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GENERAL PHYSICS

IRREDUCIBLE ANALYSIS OF REDUCIBLE SECOND-CLASS CONSTRAINTS: THE EXAMPLE OF GAUGE-FIXED THREE- AND TWO-FORMS WITH STUECKELBERG COUPLING

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Abstract. The Dirac bracket for gauge-fixed three- and two-forms with Stueckelberg coupling is derived in an irreducible fashion. It is explicitly shown that the results emerging from the irreducible context coincide with those from the reducible treatment.

Key words: second class-constraints, Dirac bracket, Stueckelberg coupling.

1. INTRODUCTION

The canonical approach to systems with reducible second-class constraints represents a difficult problem. This is because not all the second-class constraint functions are independent, hence the matrix of the Poisson brackets among them is not invertible.

In order to construct the Dirac bracket for such systems in a consistent manner we have the following options: to isolate a set of independent constraint functions [1, 2] and then build the Dirac bracket in terms of this smaller set; to construct the Dirac bracket in terms of a non-invertible matrix without separating the independent constraint functions [3–8]; to substitute the reducible second-class constraints by some equivalent irreducible ones [by appropriately enlarging the original phase-space] and further work with the Dirac bracket based on the irreducible constraints [9, 10, 11]. The split of second-class constraints into independent and dependent constraint sets may spoil some important symmetries of the theory, so it is preferably to avoid this option. The second option, although interesting in principle, is merely a link between the first and the third option, as it will be seen in the body of the paper. By contrast, the third option has the main advantage of working only with irreducible constraints, so: i) there is no need to perform the split between dependent and independent constraints and, in consequence, the symmetries of the initial theory are preserved, and ii) the construction of the associated Dirac bracket is simpler than in the second situation.

In this paper we realize an irreducible approach to gauge-fixed two- and three-forms with Stueckelberg coupling [12] (for generalized Stueckelberg models, see [13]; also, see [14] for a nice review). This is a typical model, subject to second-order reducible second-class constraints. Our approach is based on three major steps. First, we express the Dirac bracket in terms of a non-invertible matrix and compute the associated fundamental Dirac brackets. Second, we show that the previous Dirac bracket can be constructed in terms of an invertible matrix, correlated with the above mentioned non-invertible one. The possibility of working with an invertible matrix within the Dirac bracket suggests an irreducible approach to the investigated model, but on a larger phase-space. Consequently, in the third step we construct an irreducible second-class constraint set in a larger phase-space and show that the fundamental Dirac brackets from the irreducible and reducible settings coincide. These three steps emphasize that we can approach gauge-fixed two- and three-forms with Stueckelberg coupling in an irreducible fashion. In this paper we mark the main ideas of the irreducible approach and then exemplify them on the model under study, but our approach can be applied to any model subject to second-order reducible second-class constraints.

Our paper is organized in six sections. In Section 2 we briefly address the building of the Dirac bracket for a generic second-order reducible second-class system and emphasize its basic properties. Section 3 introduces the model under investigation, namely, gauge-fixed two- and three-forms with Stueckelberg coupling. In Section 4 we apply the results from Section 2 and construct the fundamental Dirac brackets for the model under study. In Section 5 we give an approach of the above mentioned model based on some irreducible second-class constraints, but on a larger phase-space. In this context we show that the fundamental Dirac brackets from the irreducible and reducible settings coincide. Section 6 ends the paper with the main conclusions.

2. REDUCIBLE SECOND-CLASS CONSTRAINTS

In this section we briefly review the construction of the Dirac bracket for a second-order reducible second-class system in terms of a non-invertible matrix [11]. Our starting point is a system with the phase-space locally parameterized by N canonical pairs $z^a = \left(q^i, p_i\right)$ and subject to the second-order reducible constraints

$$\chi_{\alpha_0}\left(z^a\right) \approx 0, \qquad \alpha_0 = 1, \cdots, M_0, \tag{1}$$

$$Z^{\alpha_0}_{\ \alpha_1}\chi_{\alpha_0}=0,\qquad \alpha_1=1,\cdots,M_1, \tag{2}$$

$$Z^{\alpha_1}_{\ \alpha_2}Z^{\alpha_0}_{\ \alpha_1}\approx 0,\qquad \alpha_2=1,\cdots,M_2. \tag{3}$$

These constraints are purely second-class if any maximal, independent set of $M_0-M_1+M_2$ constraint functions χ_A among the χ_{α_0}'s is such that the matrix

$$C_{AB}=[\chi_A,\chi_B] \tag{4}$$

is invertible. In terms of such an independent set, the Dirac bracket takes the form

$$[F,G]^*=[F,G]-[F,\chi_A]M^{AB}[\chi_B,G], \tag{5}$$

where $M^{AB}C_{BC}\approx\delta^A_{\ C}$.

We can construct the Dirac bracket even without performing such a separation. We denote the matrix of the Poisson brackets among the second-class constraint functions by

$$C_{\alpha_0\beta_0}=[\chi_{\alpha_0},\chi_{\beta_0}]. \tag{6}$$

It is easy to see, on behalf of (2), that $C_{\alpha_0\beta_0}$ is not invertible

$$Z^{\alpha_0}_{\ \alpha_1}C_{\alpha_0\beta_0}\approx 0, \tag{7}$$

but has the rank equal to $M_0-M_1+M_2$.

Let $\bar{A}_{\alpha_1}^{\ \alpha_2}$ be a solution to the equation

$$Z^{\alpha_1}_{\ \beta_2}\bar{A}_{\alpha_1}^{\ \alpha_2}\approx\delta^{\alpha_2}_{\ \beta_2} \tag{8}$$

and $\bar{\omega}_{\beta_1\gamma_1}=-\bar{\omega}_{\gamma_1\beta_1}$ a solution to

$$Z^{\beta_1}_{\ \beta_2}\bar{\omega}_{\beta_1\gamma_1}\approx 0. \tag{9}$$

Then, we can introduce an antisymmetric matrix $\hat{\omega}^{\alpha_1\beta_1}$ through the relation

$$\hat{\omega}^{\alpha_1\beta_1}\bar{\omega}_{\beta_1\gamma_1}\approx\delta^{\alpha_1}_{\ \gamma_1}-Z^{\alpha_1}_{\ \alpha_2}\bar{A}_{\gamma_1}^{\ \alpha_2}\equiv D^{\alpha_1}_{\ \gamma_1}. \tag{10}$$

The solution to (8) can be set under the form

$$\bar{A}_{\alpha_1}^{\ \alpha_2}\approx\bar{D}^{\alpha_2}_{\ \lambda_2}A_{\alpha_1}^{\ \lambda_2}, \tag{11}$$

where $A_{\alpha_1}^{\ \ \lambda_2}$ are some functions chosen such that the matrix

$$D^{\lambda_2}_{\ \ \beta_2} = Z^{\alpha_1}_{\ \ \beta_2} A_{\alpha_1}^{\ \ \lambda_2}, \tag{12}$$

is of maximum rank

$$\operatorname{rank}\left(D^{\lambda_2}_{\ \ \beta_2}\right) = M_2, \tag{13}$$

and $\bar{D}^{\alpha_2}_{\ \ \lambda_2}$ is the inverse of $D^{\lambda_2}_{\ \ \beta_2}$. Then, by means of (10) we find that $D^{\alpha_1}_{\ \ \gamma_1}$ is expressed as

$$D^{\alpha_1}_{\ \ \gamma_1} \approx \delta^{\alpha_1}_{\ \ \gamma_1} - Z^{\alpha_1}_{\ \ \alpha_2} \bar{D}^{\alpha_2}_{\ \ \lambda_2} A_{\gamma_1}^{\ \ \lambda_2}. \tag{14}$$

On the other hand, simple computation shows that the matrix $D^{\alpha_1}_{\ \ \gamma_1}$ fulfills the relations

$$\bar{A}_{\alpha_1}^{\ \ \alpha_2} D^{\alpha_1}_{\ \ \gamma_1} \approx 0, \qquad Z^{\gamma_1}_{\ \ \gamma_2} D^{\alpha_1}_{\ \ \gamma_1} \approx 0, \tag{15}$$

$$Z^{\alpha_0}_{\ \ \alpha_1} D^{\alpha_1}_{\ \ \gamma_1} \approx Z^{\alpha_0}_{\ \ \gamma_1}, \qquad D^{\alpha_1}_{\ \ \gamma_1} D^{\gamma_1}_{\ \ \lambda_1} \approx D^{\alpha_1}_{\ \ \lambda_1}. \tag{16}$$

Based on the second relation from (15) we infer that $D^{\alpha_1}_{\ \ \gamma_1}$ can be alternatively put under the form

$$D^{\alpha_1}_{\ \ \gamma_1} \approx \bar{A}_{\alpha_0}^{\ \ \alpha_1} Z^{\alpha_0}_{\ \ \gamma_1}, \tag{17}$$

for some functions $\bar{A}_{\alpha_0}^{\ \ \alpha_1}$. From the former relation in (16) and (17) it follows that

$$Z^{\gamma_0}_{\ \ \gamma_1} D^{\alpha_0}_{\ \ \gamma_0} \approx 0, \tag{18}$$

where

$$D^{\alpha_0}_{\ \ \gamma_0} \approx \delta^{\alpha_0}_{\ \ \gamma_0} - Z^{\alpha_0}_{\ \ \alpha_1} \bar{A}_{\gamma_0}^{\ \ \alpha_1}. \tag{19}$$

At this stage we can rewrite the Dirac bracket given by (5) in terms of all the second-class constraints. Taking into account formulas (7) and (18) we can introduce another matrix $M^{\alpha_0\beta_0}$ through the relation

$$M^{\alpha_0\beta_0} C_{\beta_0\gamma_0} \approx D^{\alpha_0}_{\ \ \gamma_0}, \tag{20}$$

with $M^{\alpha_0\beta_0} = -M^{\beta_0\alpha_0}$, such that

$$[F,G]^* = [F,G] - [F,\chi_{\alpha_0}]M^{\alpha_0\beta_0}[\chi_{\beta_0},G], \tag{21}$$

defines the same Dirac bracket like (5) on the surface (1).

Meanwhile, relations (19)-(21) ensure that

$$[\chi_{\alpha_0},G]^* \approx -\bar{A}_{\alpha_0}^{\ \ \alpha_1}[Z^{\beta_0}_{\ \ \alpha_1},G]\chi_{\beta_0}, \tag{22}$$

which further yields $[\chi_{\alpha_0},G]^* = 0$ for any G on the original second-class constraint surface, as required by the general properties of the Dirac bracket.

At the same time, we remark that, in spite of the fact that the matrix $C_{\alpha_0\beta_0}$ is not invertible, the Dirac bracket expressed by (21) still satisfies Jacobi's identity

$$[[F,G]^*,P]^* + [[P,F]^*,G]^* + [[G,P]^*,F]^* \approx 0 \tag{23}$$

on surface (1).

3. THE MODEL

We consider the canonical approach to gauge-fixed three- and two-forms with Stueckelerg coupling, described by the Lagrangian action

$$S_0^L[A_{\mu\nu\rho},H_{\mu\nu}] = -\int \mathrm{d}^D x\left(\frac{1}{48}F_{\mu\nu\rho\lambda}F^{\mu\nu\rho\lambda} + \right.$$
$$\left. + \frac{1}{12}\left(F_{\mu\nu\rho} - MA_{\mu\nu\rho}\right)\left(F^{\mu\nu\rho} - MA^{\mu\nu\rho}\right)\right), \tag{24}$$

where

$$F_{\mu\nu\rho\lambda} = \partial_{[\mu}A_{\nu\rho\lambda]}, F_{\mu\nu\rho} = \partial_{[\mu}H_{\nu\rho]}, \tag{25}$$

and $D > 4$. Here and in the sequel the notation $[\mu\ldots\nu]$ signifies complete antisymmetry with respect to the indices between brackets, with the conventions that the minimum number of terms is always used and the result is never divided by the number of terms. The canonical analysis of this model leads to the first-class constraints

$$G^{(1)}_{i_1} \equiv \Pi_{0i_1} \approx 0, \qquad G^{(1)}_{i_1i_2} \equiv \pi_{0i_1i_2} \approx 0, \tag{26}$$

$$\chi^{(1)}_{i_1} \equiv -2\partial^k \Pi_{ki_1} \approx 0, \tag{27}$$

$$\chi^{(1)}_{i_1 i_2} \equiv -3\partial^k \pi_{ki_1 i_2} + M\Pi_{i_1 i_2} \approx 0, \tag{28}$$

where the momenta $\Pi_{\mu\nu}$ and $\pi_{\mu\nu\rho}$ are respectively conjugated to $H^{\mu\nu}$ and $A^{\mu\nu\rho}$. In order to fix the gauge, we have to choose a set of canonical gauge conditions. An appropriate set of such gauge conditions is given by

$$G^{(2)i_1} \equiv H^{0i_1} \approx 0, \qquad G^{(2)i_1 i_2} \equiv A^{0i_1 i_2} \approx 0, \tag{29}$$

$$\chi^{(2)i_1} \equiv -2\partial_k H^{ki_1} \approx 0, \tag{30}$$

$$\chi^{(2)i_1 i_2} \equiv -\partial_k A^{ki_1 i_2} - MH^{i_1 i_2} \approx 0. \tag{31}$$

(It is easy to see that (29)–(31) define some good gauge-fixing conditions for the original first-class constraints (26)–(28) by means of computing the gauge-fixed path integral of this model (using for instance the standard BRST method and observing that it leads to a correct result). The constraints (26)–(31) are second-class and, moreover, second-order reducible. It is simple to see that (26) and (29) generate a submatrix (of the matrix of the Poisson brackets among the constraint functions) of maximum rank, therefore they form a subset of irreducible second-class constraints, so they are not relevant in view of our approach. Thus in the sequel we examine only the constraints (27)–(28) and (30)–(31). For subsequent purposes we organize these constraints as

$$\chi_{\alpha_0} \equiv \begin{pmatrix} \chi^{(1)}_{i_1} & \chi^{(1)}_{i_1 i_2} & \chi^{(2)j_1} & \chi^{(2)j_1 j_2} \end{pmatrix} \approx 0. \tag{32}$$

The second-class constraint functions from (32) are second-order reducible, with the first- and respectively second-order reducibility functions given by

$$Z^{\alpha_0}_{\alpha_1} = \begin{pmatrix} -\partial^{i_1} & -M\,\delta^{i_1}_{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\delta^{[i_1}_{k_1}\partial^{i_2]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\partial_{j_1} & M\,\delta^{l_1}_{j_1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\delta^{l_1}_{[j_1}\partial_{j_2]} \end{pmatrix}, \tag{33}$$

$$Z^{\alpha_1}_{\ \alpha_2} = \begin{pmatrix} -M & 0 \\ \partial^{k_1} & \mathbf{0} \\ 0 & M \\ \mathbf{0} & \partial_{l_1} \end{pmatrix}. \tag{34}$$

In order to simplify the notations we employ, De Witt's condensed notations and omit the continuous indices, with the convention that *all* the derivatives act on the first continuous argument of the corresponding Dirac delta functions. For instance, ∂^{i_1} means $\partial/\partial x_{i_1} \delta^{D-1}(\mathbf{x}-\mathbf{y})$. The matrix of the Poisson brackets among the constraints (32) is expressed by

$$C_{\alpha_0\beta_0} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \lambda_{i_1}^{j_1} & \lambda_{i_1}^{j_1j_2} \\ \mathbf{0} & \mathbf{0} & \lambda_{i_1i_2}^{j_1} & \lambda_{i_1i_2}^{j_1j_2} \\ -\lambda_{j_1}^{i_1} & \lambda_{j_1j_2}^{i_1} & \mathbf{0} & \mathbf{0} \\ \lambda_{j_1}^{i_1i_2} & -\lambda_{j_1j_2}^{i_1i_2} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{35}$$

where

$$\lambda_{i_1}^{j_1} = 2\Delta D_{i_1}^{j_1}, \qquad \lambda_{i_1}^{j_1j_2} = M\delta_{i_1}^{[j_1}\partial^{j_2]},$$

$$\lambda_{i_1i_2}^{j_1j_2} = \left(\Delta + M^2\right)\hat{D}_{i_1i_2}^{j_1j_2},$$

with

$$D_{i_1}^{j_1} = \delta_{i_1}^{j_1} - \frac{\partial_{i_1}\partial^{j_1}}{\Delta}, \tag{36}$$

$$\hat{D}_{i_1i_2}^{j_1j_2} = \frac{1}{2}\left(\delta_{[i_1}^{j_1}\delta_{i_2]}^{j_2} - \frac{\delta_{[i_1}^{k_1}\partial_{i_2]}\delta_{k_1}^{[j_1}\partial^{j_2]}}{\Delta + M^2}\right), \tag{37}$$

and $\Delta = \partial^k\partial_k$.

4. “REDUCIBLE” DIRAC BRACKET

Now, we construct the Dirac bracket with respect to the constraints (32). If we take $A^{\alpha_2}_{\alpha_1}$ of the form

$$A^{\alpha_2}_{\alpha_1} = \begin{pmatrix} -M & \partial_{k_1} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & M & \partial^{l_1} \end{pmatrix}, \tag{38}$$

then we obtain for $D^{\alpha_2}_{\beta_2}$ as

$$D^{\alpha_2}_{\beta_2} = \begin{pmatrix} \Delta + M^2 & 0 \\ 0 & \Delta + M^2 \end{pmatrix}, \tag{39}$$

such that relation (13) is satisfied for the model under consideration. Thus, it results that

$$\bar{D}^{\alpha_2}_{\beta_2} = \begin{pmatrix} \frac{1}{\Delta + M^2} & 0 \\ 0 & \frac{1}{\Delta + M^2} \end{pmatrix}, \tag{40}$$

so from (14) we get

$$D^{\alpha_1}_{\beta_1} = \begin{pmatrix} \rho & \rho_{k_1} & 0 & \mathbf{0} \\ \rho^{i_1} & \hat{D}^{i_1}_{k_1} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & \rho & -\rho^{l_1} \\ \mathbf{0} & \mathbf{0} & -\rho_{j_1} & \hat{D}^{l_1}_{j_1} \end{pmatrix}, \tag{41}$$

where

$$\rho = \frac{\Delta}{\Delta + M^2}, \qquad \rho_{k_1} = \frac{M}{\Delta + M^2}\partial_{k_1}, \tag{42}$$

$$\hat{D}^{i_1}_{k_1} = \delta^{i_1}_{k_1} - \frac{\partial^{i_1}\partial_{k_1}}{\Delta + M^2}. \tag{43}$$

On the other hand, by means of relation (17) we can express $D^{\alpha_1}_{\beta_1}$ as in (41) if we take $\bar{A}^{\beta_1}_{\beta_0}$ of the form

$$\bar{A}_{\beta_0}^{\ \ \beta_1} = \begin{pmatrix} \frac{-1}{M}\rho_{k_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \rho_{k_1}^{i_1} & \rho_{k_1 k_2}^{i_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{M}\rho^{l_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\rho_{j_1}^{l_1} & \rho_{j_1}^{l_1 l_2} \end{pmatrix}, \tag{44}$$

with

$$\rho_{k_1}^{i_1} = -\frac{M}{\Delta + M^2}\delta_{k_1}^{i_1}, \qquad \rho_{k_1 k_2}^{i_1} = -\frac{1}{2\left(\Delta + M^2\right)}\delta_{[k_1}^{i_1}\partial_{k_2]}. \tag{45}$$

Then, on account of (19) we find

$$D_{\ \ \beta_0}^{\alpha_0} = \begin{pmatrix} \theta_{j_1}^{i_1} & \theta_{j_1 j_2}^{i_1} & \mathbf{0} & \mathbf{0} \\ 2\theta_{j_1}^{i_1 i_2} & \hat{D}_{j_1 j_2}^{i_1 i_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \theta_{i_1}^{j_1} & -\theta_{i_1}^{j_1 j_2} \\ \mathbf{0} & \mathbf{0} & -2\theta_{i_1 i_2}^{j_1} & \hat{D}_{i_1 i_2}^{j_1 j_2} \end{pmatrix}, \tag{46}$$

with

$$\theta_{j_1}^{i_1} = \frac{\Delta}{\Delta + M^2} D_{j_1}^{i_1}, \qquad \theta_{j_1 j_2}^{i_1} = -\frac{M}{2\left(\Delta + M^2\right)}\delta_{[j_1}^{i_1}\partial_{j_2]}. \tag{47}$$

Using (35) and (46) it follows that relation (20) is satisfied for

$$M^{\alpha_0 \beta_0} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & m_{k_1}^{i_1} & m_{k_1 k_2}^{i_1} \\ \mathbf{0} & \mathbf{0} & m_{k_1}^{i_1 i_2} & m_{k_1 k_2}^{i_1 i_2} \\ -m_{j_1}^{l_1} & m_{i_1}^{j_1 j_2} & \mathbf{0} & \mathbf{0} \\ m_{j_1 j_2}^{l_1} & -m_{j_1 j_2}^{l_1 l_2} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{48}$$

where

$$m_{k_1}^{i_1} = -\frac{\Delta}{2\left(\Delta + M^2\right)^2} D_{k_1}^{i_1}, \qquad m_{k_1 k_2}^{i_1} = \frac{M}{2\left(\Delta + M^2\right)^2}\delta_{[k_1}^{i_1}\partial_{k_2]}, \tag{49}$$

$$m^{i_1 i_2}_{k_1 k_2} = -\frac{1}{\Delta + M^2} \hat{D}^{i_1 i_2}_{k_1 k_2}. \tag{50}$$

With $M^{\alpha_0 \beta_0}$ at the hand, we can construct the Dirac bracket by means of formula (21). After some computation, we find that the only non-vanishing fundamental Dirac brackets are given by

$$[A^{i_1 i_2 i_3}(x), \pi_{j_1 j_2 j_3}(y)]^*_{x^0 = y^0} = \hat{D}^{i_1 i_2 i_3}_{j_1 j_2 j_3} \delta^{D-1}(\mathbf{x} - \mathbf{y}), \tag{51}$$

$$[H^{i_1 i_2}(x), \Pi_{j_1 j_2}(y)]^*_{x^0 = y^0} = \frac{\Delta}{\Delta + M^2} \hat{D}^{i_1 i_2}_{j_1 j_2} \delta^{D-1}(\mathbf{x} - \mathbf{y}), \tag{52}$$

$$[A^{i_1 i_2 i_3}(x), \Pi_{j_1 j_2}(y)]^*_{x^0 = y^0} = \frac{-M}{2\left(\Delta + M^2\right)} \delta^{[i_1}_{j_1} \delta^{i_2}_{j_2} \partial^{i_3]} \delta^{D-1}(\mathbf{x} - \mathbf{y}), \tag{53}$$

$$[H^{i_1 i_2}(x), \pi_{j_1 j_2 j_3}(y)]^*_{x^0 = y^0} = \frac{-M}{3!\left(\Delta + M^2\right)} \delta^{i_1}_{[j_1} \delta^{i_2}_{j_2} \partial_{j_3]} \delta^{D-1}(\mathbf{x} - \mathbf{y}), \tag{54}$$

where we made the notation

$$\hat{D}^{i_1 i_2 i_3}_{j_1 j_2 j_3} = \frac{1}{3!}\left(\delta^{i_1}_{[j_1} \delta^{i_2}_{j_2} \delta^{i_3}_{j_3]} - \frac{\delta^{[i_1}_{k_1} \delta^{i_2}_{k_2} \partial^{i_3]} \delta^{k_1}_{[j_1} \delta^{k_2}_{j_2} \partial_{j_3]}}{2\left(\Delta + M^2\right)} \right). \tag{55}$$

In this way, the reducible Dirac analysis of this model is complete.

5. IRREDUCIBLE ANALYSIS

In this section we develop an irreducible analysis for the investigated model.

Initially, we investigate the problem of constructing the Dirac bracket in the original phase-space in terms of an invertible matrix. In this sense, we remark that $D^{\alpha_0}_{\ \beta_0}$ given in (19) is a projector

$$D^{\alpha_0}_{\ \beta_0} D^{\beta_0}_{\ \gamma_0} = D^{\alpha_0}_{\ \gamma_0}. \tag{56}$$

Thus, if the equations

$$M^{\alpha_0 \beta_0} = D^{\alpha_0}_{\ \gamma_0} \mu^{\gamma_0 \delta_0} D^{\beta_0}_{\ \delta_0} \tag{57}$$

possess solutions for $\mu^{\gamma_0 \delta_0}$ an invertible matrix, then, on behalf of relations (20), (21), and (57), it follows that the Dirac bracket (21) can be replaced with

$$[F,G]^{*}=[F,G]-[F,\chi_{\alpha_0}]\mu^{\alpha_0\beta_0}[\chi_{\beta_0},G]. \tag{58}$$

The brackets (21) and (58) coincide on the second-class constraint surface (1). In the case of our model, where the matrices $D^{\alpha_0}_{\beta_0}$ and $M^{\alpha_0\beta_0}$ are expressed by (46) and respectively (48), the solution to equations (57) exists and is given by

$$\mu^{\gamma_0\delta_0}=\begin{pmatrix} \mathbf{0} & \mathbf{0} & n^{i_1}_{k_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & n^{i_1i_2}_{k_1k_2} \\ -n^{l_1}_{j_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -n^{l_1l_2}_{j_1j_2} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{59}$$

with

$$n^{i_1}_{k_1}=-\frac{1}{2(\Delta+M^2)}\delta^{i_1}_{k_1},\qquad n^{i_1i_2}_{k_1k_2}=-\frac{1}{2(\Delta+M^2)}\delta^{i_1}_{[k_1}\delta^{i_2}_{k_2]}.$$

It is simple to see that the inverse of (59) reads as

$$\mu_{\delta_0\lambda_0}=\begin{pmatrix} \mathbf{0} & \mathbf{0} & \hat{n}^{p_1}_{l_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 & \hat{n}^{p_1p_2}_{l_1l_2} \\ -\hat{n}^{k_1}_{m_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\hat{n}^{k_1k_2}_{m_1m_2} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{60}$$

where

$$\hat{n}^{p_1}_{l_1}=2(\Delta+M^2)\delta^{p_1}_{l_1},\qquad \hat{n}^{p_1p_2}_{l_1l_2}=\frac{1}{2}(\Delta+M^2)\delta^{p_1}_{[l_1}\delta^{p_2}_{l_2]}.$$

The presence of the invertible matrix $\mu^{\alpha_0\beta_0}$ in (58) suggests an irreducible approach, but on a larger phase-space [11]. In view of this, we introduce some new variables $(y_{\alpha_1})_{\alpha_1=1,\cdots,M_1}$ with the Poisson brackets

$$[y_{\alpha_1},y_{\beta_1}]=\omega_{\alpha_1\beta_1}, \tag{61}$$

where $\omega_{\alpha_1\beta_1}$ is antisymmetric and invertible. Now, we consider the system subject to the constraints

$$\tilde{\chi}_{\alpha_0} \equiv \chi_{\alpha_0} + A_{\alpha_0}^{\alpha_1} y_{\alpha_1} \approx 0, \tag{62}$$

$$\tilde{\chi}_{\alpha_2} \equiv Z^{\alpha_1}_{\alpha_2} y_{\alpha_1} \approx 0, \tag{63}$$

with $A_{\alpha_0}^{\alpha_1}$ some functions of the original phase-space variables. Both $\omega_{\alpha_1\beta_1}$ and $A_{\alpha_0}^{\alpha_1}$ must be taken such that the constraint set defined by (62) and (63) is second-order and irreducible. In the case of the model under consideration the supplementary variables read as

$$y_{\alpha_1} = (p \quad P_{i_1} \quad \varphi \quad B^{j_1}), \tag{64}$$

where φ is a scalar field, B^{j_1} a vector field, and p together with P_{i_1} respectively denote their conjugated momenta. Consequently, the matrix $\omega_{\alpha_1\beta_1}$ is of the form

$$\omega_{\alpha_1\beta_1} = \begin{pmatrix} 0 & \mathbf{0} & -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\delta^{j_1}_{i_1} \\ 1 & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \delta^{i_1}_{j_1} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{65}$$

If we take the functions $A_{\alpha_0}^{\alpha_1}$ as

$$A_{\alpha_0}^{\alpha_1} = \begin{pmatrix} \partial_{k_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ M\delta^{i_1}_{k_1} & \frac{1}{2}\delta^{i_1}_{[k_1}\partial_{k_2]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2\partial^{l_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -2M\delta^{l_1}_{j_1} & \delta^{[l_1}_{j_1}\partial^{l_2]} \end{pmatrix}, \tag{66}$$

then (62)–(63) become

$$\tilde{\chi}_{\alpha_0} \equiv \left(\tilde{\chi}^{(1)}_{i_1} \quad \tilde{\chi}^{(1)}_{i_1 i_2} \quad \tilde{\chi}^{(2)j_1} \quad \tilde{\chi}^{(2)j_1 j_2}\right) \approx 0, \tag{67}$$

$$\tilde{\chi}_{\alpha_2} \equiv \left(\tilde{\chi}^{(1)} \quad \tilde{\chi}^{(2)}\right) \approx 0, \tag{68}$$

with

$$\tilde{\chi}_{i_1}^{(1)} = 2\partial^k \Pi_{ki_1} - \partial_{i_1} p + MP_{i_1}, \tag{69}$$

$$\tilde{\chi}_{i_1 i_2}^{(1)} = -3\partial^k \pi_{ki_1 i_2} + M\Pi_{i_1 i_2} + \frac{1}{2}\partial_{[i_1} P_{i_2]}, \tag{70}$$

$$\tilde{\chi}^{(2)j_1} = -2\partial_l H^{lj_1} - 2\partial^{j_1}\varphi - 2MB^{j_1}, \tag{71}$$

$$\tilde{\chi}^{(2)j_1 j_2} = -\partial_l A^{lj_1 j_2} - MH^{j_1 j_2} + \partial^{[j_1} B^{j_2]}, \tag{72}$$

$$\tilde{\chi}^{(1)} = -Mp - \partial^k P_k, \tag{73}$$

$$\tilde{\chi}^{(2)} = M\varphi - \partial_l B^l. \tag{74}$$

By using the collective notation $\tilde{\chi}_\Delta = (\tilde{\chi}_{\alpha_0}, \tilde{\chi}_{\alpha_2})$, we find that the matrix $C_{\Delta\Delta'} = [\tilde{\chi}_\Delta, \tilde{\chi}_{\Delta'}]$ decomposes as

$$C_{\Delta\Delta'} = \begin{pmatrix} \mu_{\alpha_0\beta_0} & A_{\alpha_0}^{\ \ \alpha_1} \omega_{\alpha_1\beta_1} Z^{\beta_1}_{\ \ \beta_2} \\ Z^{\alpha_1}_{\ \ \alpha_2} \omega_{\alpha_1\beta_1} A^{\ \ \beta_1}_{\beta_0} & Z^{\alpha_1}_{\ \ \alpha_2} \omega_{\alpha_1\beta_1} Z^{\beta_1}_{\ \ \beta_2} \end{pmatrix}. \tag{75}$$

Matrix (75) is invertible, with the inverse

$$C^{\Delta'\Delta''} = \begin{pmatrix} \mu^{\beta_0\rho_0} & Z^{\beta_0}_{\ \ \gamma_1} \hat{e}^{\gamma_1}_{\ \ \sigma_1} \omega^{\sigma_1\lambda_1} A_{\lambda_1}^{\ \ \tau_2} \bar{D}^{\rho_2}_{\ \ \tau_2} \\ \bar{D}^{\beta_2}_{\ \ \lambda_2} A_{\sigma_1}^{\ \ \lambda_2} \omega^{\sigma_1\lambda_1} \hat{e}^{\gamma_1}_{\ \ \lambda_1} Z^{\rho_0}_{\ \ \gamma_1} & \bar{D}^{\beta_2}_{\ \ \lambda_2} A_{\sigma_1}^{\ \ \lambda_2} \omega^{\sigma_1\lambda_1} A_{\lambda_1}^{\ \ \tau_2} \bar{D}^{\rho_2}_{\ \ \tau_2} \end{pmatrix}, \tag{76}$$

where

$$\hat{e}^{\beta_1}_{\ \ \gamma_1} = \frac{1}{\Delta + M^2} \begin{pmatrix} -1 & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & -\delta^{k_1}_{m_1} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & -\frac{1}{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2}\delta^{n_1}_{l_1} \end{pmatrix}. \tag{77}$$

The invertibility of (75) emphasizes that (67)–(68) define an irreducible set of second-class constraints. The Dirac bracket built with respect to these irreducible constraints reads as

$$\begin{aligned}
[F,G]^{*}\Big|_{\mathrm{ired}} &= [F,G]-[F,\tilde{\chi}_{\alpha_0}]\mu^{\alpha_0\beta_0}[\tilde{\chi}_{\beta_0},G]- \\
&-[F,\tilde{\chi}_{\alpha_0}]Z^{\alpha_0}_{\ \ \gamma_1}\hat{e}^{\gamma_1}_{\ \ \sigma_1}\omega^{\sigma_1\lambda_1}A_{\lambda_1}^{\ \ \tau_2}\bar{D}^{\beta_2}_{\ \ \tau_2}[\tilde{\chi}_{\beta_2},G]- \\
&-[F,\tilde{\chi}_{\alpha_2}]\bar{D}^{\alpha_2}_{\ \ \lambda_2}A_{\sigma_1}^{\ \ \lambda_2}\omega^{\sigma_1\lambda_1}\hat{e}^{\gamma_1}_{\ \ \lambda_1}Z^{\beta_0}_{\ \ \gamma_1}[\tilde{\chi}_{\beta_0},G]- \\
&-[F,\tilde{\chi}_{\alpha_2}]\bar{D}^{\alpha_2}_{\ \ \lambda_2}A_{\sigma_1}^{\ \ \lambda_2}\omega^{\sigma_1\lambda_1}A_{\lambda_1}^{\ \ \tau_2}\bar{D}^{\beta_2}_{\ \ \tau_2}[\tilde{\chi}_{\beta_2},G].
\end{aligned} \tag{78}$$

If we compute the Dirac brackets among the original field/momenta on behalf of (78), we reobtain the same fundamental non-vanishing Dirac brackets like in the reducible situation, namely, (51)–(54). Thus, for any functions depending on the original phase-space variables we have that

$$[F,G]^{*} \approx [F,G]^{*}\Big|_{\mathrm{ired}}. \tag{79}$$

Relations (79) stand for the main result of our paper and enable us to state that an equivalent manner of constructing the Dirac bracket with respect to the second-order reducible second-class constraints of gauge-fixed three- and two-forms with Stueckelberg coupling is to transform them into some irreducible second-class ones and compute the Dirac bracket in terms of the irreducible set. Moreover, we obtain that

$$[\varphi,F]^{*}\Big|_{\mathrm{ired}} = 0 = [p,F]^{*}\Big|_{\mathrm{ired}}, \tag{80}$$

$$[B^{j_1},F]^{*}\Big|_{\mathrm{ired}} = 0 = [P_{i_1},F]^{*}\Big|_{\mathrm{ired}}. \tag{81}$$

Based on (79), it results that the equations of motion arising from the reducible and irreducible settings coincide in the sector of original field/momenta. Related to the newly introduced variables, from (80)-(81) we have that their equations of motion are trivial, i.e.,

$$\dot{\varphi}=0, \qquad \dot{p}=0, \qquad \dot{B}^{j_1}=0, \qquad \dot{P}_{i_1}=0, \tag{82}$$

which lead to $\varphi=0=p$, $B^{j_1}=0=P_{i_1}$ by taking some convenient boundary conditions (vacuum to vacuum) with respect to these non-physical variables.

6. CONCLUSION

To conclude with, in this paper we developed an irreducible approach to gauge-fixed two- and three-forms with Stueckelberg coupling. The main feature of this model is that it is subject to some second-order reducible second-class constraints. Our results emerge after three main steps. In the first step we write the original Dirac bracket in terms of a non-invertible matrix and compute the associated fundamental Dirac brackets. The second step relates the previous Dirac bracket to an equivalent expression, but constructed in terms of an invertible matrix. Based on this result, in the third step we generate an irreducible second-class constraint set on a larger phase-space and show that the fundamental Dirac brackets from the irreducible and reducible settings indeed coincide. All these steps underline that gauge-fixed two- and three-forms with Stueckelberg coupling can be tackled in an irreducible fashion. The generalization of the procedure developed so far to the case of Stueckelberg-coupled p- and $(p+1)$-forms is under consideration.

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