phi=\frac{1}{t^{\frac{m}{2}}}\Psi \tag{20}$$

and then Ψ satisfies the equation

$$\mathrm{i}\frac{\partial \Psi}{\partial t}+\alpha\frac{\partial^2 \Psi}{\partial r^2}+\frac{\beta}{t^m}\left|\Psi\right|^2\Psi=0. \tag{21}$$

This is a NLS equation having β replaced by β/t^m. Therefore all the results found in the NLS case will also be valid for (21) if one uses β/t^m instead of β. A deterministic approach of the modulational instability of equation (19) was given by Xue and Land [34] with the result

$$\Omega_i=\left|\alpha\right|Q\sqrt{2\frac{\beta}{\partial}\frac{\left|a\right|^2}{t^m}-Q^2} \tag{22}$$

and it is easily seen that it is obtained from the NLS results (5) by replacing β with $\frac{\beta}{t^m}$. The instability is possible if (α,β) have the same sign (focusing condition for NLS equation) and if

$$Q^2 \le Q_C^2\left(t\right) = 2\frac{\beta}{\partial}\frac{\left|a\right|^2}{t^m}. \tag{23}$$

Unlike the NLS case, the instability growth will cease for a given Q if

$$t \ge t_{\max} = \left(2\frac{\alpha}{\beta}\frac{\left|a\right|^2}{Q^2}\right)^{1/m}. \tag{24}$$

This is a special feature of MI in cylindrical and spherical NLS equations, which does not exist in the 1-D case (ordinary NLS equation, m = 0). One can define a total growth of the modulation by exp(G) where ($\gamma = t/t_0$ – see details in [34])

$$G = \int_{t_0}^{t_{\max}} \Omega_i\left(t\right)\mathrm{d}t = \left|\alpha\right| Q^2 t_0 \int_1^{R^{1/m}} \sqrt{\frac{R}{\lambda^m} - 1}\,\mathrm{d}\gamma\,, \tag{25}$$

where $R = 2\frac{\beta}{\alpha}\frac{\left|a\right|^2}{Q^2}\frac{1}{t_0^m} > 1$. The integration is straightforward both for the cylindrical (m = 1) and spherical (m = 2) geometries, and as a general conclusion the spherical waves are more structurally stable to perturbations than the cylindrical ones [34].

In the statistical approach of modulational instability for cylindrical and spherical NLS equations we follow the same steps as in the case of the onedimensional one. As mentioned before the results will have the same form, but with β replaced by β/t^m. Then the integral form of the stability equation will be

$$1 = \frac{\beta}{\alpha}\frac{1}{Q}\frac{1}{t^m}\int_{-\infty}^{+\infty} \frac{f\left(k + \frac{Q}{2}\right) - f\left(k - \frac{Q}{2}\right)}{\frac{\Omega}{2\alpha q} - k}\mathrm{d}k = 0. \tag{26}$$

This will be solved for the different initial conditions (16).

2.1. δ-SPECTRUM

$$f_0\left(k\right) = \bar{a}_0^2\delta\left(k\right), \quad \rho_0\left(x\right) = \bar{a}_0^2.$$

Assuming Ω purely imaginary, $\Omega = \mathrm{i}\Omega_i$ we find

$$\Omega_i = |\alpha| Q \sqrt{4 \frac{\alpha}{\beta} \frac{\overline{a}_0^2}{t^m} - Q^2}\,, \tag{27}$$

which is of the same form as the DAMI result (22) if we make the correspondence D.A.M.I.

$$\begin{array}{ccc} \text{D.A.M.I.} & & \text{S.A.M.I.} \\ |a|^2 & \leftrightarrow & 2\overline{a}_0^2 \end{array} \tag{28}$$

All the conclusions given in the previous paragraph remain valid also in this case.

2.2. LORENTZIAN SPECTRUM

$$f_0(k) = \frac{a_0^2}{\pi} \frac{p}{k^2 + p^2}, \quad \rho_0(x) = \overline{a}_0^2 \mathrm{e}^{-p|x|}.$$

In the real space this situation corresponds to a decreasing exponential initial correlation function, $\rho_0(x)$. The final results will be

$$\Omega_i = 2|\alpha| Q \left[\sqrt{\frac{\alpha}{\beta} \frac{\overline{a}_0^2}{t^m} - \frac{Q^2}{4}} - p \right] \tag{29}$$

and as in SAMI case the instability exists if (α, β) have the same sign, and if Q is in the long wave length limit, namely

$$\frac{Q^2}{4} < \frac{\overline{a}_0^2}{t^m} \frac{\alpha}{\beta} - p^2. \tag{30}$$

Also in this case, for given Q and p, we can define a maximum time value t_{max}

$$t_{\max} = \left(4 \frac{\alpha}{\beta} \frac{\overline{a}_0^2}{Q^2} \frac{1}{1 + 4p^2/Q^2} \right)^{1/m} \tag{31}$$

and the instability can exist only for $t < t_{\max}$. The total growth of the instability is given by the same relation (25), but with $\Omega_i(t)$ defined in (29) as well as $t_{\max}$ from (31). We get

$$G = |\alpha| Q^2 t_0 \left\{ \int_1^{t_{\max}/t_0} \sqrt{\frac{R}{\gamma^m} - 1}\, \mathrm{d}\gamma - \frac{2p}{Q} \left(\frac{t_{\max}}{t_0} - 1 \right) \right\},$$

where we denoted $R = 4\frac{\alpha}{\beta}\frac{\overline{a}_0^2}{Q^2}\frac{1}{t_0^m}$ and $\frac{t_{\max}}{t_0} = \left(\frac{R}{1+4p^2/Q^2}\right)^{1/m}$. The integration is easily done for the cylindrical case (m = 1) and for the spherical one (m = 2). For the cylindrical case one obtains

$$G = 4|\alpha|\frac{\alpha}{\beta}\overline{a}_0^2\left\{\left(\arctan\sqrt{R-1} - \frac{\sqrt{R-1}}{R}\right) - \left(\arctan\frac{2p}{Q} - \frac{\frac{2p}{Q}}{R}\right)\right\} \tag{32}$$

For $p \to 0$ the equation (32) transforms into the result obtained by Xue and Lang [34].

2.3. GAUSSIAN SPECTRUM

$$f_0(k) = \frac{\overline{a}_0^2}{\sqrt{2\pi}\sigma}\exp\left(-\frac{k_2}{2\sigma^2}\right),\quad \rho_0(x) = \overline{a}_0^2\exp\left(-\frac{\sigma^2 x^2}{2}\right).$$

It represents a more realistic assumption, in accordance with the Gaussian approximation used in the decoupling procedure. It is the Fourier transform of an initial Gaussian distribution in space $\rho_0(x)$.

The stability equation (26) writes

$$1 = \frac{2}{t^m}\frac{\alpha}{\beta}\frac{\overline{a}_0^2}{\sqrt{2\pi}\sigma Q}\int_{-\infty}^{+\infty} e^{-\xi^2}\left[\frac{1}{z-\xi} - \frac{1}{-z^*-\xi}\right]d\xi, \tag{33}$$

where $\xi = \frac{k}{\sqrt{2}\sigma}$ and $z = \frac{1}{2\sqrt{2}\sigma}\left(Q + i\frac{\Omega_i}{\alpha Q}\right)$ (we considered $\Omega = i\Omega_i > 0$). Using the complex integral function [38]

$$w(z) = \frac{i}{\pi}\int_{-\infty}^{+\infty} e^{-\xi^2}\frac{e^{-\xi^2}d\xi}{z-\xi} = \frac{2iz}{\pi}\int_0^{+\infty} e^{-\xi^2}\frac{e^{-\xi^2}d\xi}{z^2-\xi^2} = e^{-z^2}\operatorname{erfc}(-iz), \tag{34}$$

the relation (33) becomes

$$1 = \frac{2}{t^m}\frac{\beta}{\alpha}\frac{\sqrt{2\pi}\overline{a}_0^2}{Q\sigma}\operatorname{Im} w(z). \tag{35}$$

The function $w(z)$ is tabulated [38] for $x \in [0, 3.9]$ and $y \in [0, 3]$ where $z = x + \mathrm{i}y \quad \left(x = \frac{1}{2\sqrt{2}} \frac{Q}{\sigma}, \quad y = \frac{1}{2\sqrt{2}\alpha} \frac{\Omega_i}{\sigma Q} \right)$. For $\sigma \to 0$ and finite values of Q, Ω_i, we have $z \to \infty$ and one can use the asymptotic expansion of $w(z)$ [38]

$$w(z) \simeq \frac{\mathrm{i}}{\pi} \frac{1}{z} \left(1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \ldots \right) \tag{36}$$

and in leading order of z

$$\operatorname{Im} w(z) = \frac{1}{\sqrt{\pi}} \frac{x}{x^2 + y^2}. \tag{37}$$

Using this in (35) on immediately recovers the result (27) of a δ-spectrum. In Figure 1, Im$w(z)$ is represented for positive values of x and y ($z = x + \mathrm{i}y$).

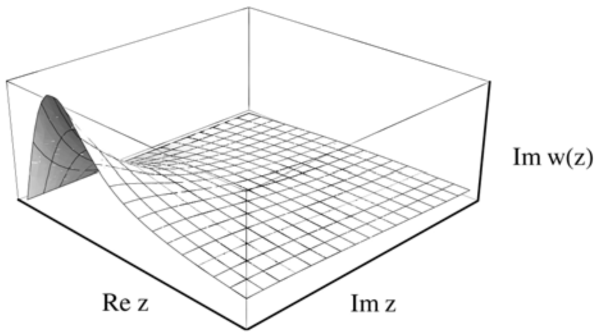

Fig. 1 – Im $w(z)$ as function of Rez and Imz.

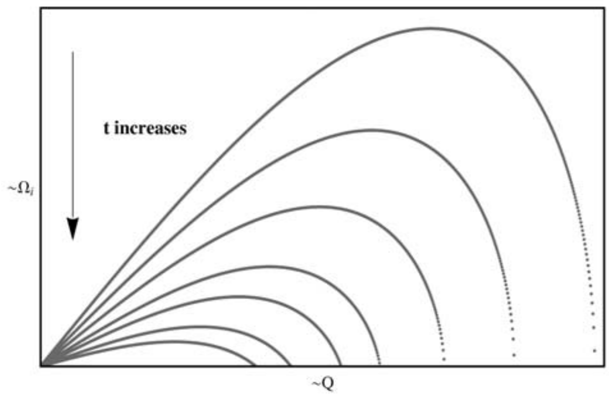

Fig. 2 –Ω_i i as function of Q for different values of t.

The numerically calculated Ω_i, function of Q, is represented in Fig. 2 for different values of t (t is increasing from upper to lower curves).

3. CONCLUSIONS

The problem of modulational instability for cylindrical and spherical NLS equations was discussed from deterministic and statistical point of view. Using the fact that both cNLS and sNLS are equivalent to an usual NLS equation if β is replaced by β / t^m, the results already known for the NLS case were immediately transposed for cNLS and sNLS equations. The integral stability equation of the SAMI analysis was solved for different initial distributions(δ spectrum, Lorentzian and Gaussian spectra) and in the case of a initial Lorentzian spectrum the total growth of the instability was calculated.

Although less studied in the literature, these extended NLS equations have been used lately to describe the evolution of a quasi-monochromatic pulse in weakly nonlinear media in restricted geometries. Recently a correspondence between a family of cNLS equations and cylindrical KdV ones, at least in the class of solitary waves, was established using a Madelung approach [39]. Several solitary type solutions (bright, dark and gray solitons, periodic solutions) can be found using this correspondence.

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