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SHORT NOTE

EFFECTIVE WAVE EQUATION FOR THE DYNAMICS OF HIGH-DENSITY DISK-SHAPED BOSE-EINSTEIN CONDENSATES

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Abstract. Starting from the three-dimensional (3D) Gross-Pitaevskii equation (GPE) and employing a q-Gaussian ansatz we derive a novel non-polynomial Schrödinger equation that models high-density disk-shaped Bose-Einstein condensates (BECs).

Key words: Bose-Einstein condensates, effective wave equation, q-Gaussian ansatz.

Due to their unprecedented experimental maneuverability [1] and to a solid theoretical framework [2, 3] BECs have become one of the most appealing systems for nonlinear scientists [4]. A close look at the literature shows that there is good quantitative agreement between the full GPE and experimental results on a wide range of topics which includes the dynamics of dark solitons inside magnetic traps [5, 6], the interactions of bright solitons [7], nonequilibrium oscillations in binary BECs [8], and Faraday waves in periodically driven BECs [9]. Despite this quantitative agreement, there are, however, very few effective wave equations for specific geometries [10, 11]. In this paper we derive an effective two-dimensional (2D) non-polynomial Schrödinger equation that models high-density disk-shaped BECs.

The starting point of our brief investigation is the 3D GPE which describes the $T = 0$ dynamics of the condensate, namely

$$\mathrm{i}\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})+gN\left|\psi(r,t)\right|^2\right]\psi(\mathbf{r},t), \tag{1}$$

where ψ represents the BEC wave function, N denotes the number of atoms, $g = 4\pi\hbar^2 a_s / m$ is proportional to a_s, the scattering length of the interatomic interactions, and m is the boson mass. The trapping potential is taken to be

$$V(\mathbf{r}) = \frac{1}{2} m\omega_\perp^2 r^2 + \frac{1}{2} m\omega_z^2 z^2, \tag{2}$$

where $r^2 = x^2 + y^2$. The 3D GPE equation can be derived through the least action principle from the following action functional:

$$S[\psi(\mathbf{r},t)] = \int \mathrm{d}t\mathrm{d}r \psi^*(\mathbf{r},t) \left[\mathrm{i}\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - V(\mathbf{r}) - \frac{1}{2} gN |\psi(\mathbf{r},t)|^2 \right] \psi(\mathbf{r},t).$$

Due to the two-dimensional nature of the condensate under scrutiny we can neglect the first term of $V(\mathbf{r})$ and decompose the original wave function as

$$\psi(\mathbf{r},t) = \phi(z,t; w(r,t), q(r,t)) f(r,t), \tag{3}$$

where ϕ and f represent the axial and transverse component of the wave function. Next, we consider a q-Gaussian ansatz for the axial component, namely

$$\phi(z,t;w,q) = c\left(1 - \frac{z^2(1-q)}{2w^2}\right)^{1/(1-q)}, \tag{4}$$

where both w and q are functions of r and t. This ansatz is extremely versatile and can describe both low- and high-density radial profiles (see Ref. [12] for an extensive discussion) as $q = 1$ corresponds to the usual Gaussian ansatz used by Salasnich *et al.* [10], while $q = -1$ recovers the Tomas-Fermi regime. Notice that in the vicinity of $q = -1$ our ansatz describes both the bulk part of the condensate and its surface, *i.e.*, the trial wave function is well-behaved at the surface of the condensate and does not require any cutoff. Finally, let us impose that the axial component of the ansatz is normalized to one, which then yields

$$c = \left(\frac{1-q}{2}\right)^{1/4} \left(w\mathbf{B}\left(\frac{1}{2}, \frac{q-3}{q-1}\right)\right)^{1/2}, \tag{5}$$

where $\mathbf{B}(\cdot,\cdot)$ is the usual beta function.

Assuming that ϕ is slowly varying along the transverse direction with respect to the axial direction, *i.e.*, $\nabla^2\phi \approx \partial^2\phi / \partial z^2$, and performing the integration on the z axis, the action functional simplifies to

$$S\left[f(r,t)\right]=\int \mathrm{d}t\mathrm{d}rf^{*}(r,t)\left[\mathrm{i}\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}-\frac{mw^2\omega_{\perp}^2}{7-3q}-\frac{gN\left|f(r,t)\right|^2}{2}\cdot\right.$$

$$\left.\cdot\frac{\sqrt{1-q}\,\mathbf{B}\left(\frac{1}{2},\frac{q-5}{q-1}\right)}{w\sqrt{2}\,\mathbf{B}\left(\frac{1}{2},\frac{q-3}{q-1}\right)^2}-\frac{\hbar^2}{m}\frac{U_2\left(\frac{1}{2},2,\frac{3}{2}-\frac{2}{q-1},1\right)}{w^2(3+q)}\right]f(r,t),$$

which can be accurately approximated in the high-density regime by

$$S\left[f(r,t)\right]\approx\int \mathrm{d}t\mathrm{d}rf^{*}(r,t)\left[\mathrm{i}\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}-\frac{mw^2\omega_{\perp}^2}{7-3q}+\right.$$
$$\left.+\frac{\hbar^2}{2m}\frac{1}{w^2}\left(\frac{1}{4}-\frac{3}{2(q+1)}\right)-\frac{gN\left|f(r,t)\right|^2}{2}\frac{a-b(q+1)}{w}\right]f(r,t). \tag{6}$$

This latter functional can be easily minimized through the Euler-Lagrange equations for $\left\{f,f^{*},a,q\right\}$. We then have

$$\mathrm{i}\hbar\frac{\partial f(r,t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}+gN\left|f(r,t)\right|^2\frac{a-b(q+1)}{w}-\right.$$
$$\left.-\frac{\hbar^2}{2m}\frac{1}{w^2}\left(\frac{1}{4}-\frac{3}{2(q+1)}\right)+\frac{mw^2\omega_{\perp}^2}{7-3q}\right]f(r,t) \tag{7}$$

for f^{*} (along with its complex conjugate for f),

$$\frac{gN|f|^2}{2}\frac{a-b(q+1)}{w^2}-\frac{\hbar^2}{mw^3}\left(\frac{1}{4}-\frac{3}{2(q+1)}\right)-\frac{2mw\omega_{\perp}^2}{7-3q}=0 \tag{8}$$

for w, and

$$\frac{gN|f|^2 b}{2\omega}+\frac{\hbar^2}{2mw^2}\frac{3}{2(q+1)^2}-\frac{3mw^2\omega_{\perp}^2}{(7-3q)^2}=0 \tag{9}$$

for q. While the last two algebraic equations cannot be solved analytically for an arbitrary value of N, it can easily shown that for $N\gg 1$ we have

$$q \approx -1 + \left(\frac{2^4 5^5 \hbar^2 \omega_\perp^2}{27 |f|^8 m^2 a^4 g^4 N^4} \right)^{1/9} \tag{10}$$

and

$$w \approx \left(\frac{5}{2} \frac{a |f|^2 gN}{m\omega_\perp^2} \right)^{1/3}, \tag{11}$$

which then yield

$$\begin{aligned} \mathrm{i}\hbar \frac{\partial f(r,t)}{\partial t} = \Bigg[& -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \left(\frac{m}{8}\right)^{1/3} \left(\frac{5|f|^2 ag\omega_\perp N}{2} \right)^{2/3} \\ & + (3a - 40b) \left(\frac{g|f|^2 Nh^3 \omega_\perp^4}{2} \right)^{2/9} \frac{m^{1/9}}{4 \cdot a^{7/9} 3^{1/3} 5^{7/9}} \Bigg] f(r,t), \end{aligned} \tag{12}$$

where we have neglected terms $\mathcal{O}\left(N^{-2/9}\right)$ and smaller. This novel high-density equation is the chief result of our investigation and will serve as starting point for a future study on the emergence of Faraday waves in dense pancake-shaped BECs.

Summarizing, we have introduced a novel two-dimensional non-polynomial Schrödinger equation specifically designed for high-density cigar-shaped condensates. The key ingredient of our recipe is a q-Gaussian ansatz for the transverse component of the wave function of the condensate which correctly accounts for the high density.

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