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ON SOME EXACTLY SOLVABLE SCHRÖDINGER TYPE EQUATIONS*

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Abstract. Hypergeometric type operators are shape invariant, and a factorization into a product of first order differential operators can be explicitly described in the general case. Some additional shape invariant operators, directly related to certain Schrödinger type operators, are obtained by using a deformation of the operators occurring in this general factorization. The mathematical properties of the eigenvalues and eigenfunctions of the operators thus obtained depend on the values of parameters involved. We investigate the square integrability of eigenfunctions and the monotony of the eigenvalue sequence.

Key words: Schrödinger equation, hypergeometric type operators, shape invariance.

1. INTRODUCTION

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0\,, \tag{1}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively, and λ is a constant. These equations are usually called *equations of hypergeometric type* [9], and each of them can be reduced to the self-adjoint form

$$[\sigma(s)\rho(s)y'(s)]' + \lambda\rho(s)y(s) = 0, \tag{2}$$

by choosing a function ρ such that $(\sigma\rho)' = \tau\rho$. The equation (1) is usually considered on an interval (a,b), chosen such that $\lim\limits_{s\to a,b}\sigma(s)\rho(s) = 0$ and

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$\sigma(s)>0, \rho(s)>0$ for all $s\in(a,b)$. Since the form of the equation (1) is invariant under a change of variable $s\mapsto cs+d$, it is sufficient to analyze the cases presented in Table 1. Some restrictions are imposed on α and β in order that the interval (a,b) exist.

Table 1

The main cases

$\sigma(s)$	$\tau(s)$	$\rho(s)$	(a,b)	α,β
1	$\alpha s+\beta$	$e^{\alpha s^2/2+\beta s}$	$(-\infty,\infty)$	$\alpha<0$
s	$\alpha s+\beta$	$s^{\beta-1}e^{\alpha s}$	$(0,\infty)$	$\alpha<0, \beta>0$
$1-s^2$	$\alpha s+\beta$	$(1+s)^{-(\alpha-\beta)/2-1}(1-s)^{-(\alpha+\beta)/2-1}$	$(-1,1)$	$\alpha<\beta<-\alpha$
s^2-1	$\alpha s+\beta$	$(s+1)^{(\alpha-\beta)/2-1}(s-1)^{(\alpha+\beta)/2-1}$	$(1,\infty)$	$-\beta<\alpha<0$
s^2	$\alpha s+\beta$	$s^{\alpha-2}e^{-\beta/s}$	$(0,\infty)$	$\alpha<0, \beta>0$
s^2+1	$\alpha s+\beta$	$(1+s^2)^{\alpha/2-1}e^{\beta\arctan s}$	$(-\infty,\infty)$	$\alpha<0$

It is well-known [9] that for $\lambda=\lambda_l$, where $l\in\mathbb{N}$ and $\lambda_l=-\frac{\sigma''(s)}{2}l(l-1)-\tau'(s)l$ the equation (1) admits a polynomial solution $\Phi_l=\Phi_l^{(\alpha,\beta)}$ of at most l degree

$$\sigma(s)\Phi_l''+\tau(s)\Phi_l'+\lambda_l\Phi_l=0. \tag{3}$$

The function $\Phi_l(s)\sqrt{\rho(s)}$ is square integrable [2, 3, 9] on (a,b) and $0=\lambda_0<\lambda_1<\ldots<\lambda_l$ for any $l<\Lambda$, where $\Lambda=\infty$ for $\sigma''(s)\in\{0,-2\}$ and $\Lambda=(1-\alpha)/2$ for $\sigma''(s)=2$. The system of polynomials $\{\Phi_l\,|\,l<\Lambda\}$ is orthogonal with weight function $\rho(s)$ in (a,b).

2. FUNCTIONS OF HYPERGEOMETRIC TYPE

Let $l\in\mathbb{N}$, $l<\Lambda$, and let $m\in\{0,1,...,l\}$. If we differentiate (4) m times then we get

$$\sigma(s)\frac{d^{m+2}}{ds^{m+2}}\Phi_l+[\tau(s)+m\sigma'(s)]\frac{d^{m+1}}{ds^{m+1}}\Phi_l+(\lambda_l-\lambda_m)\frac{d^m}{ds^m}\Phi_l=0. \tag{4}$$

The equation obtained by multiplying this relation by $\sqrt{\sigma^m(s)}$ can be written as $H_m\Phi_{l,m}=\lambda_l\Phi_{l,m}$, where H_m is the differential operator

$$H_m = -\sigma(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \tau(s)\frac{\mathrm{d}}{\mathrm{d}s} + \frac{m(m-2)}{4}\frac{(\sigma'(s))^2}{\sigma(s)} + \\ + \frac{m\tau(s)}{2}\frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2}m(m-2)\sigma''(s) - m\tau'(s), \tag{5}$$

and the functions $\Phi_{l,m}(s) = \kappa^m(s)\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Phi_l(s)$ defined by using $\kappa(s) = \sqrt{\sigma(s)}$ are called the *associated special functions*. If $0 \le m \le l < \Lambda$ then $\Phi_{l,m}(s)\sqrt{\rho(s)}$ is square integrable [2, 3] on (a,b). One can prove that the functions $\Phi_{l,m}$ are related through the first order differential operators

$$A_m = \kappa(s)\frac{\mathrm{d}}{\mathrm{d}s} - m\kappa'(s) \\ A_m^+ = -\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s} - \frac{\tau(s)}{\kappa(s)} - (m-1)\kappa'(s), \tag{6}$$

namely, we have

$$A_m\Phi_{l,m} = \begin{cases} 0 & \text{for} \quad l = m \\ \Phi_{l,m+1} & \text{for} \quad m < l < \Lambda \end{cases} \\ A_m^+\Phi_{l,m+1} = (\lambda_l - \lambda_m)\Phi_{l,m} \quad \text{for} \quad 0 \le m < l < \Lambda, \tag{7}$$

and

$$\Phi_{l,m}(s) = \begin{cases} \kappa^l(s) & \text{for} \quad m = l \\ \frac{A_m^+}{\lambda_l - \lambda_m}\frac{A_{m+1}^+}{\lambda_l - \lambda_{m+1}}\cdots\frac{A_{l-1}^+}{\lambda_l - \lambda_{l-1}}\kappa^l(s) & \text{for} \quad 0 < m < l < \Lambda. \end{cases} \tag{8}$$

The operators H_m satisfy the intertwining relations [2, 7, 8]

$$H_m A_m^+ = A_m^+ H_{m+1} \quad A_m H_m = H_{m+1} A_m \tag{9}$$

and are shape invariant.

$$H_m - \lambda_m = A_m^+ A_m \quad H_{m+1} - \lambda_m = A_m A_m^+. \tag{10}$$

For each $m < \Lambda$, the functions $\Phi_{l,m}$ with $m \le l < \Lambda$ are orthogonal [2, 3] with weight function $\rho(s)$ in (a,b), and $\|\Phi_{l,m+1}\| = \sqrt{\lambda_l - \lambda_m}\,\|\Phi_{l,m}\|$.

The *normalized associated special functions* $\phi_{l,m} = \Phi_{l,m}/\|\Phi_{l,m}\|$ satisfy the relations

$$A_m\ \phi_{l,m} = \begin{cases} 0 & \text{for}\quad l=m \\ \sqrt{\lambda_l-\lambda_m}\ \ \phi_{l,m+1} & \text{for}\quad m<l<\Lambda \end{cases}$$
$$A_m^+\ \phi_{l,m+1} = \sqrt{\lambda_l-\lambda_m}\ \ \phi_{l,m}\ \ \text{for}\ \ 0\le m<l<\Lambda \tag{11}$$
$$\phi_{l,m} = \frac{A_m^+}{\sqrt{\lambda_l-\lambda_m}}\frac{A_{m+1}^+}{\sqrt{\lambda_l-\lambda_{m+1}}}\cdots\frac{A_{l-1}^+}{\sqrt{\lambda_l-\lambda_{l-1}}}\phi_{l,l}.$$

3. SHAPE INVARIANT OPERATORS RELATED TO H_m

Some additional shape invariant operators directly related to H_m can be obtained in the cases when α and β are such that there exists $k\in\mathbb{R}$ with $\rho(s)=\sigma^k(s)$ (Table 2).

Table 2

The cases when $\rho(s)$ is a power of $\sigma(s)$

$\sigma(s)$	$\tau(s)$	$\rho(s)$	k	(a,b)	α,β
s	β	$s^{\beta-1}$	$\beta-1$	$(0,\infty)$	$\beta>0$
$1-s^2$	αs	$(1-s^2)^{-\alpha/2-1}$	$-\frac{\alpha}{2}-1$	$(-1,1)$	$\alpha<0$
s^2-1	αs	$(s^2-1)^{\alpha/2-1}$	$\frac{\alpha}{2}-1$	$(1,\infty)$	$\alpha<0$
s^2	αs	$s^{\alpha-2}$	$\frac{\alpha}{2}-1$	$(0,\infty)$	$\alpha<0$
s^2+1	αs	$(s^2+1)^{\alpha/2-1}$	$\frac{\alpha}{2}-1$	$(-\infty,\infty)$	$\alpha<0$

From $(\sigma\rho)'=\tau\rho$ we get $\tau(s)=(k+1)\sigma'(s)=2(k+1)\kappa(s)\kappa'(s)$, and

$$A_m=\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s}-m\kappa'(s)\quad A_m^+=-\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s}-(2k+m+1)\kappa'(s). \tag{12}$$

For any constants ε_m the deformed operators $\tilde{A}_m^+=A_m^++\varepsilon_m$ and $\tilde{A}_m=A_m+\varepsilon_m$ which we can consider for any $m\in\mathbb{R}$ satisfy the relations

$$(A_m^++\varepsilon_m)(A_m+\varepsilon_m)=H_m-\lambda_m-\varepsilon_m(2m+2k+1)\kappa'(s)+\varepsilon_m^2$$
$$(A_m+\varepsilon_m)(A_m^++\varepsilon_m)=H_{m+1}-\lambda_m-\varepsilon_m(2m+2k+1)\kappa'(s)+\varepsilon_m^2.$$

If we choose $\varepsilon_m=\delta/(2m+2k+1)$ with δ an arbitrary constant, then the operator $\tilde{H}_m=H_m-\delta\frac{\mathrm{d}\kappa}{\mathrm{d}s}$ satisfy the intertwining relations [4, 5]

$$\tilde{A}_m \tilde{H}_m = \tilde{H}_{m+1} \tilde{A}_m ,\ \ \tilde{H}_m \tilde{A}_m^+ = \tilde{A}_m^+ \tilde{H}_{m+1} \tag{13}$$

and is shape invariant, namely, we have

$$\tilde{A}_m^+ \tilde{A}_m = \tilde{H}_m - \tilde{\lambda}_m ,\ \ \tilde{A}_m \tilde{A}_m^+ = \tilde{H}_{m+1} - \tilde{\lambda}_m , \tag{14}$$

where $\tilde{\lambda}_m = \lambda_m - \dfrac{\delta^2}{(2m+2k+1)^2}$. Following the analogy with (7) we consider the function $\tilde{\Phi}_{m,m}$ obtained, up to a multiplicative constant, by solving the equation $\tilde{A}_m \tilde{\Phi}_{m,m} = 0$,

$$\tilde{\Phi}_{m,m}(s) = \begin{cases} (\sqrt{s})^m \, \mathrm{e}^{-\frac{2\delta\sqrt{s}}{2m+2\beta-1}} & \text{if} \quad \sigma(s) = s \\ (\sqrt{1-s^2})^m \, \mathrm{e}^{-\frac{\delta \arcsin s}{2m-\alpha-1}} & \text{if} \quad \sigma(s) = 1-s^2 \\ (\sqrt{s^2-1})^m (s+\sqrt{s^2-1})^{-\frac{\delta}{2m+\alpha-1}} & \text{if} \quad \sigma(s) = s^2-1 \\ s^{m-\frac{\delta}{2m+\alpha-1}} & \text{if} \quad \sigma(s) = s^2 \\ (\sqrt{s^2+1})^m (s+\sqrt{s^2+1})^{-\frac{\delta}{2m+\alpha-1}} & \text{if} \quad \sigma(s) = s^2+1. \end{cases} \tag{15}$$

The mapping $m \mapsto \tilde{\lambda}_m$ is an increasing function on the set $\{m \mid \frac{\mathrm{d}}{\mathrm{d}m}\tilde{\lambda}_m > 0\}$.

The set $\mathsf{M} = \{\ m \mid \frac{\mathrm{d}}{\mathrm{d}m}\tilde{\lambda}_m > 0 \ \text{and} \ \int_a^b \tilde{\Phi}_{m,m}^2(s)\rho(s)\mathrm{d}s < \infty \ \}$ of all the values of m for which $\frac{\mathrm{d}}{\mathrm{d}m}\tilde{\lambda}_m > 0$ and $\tilde{\Phi}_{m,m}\sqrt{\rho}$ is square integrable on (a,b) is presented in Table 3.

Table 3

The set M

$\sigma(s)$	$\tau(s)$	M
s	β	$\begin{cases} \varnothing & \text{for} \quad \delta \le 0 \\ (-\beta+\frac{1}{2},\infty) & \text{for} \quad \delta > 0 \end{cases}$
$1-s^2$	αs	$\left(\frac{1+\alpha}{2},\infty\right)$ for any $\delta \in \mathbb{R}$

Table 3 (continued)

s^2-1	αs	$\begin{cases} \varnothing & \text{for} \quad \delta \le -\frac{1}{2} \\ \left(-\frac{\alpha}{2}, \frac{1-\alpha}{2} - \sqrt{-\frac{\delta}{2}}\right) & \text{for} \quad \delta \in (-\frac{1}{2}, 0] \\ \left(-\frac{\alpha}{2}, \frac{1-\alpha}{2} - \sqrt{\frac{\delta}{2}}\right) \cup \left(\frac{1-\alpha}{2}, \frac{1-\alpha}{2} + \sqrt{\frac{\delta}{2}}\right) & \text{for} \quad \delta \in (0, \frac{1}{2}) \\ \left(\frac{1-\alpha}{2}, \frac{1-\alpha}{2} + \sqrt{\frac{\delta}{2}}\right) & \text{for} \quad \delta \in [\frac{1}{2}, \infty) \end{cases}$
s^2	αs	$\varnothing$ for any $\delta \in \mathbb{R}$
s^2+1	αs	$\left(-\infty, \frac{1-\alpha}{2} - \sqrt{\frac{\lvert\delta\rvert}{2}}\right)$ for any $\delta \in \mathbb{R}$

If $l \in \mathrm{M}$ and $n \in \mathbb{N}$ are such that $\{l-n, l-n+1, ..., l\} \subset \mathrm{M}$ then for each $m \in \{l-n, l-n+1, ..., l-1\}$ the function

$$\tilde{\Phi}_{l,m} = \frac{\tilde{A}_m^+}{\tilde{\lambda}_l - \tilde{\lambda}_m} \frac{\tilde{A}_{m+1}^+}{\tilde{\lambda}_l - \tilde{\lambda}_{m+1}} \cdots \frac{\tilde{A}_{l-2}^+}{\tilde{\lambda}_l - \tilde{\lambda}_{l-2}} \frac{\tilde{A}_{l-1}^+}{\tilde{\lambda}_l - \tilde{\lambda}_{l-1}} \tilde{\Phi}_{l,l} \tag{16}$$

has the form

$$\tilde{\Phi}_{l,m} = \begin{cases} \sum_{j=0}^{l-m} c_j (\sqrt{s})^{l-j} \, \mathrm{e}^{-\frac{2\delta\sqrt{s}}{2l+2\beta-1}} & \text{if} \quad \sigma(s) = s \\ \sum_{j=0}^{l-m} c_j s^j (\sqrt{1-s^2})^{l-j} \, \mathrm{e}^{-\frac{\delta \arcsin s}{2l-\alpha-1}} & \text{if} \quad \sigma(s) = 1-s^2 \\ \sum_{j=0}^{l-m} c_j s^j (\sqrt{s^2-1})^{l-j} (s+\sqrt{s^2-1})^{-\frac{\delta}{2l+\alpha-1}} & \text{if} \quad \sigma(s) = s^2-1 \\ \sum_{j=0}^{l-m} c_j s^j (\sqrt{s^2+1})^{l-j} (s+\sqrt{s^2+1})^{-\frac{\delta}{2l+\alpha-1}} & \text{if} \quad \sigma(s) = s^2+1, \end{cases} \tag{17}$$

where c_j are real constants. If $l \in \mathrm{M}$ and $n \in \mathbb{N}$ are such that $\{l, l-1, ..., l-n\} \subset \mathrm{M}$, then the function $\tilde{\Phi}_{l,m}\sqrt{\rho}$ is square integrable for any

$m \in \{l, l-1, ..., l-n\}$. The definition (16) of $\tilde{\Phi}_{l,m}$ can be re-written as $\tilde{\Phi}_{l,m} = \frac{\tilde{A}_m^+}{\tilde{\lambda}_l - \tilde{\lambda}_m} \tilde{\Phi}_{l,m+1}$, whence $\tilde{A}_m^+ \tilde{\Phi}_{l,m+1} = (\tilde{\lambda}_l - \tilde{\lambda}_m) \tilde{\Phi}_{l,m}$.

$$\begin{array}{ccccccc}
\tilde{\Phi}_{l,l-3} & \xleftarrow{\tilde{A}_{l-3}^+} & \tilde{\Phi}_{l,l-2} & \xleftarrow{\tilde{A}_{l-2}^+} & \tilde{\Phi}_{l,l-1} & \xleftarrow{\tilde{A}_{l-1}^+} & \tilde{\Phi}_{l,l} \\
\tilde{\Phi}_{l-1,l-3} & \xleftarrow{\tilde{A}_{l-3}^+} & \tilde{\Phi}_{l-1,l-2} & \xleftarrow{\tilde{A}_{l-2}^+} & \tilde{\Phi}_{l-1,l-1} & & \\
\tilde{\Phi}_{l-2,l-3} & \xleftarrow{\tilde{A}_{l-3}^+} & \tilde{\Phi}_{l-2,l-2} & & & & \\
\tilde{\Phi}_{l-3,l-3} & & & & & &
\end{array}$$

Fig. 1 – The functions $\tilde{\Phi}_{l,m}$.

Since $\tilde{H}_l \tilde{\Phi}_{l,l} = (\tilde{A}_l^+ \tilde{A}_l + \tilde{\lambda}_l) \tilde{\Phi}_{l,l} = \tilde{\lambda}_l \tilde{\Phi}_{l,l}$ and

$$\tilde{H}_{m+1} \tilde{\Phi}_{l,m+1} = \tilde{\lambda}_l \tilde{\Phi}_{l,m+1} \Rightarrow \tilde{H}_m \tilde{\Phi}_{l,m} = \frac{\tilde{H}_m \tilde{A}_m^+}{\tilde{\lambda}_l - \tilde{\lambda}_m} \tilde{\Phi}_{l,m+1} = \frac{\tilde{A}_m^+ \tilde{H}_{m+1}}{\tilde{\lambda}_l - \tilde{\lambda}_m} \tilde{\Phi}_{l,m+1} = \tilde{\lambda}_l \tilde{\Phi}_{l,m},$$

we get by recurrence $\tilde{H}_m \tilde{\Phi}_{l,m} = \tilde{\lambda}_l \tilde{\Phi}_{l,m}$. We have

$$\tilde{A}_m \tilde{\Phi}_{l,m} = \frac{\tilde{A}_m \tilde{A}_m^+}{\tilde{\lambda}_l - \tilde{\lambda}_m} \tilde{\Phi}_{l,m+1} = \frac{\tilde{H}_{m+1} - \tilde{\lambda}_m}{\tilde{\lambda}_l - \tilde{\lambda}_m} \tilde{\Phi}_{l,m+1} = \tilde{\Phi}_{l,m+1}$$

that is, $\tilde{A}_m \tilde{\Phi}_{l,m} = \tilde{\Phi}_{l,m+1}$.

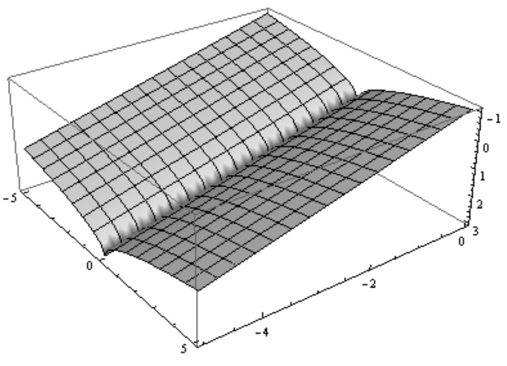

Fig. 2 – The boundary of M in the case $\sigma(s) = s^2 + 1$.

4. APPLICATION TO SCHRÖDINGER TYPE EQUATIONS

If we use in equation $\tilde{H}_m\tilde{\Phi}_{l,m}=\tilde{\lambda}_l\tilde{\Phi}_{l,m}$ a change of variable $(a',b')\to(a,b):x\mapsto s(x)$ such that $\mathrm{d}s/\mathrm{d}x=\pm\kappa(s(x))$ and define the new functions $\tilde{\Psi}_{l,m}(x)=\sqrt{\kappa(s(x))\rho(s(x))}\tilde{\Phi}_{l,m}(s(x))$ then we get an equation of Schrödinger type

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{\Psi}_{l,m}(x)+\tilde{\mathsf{V}}_m(x)\tilde{\Psi}_{l,m}(x)=\tilde{\lambda}_l\tilde{\Psi}_{l,m}(x). \tag{18}$$

The operators corresponding to $\tilde{A}_m$ and $\tilde{A}_m^+$ are

$$\begin{aligned}\tilde{\mathsf{A}}_m&=[\kappa(s)\rho(s)]^{1/2}\,\tilde{A}_m[\kappa(s)\rho(s)]^{-1/2}\,|_{s=s(x)}=\pm\frac{\mathrm{d}}{\mathrm{d}x}+\tilde{W}_m(x),\\ \tilde{\mathsf{A}}_m^+&=[\kappa(s)\rho(s)]^{1/2}\,\tilde{A}_m^+[\kappa(s)\rho(s)]^{-1/2}\,|_{s=s(x)}=\mp\frac{\mathrm{d}}{\mathrm{d}x}+\tilde{W}_m(x),\end{aligned} \tag{19}$$

where the *superpotential* $\tilde{W}_m(x)$ is given by the formula

$$\tilde{W}_m(x)=-\frac{\tau(s(x))}{2\kappa(s(x))}-\left(m-\frac{1}{2}\right)\frac{\mathrm{d}\kappa}{\mathrm{d}s}(s(x))+\frac{\delta}{2m+2k+1} \tag{20}$$

and $\tilde{\mathsf{V}}_m(x)=\tilde{W}_m^2(x)\mp\dot{\tilde{W}}_m(x)+\tilde{\lambda}_m$. Since

$$\int_{a'}^{b'}\tilde{\Psi}_{l,m}(x)\tilde{\Psi}_{k,m}(x)\mathrm{d}x=\int_a^b\tilde{\Phi}_{l,m}(s)\tilde{\Phi}_{k,m}(s)\rho(s)\mathrm{d}s$$

the functions $\tilde{\Psi}_{l,m}(x)$ are square integrable (resp. orthogonal) if and only if the corresponding functions $\tilde{\Phi}_{l,m}(s)$ are square integrable (resp. orthogonal).

Particular cases [1, 6, 8]. Let $\alpha_m=-(2m+\alpha-1)/2$ and $\alpha'_m=(2m-\alpha-1)/2$.

1. *Coulomb type potential.* In the case $\sigma(s)=s$, the change of variable $(0,\infty)\to(0,\infty):x\mapsto s(x)=x^2/4$ leads to

$$\begin{aligned}\tilde{W}_m(x)&=-\left(\beta+m-\frac{1}{2}\right)\frac{1}{x}+\frac{\delta}{2m+2\beta-1},\\ \tilde{\mathsf{V}}_m(x)&=\left(\beta+m-\frac{1}{2}\right)\left(\beta+m-\frac{3}{2}\right)\frac{1}{x^2}-\delta\frac{1}{x},\\ \tilde{\lambda}_m&=-\frac{\delta^2}{(2m+2\beta-1)^2}.\end{aligned} \tag{21}$$

2. *Trigonometric Rosen-Morse type potential.* In the case $\sigma(s) = 1 - s^2$, the change of variable $(0,\pi) \to (-1,1) : x \mapsto s(x) = \cos\ x$ leads to

$$\begin{aligned}
\tilde{W}_m(x) &= \alpha'_m\ \text{cotan}x + \frac{\delta}{2m-\alpha-1}, \\
\tilde{V}_m(x) &= \left(\alpha_m'^{\,2} - \alpha'_m\right)\text{cosec}^2 x + \delta\,\text{cotan}\,x - \alpha_m'^{\,2} + m(m-\alpha-1), \qquad (22)\\
\tilde{\lambda}_m &= m(m-\alpha-1) - \frac{\delta^2}{(2m-\alpha-1)^2}.
\end{aligned}$$

3. *Eckart type potential.* In the case $\sigma(s) = s^2 - 1$, the change of variable $(0,\infty) \to (1,\infty) : x \mapsto s(x) = \cosh x$ leads to

$$\begin{aligned}
\tilde{W}_m(x) &= \alpha_m \text{cotanh}x + \frac{\delta}{2m+\alpha-1}, \\
\tilde{V}_m(x) &= \left(\alpha_m^2 + \alpha_m\right)\ \text{cosech}^2 x - \delta\text{cotanh}x + \alpha_m^2 - m(m-\alpha-1), \qquad (23)\\
\tilde{\lambda}_m &= -m(m-\alpha-1) - \frac{\delta^2}{(2m+\alpha-1)^2}.
\end{aligned}$$

4. *Hyperbolic Rosen-Morse type potential.* In the case $\sigma(s) = s^2 + 1$, the change of variable $\mathbb{R} \to \mathbb{R} : x \to s(x) = \sinh x$ leads to

$$\begin{aligned}
\tilde{W}_m(x) &= \alpha_m \tanh x + \frac{\delta}{2m+\alpha-1}, \\
\tilde{V}_m(x) &= -\left(\alpha_m^2 + \alpha_m\right)\text{sech}^2 x - \delta\tanh x + \alpha_m^2 - m(m-\alpha-1), \qquad (24)\\
\tilde{\lambda}_m &= -m(m-\alpha-1) - \frac{\delta^2}{(2m+\alpha-1)^2}.
\end{aligned}$$

The exactly solvable Schrödinger type equations play an important role in quantum mechanics. In this paper, we have explored the properties of some additional shape invariant operators directly related to hypergeometric type operators. Particularly, we have identified new exactly solvable Schrödinger type equations satisfying the usual requirements concerning their eigenvalues and eigenfunctions. They may play a role in some future applications.

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