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HOMOGENIZATION RESULTS FOR HYPERBOLIC-PARABOLIC EQUATIONS*

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Abstract. The asymptotic behavior of the solution of a hyperbolic-parabolic equation with nonlinear sources and suitable boundary and initial conditions, defined in a perforated medium, is analyzed. We prove that the effective behavior of the solution of such a problem is governed by a parabolic equation, defined on a nonperforated domain.

Key words: homogenization, perforated domain, changing-type equation.

1. INTRODUCTION AND SETTING OF THE PROBLEM

The aim of this paper is to study the asymptotic behavior of the solution of a boundary-value problem modeling various phenomena arising, for instance, in electricity, magnetism, in the theory of elasticity, in vibrations theory or in hydrodynamics (see [10] and [11] and the references therein).

Let Ω be an open bounded set in $\mathbb{R}^n\left(n\geq 2\right)$ and let us perforate it by removing, with period ε, a number of holes of characteristic size ε. As a result, we get a perforated domain Ω^ε, ε being a small parameter related to the characteristic size of the perforations.

If we denote by (0, *T*) the time interval of interest, we shall analyze the asymptotic behaviour, as $\varepsilon \to 0$, of the solution of the following problem:

$$\varepsilon\frac{\partial^2 u^\varepsilon}{\partial t^2}+\alpha^\varepsilon\frac{\partial u^\varepsilon}{\partial t}-D_0\Delta u^\varepsilon+g\left(u^\varepsilon\right)=f\left(t,x\right),\quad \text{in}\quad \Omega^\varepsilon\times\left(0,T\right), \tag{1.1}$$

$$\frac{\partial u^\varepsilon}{\partial \nu}=0,\quad \text{on } S^\varepsilon\times\left(0,T\right), \tag{1.2}$$

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$$u^{\varepsilon}(0,x)=u_0^{\varepsilon}(x), \quad \text{in} \quad \Omega^{\varepsilon}, \tag{1.3}$$

$$\frac{\partial u^{\varepsilon}}{\partial t}(0,x)=\mathrm{v}_0^{\varepsilon}(x), \quad \text{in} \quad \Omega^{\varepsilon}, \tag{1.4}$$

$$u^{\varepsilon}=0, \quad \text{on} \quad \partial\Omega\times(0,T). \tag{1.5}$$

Here, v is the exterior unit normal to Ω^{ε}, $f\in L^2\left(0,T;L^2(\Omega)\right)$, $\mathrm{v}_0^{\varepsilon}\in L^2(\Omega)$, $u_0^{\varepsilon}\in H_0^1(\Omega)$, $\alpha^{\varepsilon}\in L^{\infty}(\Omega)$, $\alpha^{\varepsilon}>0$, $D_0>0$ and S^{ε} is the boundary of the perforations. Also, we shall assume that the nonlinear function g is given (see Section 2).

The existence and uniqueness of a weak solution of problem (1.1–1.5) can be proven by applying the theory of nonlinear monotone problems. We shall be interested in getting the asymptotic behavior, when $\varepsilon\to 0$, of the solution of problem (1.1–1.5). Using a classical homogenization procedure, *i.e.* Tartar's method of oscillating test functions, coupled with monotonicity methods and results from the theory of semilinear problems, we can prove that the solution of problem (1.1–1.5), properly extended to the whole of Ω, converges to the unique solution of a new problem, defined all over the domain Ω and given by a new parabolic operator (see Section 2).

If we do not scale the second derivative with respect to time in (1.1) by ε, we get, at the macroscale, a wave equation with interior damping.

Also, let us mention that we can get similar results if the term $-D_0\Delta u^{\varepsilon}$ in (1.1) is replaced by a general strong elliptic operator $-\operatorname{div}\left(A^{\varepsilon}\nabla u^{\varepsilon}\right)$, where A^{ε} is Y-periodic and satisfies strong ellipticity conditions. The positive parameter ε will also define a length scale measuring how densely the inhomogeneities are distributed in Ω. In fact, this means to assume that we are dealing with heterogeneous media. From a mathematical point of view, we can consider the case of a general medium, having discontinuous properties, represented by a coercive periodic matrix with rapidly oscillating coefficients. The matrix A^{ε} is defined in terms of a symmetric matrix $A\in L_{\#}^{\infty}(\Omega)^{n\times n}$, whose entries are Y-periodic, bounded and measurable real functions. Here, we have used the symbol # to denote periodicity properties. We assume that for some $0<\delta<\gamma$,

$$\delta|\xi|^2\le A(y)\xi\cdot\xi\le\gamma|\xi|^2 \quad \forall\xi, \quad y\in\mathbb{R}^n,$$

and we denote by $A^{\varepsilon}(x)$ the value of $A(y)$ at the point $y=x/\varepsilon$, *i.e.*

$$A^{\varepsilon}(x)=A\left(\frac{x}{\varepsilon}\right).$$

For the corresponding homogenization result, see also Section 2, Remark 2.

Other nonlinear problems modeling various physical phenomena arising in other domains of mechanics, physics and technology can benefit from a similar effective medium approach (see, for instance, the monograph [7]).

The results of this paper constitute a generalization of some of the results contained in [8] and [9], by considering nonlinear source terms.

Also, we can treat in the same manner the case in which we consider that we have a boundary damping term (*i.e.* we deal with a dynamic boundary condition on the surface of the perforations) or the case in which we assume that we have a nonlinear interior damping, given by a suitable nonlinear function $h\left(u_t^{\varepsilon}\right)$. In this case, we have a damping-source interaction (see [16]).

Problems closed to this one have been considered by D. Cioranescu, P. Donato, F. Murat and E. Zuazua [4], D. Cioranescu, P. Donato and H.I. Ene [3], D. Cioranescu, P. Donato and R. Zaki [5], C. Conca, J.I. Dìaz and C. Timofte [6], M Amar, Dall'Aglio and F. Paronetto [1], C. Timofte [12-16].

The plan of the paper is as follows: in the second section we introduce some useful notations and assumptions and we give the main convergence result of this paper. For obtaining it, we need some preliminary results, which are given in Section 3. The last section is devoted to the proof of the convergence result.

2. NOTATION AND ASSUMPTIONS

Let Ω be a bounded connected open subset of $\mathbb{R}^n$ $(n \geq 2)$, with $\partial\Omega$ of class C^2 and let $[0, T]$ be the time interval of interest. Let $Y = [0, l_1] \times \ldots [0, l_n]$ be the representative cell in $\mathbb{R}^n$ and F an open subset of Y with boundary ∂F of class C^2, such that $\bar{F} \subset Y$.

We shall denote by $F(\varepsilon, \mathbf{k})$ the translated image of εF by the vector $\varepsilon\mathbf{kl}$, $\mathbf{k} \in \mathbb{Z}^n$, $\mathbf{kl} = (k_1 l_1, \ldots, k_n l_n)$:

$$F(\varepsilon, \mathbf{k}) = \varepsilon(\mathbf{kl} + F).$$

Also, we shall denote by F^{ε} the set of all the holes contained in Ω. So

$$F^{\varepsilon} = \bigcup_{\mathrm{k} \in \mathbb{Z}^n} \left\{ F(\varepsilon, \mathbf{k}) \middle| \bar{F}(\varepsilon, \mathbf{k}) \subset \Omega \right\}.$$

Let $\Omega^{\varepsilon} = \Omega \setminus \overline{F^{\varepsilon}}$. Hence, Ω^{ε} is a periodically perforated domain with holes of the same size as the period. Let us remark that the holes do not intersect the boundary $\partial\Omega$.

We shall use the following notations:

$$Y^* = Y \setminus \bar{F}, \tag{2.1}$$

$$\theta = \frac{|Y^*|}{|Y|}. \tag{2.2}$$

Also, we shall denote by χ^ε the characteristic function of the domain Ω^ε and throughout the paper, by C we shall denote a generic fixed strictly positive constant, whose value can change from line to line.

As already mentioned, we are interested in studying the asymptotic behavior, as $\varepsilon \to 0$, of the solution of the hyperbolic-parabolic problem (1.1–1.5).

We shall consider that the function g in (1.1) is continuous, monotonously non-decreasing and such that $g(0) = 0$. Moreover, we shall assume that there exist $C \geq 0$ and an exponent q such that

$$|g(\mathrm{v})| \leq C\left(1 + |\mathrm{v}|^q\right), \tag{2.3}$$

with $0 \leq q < n/(n-2)$ if $n \geq 3$ and $0 \leq q < +\infty$ if $n = 2$.

For the highly oscillating coefficient in front of the first derivative of u with respect to time we shall assume that $\alpha^\varepsilon \in L^\infty(\Omega)$ and $\alpha^\varepsilon > 0$ in Ω. Moreover, we assume that there exists $\alpha > 0$ such that

$$\alpha^\varepsilon \to \alpha \quad \text{strongly in } L^\infty(\Omega). \tag{2.4}$$

For the initial data, we assume that $u_0^\varepsilon \in H_0^1(\Omega)\, \mathrm{v}_0^\varepsilon \in L^2(\Omega)$. Also, we suppose that

$$u_0^\varepsilon \to u_0 \quad \text{weakly in } H_0^1(\Omega) \tag{2.5}$$

and

$$\mathrm{v}_0^\varepsilon \to {}_0 \quad \text{strongly in } L^2(\Omega). \tag{2.6}$$

Remark 1. The results of this paper will be obtained for the case $n \geq 3$. All of them are still valid, under our assumptions, in the case in which $n = 2$. Of course, for this case, $n/(n-2)$ has to be replaced by $+\infty$.

The existence and uniqueness of a weak solution of (1.1–1.5) can be settled by using the classical theory of semilinear monotone problems (see [2, 4, 8, 16]). As a result, we know that there exists a unique weak solution

$$u^\varepsilon \in L^\infty \rho(0, T; V_\varepsilon),$$

with

$$\frac{\partial u^{\varepsilon}}{\partial t} \in L^{\infty}\left(0,T;L^{2}\left(\Omega^{\varepsilon}\right)\right),$$

and

$$\frac{\partial^{2} u^{\varepsilon}}{\partial t^{2}} \in L^{2}\left(0,T;V_{\varepsilon}'\right).$$

Here,

$$V_{\varepsilon}=\left\{\mathrm{v}\in H^{1}\left(\Omega^{\varepsilon}\right)\middle|\mathrm{v}=0 \text{ on } \partial\Omega\right\}.$$

The main convergence result of this paper is given by the following theorem:

THEOREM 1. *One can construct an extension* $P^{\varepsilon}u^{\varepsilon}$ *of the solution* u^{ε} *of the problem* (1.1–1.5) *such that* $P^{\varepsilon}u^{\varepsilon} \rightharpoonup u$ *weakly in* $L^{2}\left(0,T;H_{0}^{1}\left(\Omega\right)\right)$, *where u is the unique solution of the following problem*:

$$\begin{cases} \alpha\dfrac{\partial u}{\partial t}-\displaystyle\sum_{i,j=1}^{n} q_{ij}\,\dfrac{\partial^{2} u}{\partial x_i \partial x_j}+g\left(u\right)=f \quad x\in\Omega, \quad t\in\left(0,T\right), \\ u=0 \quad x\in\partial\Omega, \quad t\in\left(0,T\right), \\ u\left(0,x\right)=u_0\left(x\right), \quad \dfrac{\partial u}{\partial t}\left(0,x\right)=\mathrm{v}_0 \quad x\in\Omega. \end{cases} \tag{2.7}$$

Here, $Q=\left(\left(q_{ij}\right)\right)$ is the homogenized matrix entries are defined by:

$$q_{ij}=D_0\left(\delta_{ij}+\frac{1}{\left|Y^{*}\right|}\int_{Y^{*}}\frac{\partial\chi_j}{\partial y_i}\,\mathrm{d}y\right), \tag{2.8}$$

in terms of the functions χ_i, $i=1,\dots,n$, solutions of the cell problems

$$\begin{cases} -\Delta\chi_i=0 \quad \text{in} \quad Y^{*}, \\ \dfrac{\partial\left(\chi_i+y_i\right)}{\partial\nu}=0 \quad \text{on} \quad \partial F, \\ \chi_{\mathrm{i}} \quad Y-\text{periodic}. \end{cases}$$

Thus, in the limit, when $\varepsilon \to 0$, we get a constant coefficient heat equation, with a Dirichlet boundary condition.

Let us notice that there exists a unique solution of the macromodel problem (2.7).

Remark 2. As mentioned in the Introduction, in the general case of an heterogeneous medium, it is not difficult to see that the limit problem is the following one:

$$\begin{cases} \alpha \dfrac{\partial u}{\partial t} - \operatorname{div}\left(A^0 \nabla u\right) + g\left(u\right) = f \quad x \in \Omega, \quad t \in \left(0, T\right), \\ u = 0 \quad x \in \partial\Omega, \quad t \in \left(0, T\right), \\ u\left(0, x\right) = u_0\left(x\right), \quad \dfrac{\partial u}{\partial t}\left(0, x\right) = \mathrm{v}_0 \quad x \in \Omega. \end{cases} \tag{2.9}$$

Here, $A^0 = \left(a_{ij}^0\right)$ is the classical homogenized matrix, whose entries are defined as follows:

$$a_{ij}^0 = \frac{1}{\left|Y^*\right|} \int_{Y^*} \left(a_{ij}\left(y\right) + a_{ik}\left(y\right) \frac{\partial \chi_j}{\partial y_k} \right) \mathrm{d}y,$$

in terms of the functions χ_i, $j = 1, \ldots, n,$ solutions of the cell problems

$$\begin{cases} -\operatorname{div}_y A\left(y\right)\left(D_y \chi_j + \mathbf{e}_j\right) = 0 \quad \text{in} \quad Y^*, \\ A\left(y\right)\left(D\chi_j + \mathbf{e}_j\right) \cdot \nu = 0 \quad \text{on} \quad \partial F, \\ \chi_j \in H^1_{\#Y}\left(Y^*\right), \quad \int_{Y^*} \chi_j = 0, \end{cases}$$

where $\mathbf{e}_i$, $1 \le i \le n$, are the elements of the canonical basis in $\mathbb{R}^n$. The constant matrix A^0 is symmetric and positive-definite.

In this limit problem, the periodic heterogeneous structure of our medium is reflected by the presence of the homogenized matrix A^0.

3. PRELIMINARY RESULTS

As already mentioned, there exists a unique solution for the nonlinear problem (1.1–1.5),

$$u^\varepsilon \in L^\infty\left(0, T; V_\varepsilon\right),$$

with

$$\frac{\partial u^\varepsilon}{\partial t} \in L^\infty\left(0, T; L^2\left(\Omega^\varepsilon\right)\right)$$

and

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} \in L^2\left(0,T;V'_\varepsilon\right).$$

In order to extend it to the whole of Ω, let us recall the following result (see [6] and [8]).

LEMMA 1. *There exists a linear continuous extension operator* $P^\varepsilon \in \mathcal{L}\left(L^2\left(0,T;H^1\left(\Omega^\varepsilon\right)\right),\ L^2\left(0,T;H^1\left(\Omega\right)\right)\right) \cap \mathcal{L}\left(L^2\left(\Omega^\varepsilon \times \left(0,T\right)\right);\ L^2\left(\Omega \times \left(0,T\right)\right)\right)$ *and a positive constant C, independent of* ε, *such that*:

$$P^\varepsilon \mathrm{v} = \mathrm{v} \textit{ for any } \mathrm{v} \textit{ defined on } \Omega^\varepsilon \times \left(0,T\right);$$

$$P^\varepsilon \mathrm{v}' = \left(P^\varepsilon \mathrm{v}\right)' \text{ in } \Omega \times \left(0,T\right);$$

$$\left\|P^\varepsilon \mathrm{v}\right\|_{L^2\left(\Omega\times\left(0,T\right)\right)} \le C\left\|\mathrm{v}\right\|_{L^2\left(\Omega^\varepsilon\times\left(0,T\right)\right)}, \textit{for any } \mathrm{v} \in L^2\left(\Omega^\varepsilon \times \left(0,T\right)\right)$$

$$\left\|P^\varepsilon \mathrm{v}'\right\|_{L^2\left(\Omega\times\left(0,T\right)\right)} \le C\left\|\mathrm{v}'\right\|_{L^2\left(\Omega^\varepsilon\times\left(0,T\right)\right)} \textit{ for any } \mathrm{v} \in L^2\left(0,T;H^1\left(\Omega^\varepsilon\right)\right);$$

$$\left\|\nabla P^\varepsilon \mathrm{v}\right\|_{L^2\left(0,T;\left(L^2(\Omega)\right)^n\right)} \le C\left\|\nabla \mathrm{v}\right\|_{L^2\left(0,T;\left(L^2(\Omega^\varepsilon)\right)^n\right)}$$

where by ' we denoted the first derivative of v *with respect to time*.

For getting the effective behavior of our solution u^ε, we have to pass to the limit in the variational formulation of problem (1.1–1.5) (see (4.1)), which contains a nonlinear source term. To this end, let us recall a result from [6].

Let *H* be a continuous function, monotonously non-decreasing and such that $H\left(\mathrm{v}\right)=0$ if and only if $\mathrm{v}=0$. We shall suppose that there exist a positive constant *C* and an exponent *q*, with $0 \le q < n/(n-2)$, such that

$$\left|H\left(\mathrm{v}\right)\right| \le C\left(1+\left|\mathrm{v}\right|^q\right).$$

One can prove that for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1\left(\Omega\right)$, we have

$$H\left(z^\varepsilon\right) \to H\left(z\right) \quad \text{strongly in} \quad L^{\bar{q}}\left(\Omega\right), \tag{3.1}$$

where

$$\bar{q} = \frac{2n}{q\left(n-2\right)+n}.$$

4. PROOF OF THE MAIN RESULT

Let us consider the variational formulation of problem (1.1)–(1.5):

$$\begin{aligned}&\varepsilon\int_0^T\int_{\Omega^\varepsilon}\frac{\partial^2 u^\varepsilon}{\partial t^2}\varphi\psi\mathrm{d}x\mathrm{d}t+\beta^\varepsilon\int_0^T\int_{\Omega^\varepsilon}\frac{\partial u^\varepsilon}{\partial t}\varphi\psi\mathrm{d}x\mathrm{d}t+\\&+D_0\int_0^T\int_{\Omega^\varepsilon}\nabla u^\varepsilon\cdot\nabla\varphi\psi\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\Omega^\varepsilon}g\varphi\psi\mathrm{d}x\mathrm{d}t=\int_0^T\int_{\Omega^\varepsilon}f\varphi\psi\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{4.1}$$

for any $\varphi\in C_0^\infty\left((0,T)\right)$ and $\psi\in C_0^\infty\left(\Omega\right)$.

By classical existence and uniqueness results, we know that there exists a unique weak solution of (4.1). Following a standard procedure, *i.e.* multiplying the equation (1.1) by $\left(u^\varepsilon\right)'$, integrating in time and using our assumptions on the data and Cauchy-Schwartz, Poincaré's and Young's inequalities, we can obtain suitable energy estimates, independent of ε, for our solution (see [6, 8, 9, 14, 16]). More precisely, we have:

$$\begin{cases}\left\|u^\varepsilon\right\|_{L^\infty\left(0,T;V_\varepsilon\right)}\le C,\\ \left\|\left(u^\varepsilon\right)'\right\|_{L^\infty\left(0,T;L^2\left(\Omega^\varepsilon\right)\right)}\le C.\end{cases}\tag{4.2}$$

Denoting by $P^\varepsilon u^\varepsilon$ the extension of u^ε given by Lemma 1, we can easily prove that $P^\varepsilon u^\varepsilon$ is bounded in $L^2\left(0,T;H_0^1\left(\Omega\right)\right)$ and $\dfrac{\partial P^\varepsilon u^\varepsilon}{\partial t}$ is bounded in $L^2\left(0,T;L^2\left(\Omega\right)\right)$ (see, for details, [6, 14, 16]). So, by passing to a subsequence, we have $P^\varepsilon u^\varepsilon\rightharpoonup u$ weakly in $L^2\left(0,T;H_0^1\left(\Omega\right)\right)$ and strongly in $L^2\left(0,T;L^2\left(\Omega\right)\right)$ and $\dfrac{\partial P^\varepsilon u^\varepsilon}{\partial t}\rightharpoonup\dfrac{\partial u}{\partial t}$ weakly in $L^2\left(0,T;L^2\left(\Omega\right)\right)$.

It is well-known by now how to pass to the limit, with ε → 0, in the linear terms of (4.1) defined on Ω^ε (see, for instance [6, 8, 15]). Also, recall that θ is the weak-⋆ limit in $L^\infty\left(\Omega\right)$ of χ^ε. Thus, we get:

$$\varepsilon\int_0^T\int_{\Omega^\varepsilon}\frac{\partial^2 u^\varepsilon}{\partial t^2}\varphi\psi\mathrm{d}x\mathrm{d}t\to 0,\tag{4.3}$$

$$\alpha^{\varepsilon}\int_0^T\int_{\Omega^{\varepsilon}}\frac{\partial u^{\varepsilon}}{\partial t}\varphi\psi\,\mathrm{d}x\mathrm{d}t \to \alpha\int_0^T\int_{\Omega}\theta\frac{\partial u}{\partial t}\varphi\psi\,\mathrm{d}x\mathrm{d}t, \tag{4.4}$$

$$D_0\int_0^T\int_{\Omega^{\varepsilon}}\nabla u^{\varepsilon}\cdot\nabla\varphi\psi\,\mathrm{d}x\mathrm{d}t \to \int_0^T\int_{\Omega}\theta Q\nabla u\cdot\nabla\varphi\psi\,\mathrm{d}x\mathrm{d}t, \tag{4.5}$$

$$\int_0^T\int_{\Omega^{\varepsilon}} f\varphi\psi\,\mathrm{d}x\mathrm{d}t \to \int_0^T\int_{\Omega}\theta f\varphi\psi\,\mathrm{d}x\mathrm{d}t. \tag{4.6}$$

Our goal now is to pass to the limit in the nonlinear term in (4.1). To this end, using, as already mentioned, a result from [6] (see (3.1)) and Lebesgue's convergence theorem, it is not difficult to see that

$$\int_0^T\int_{\Omega^{\varepsilon}} g\left(u^{\varepsilon}\right)\varphi\psi\,\mathrm{d}x\mathrm{d}t \to \int_0^T\int_{\Omega} g\left(u\right)\theta\varphi\psi\,\mathrm{d}x\mathrm{d}t\,.$$

Putting together (4.3)–(4.7), we can pass to the limit in all the terms in (4.1) and we obtain exactly the variational formulation of the limit problem (2.7).

Since, exactly like in [8], we can pass to the limit, with $\varepsilon \to 0$, in the initial conditions, we have

$$u\left(0,x\right)-u_0\left(x\right),\quad \frac{\partial u}{\partial t}\left(0,x\right)=\mathrm{v}_0 \quad x\in\Omega\,.$$

As u is uniquely determined, the whole sequence $P^{\varepsilon}u^{\varepsilon}$ converges to u and this completes the proof of Theorem 1.

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