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ELASTIC WAVES PRODUCED BY LOCALIZED FORCES IN A SEMI-INFINITE BODY

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Abstract. The propagation of elastic waves in isotropic bodies is investigated by a new method, based on the electromagnetic Kirchhoff potentials. This method is applied herein to elastic waves produced in a semi-8 infinite (half-space) isotropic body by the action of an external force localized beneath, or on the body surface. The method leads to coupled integral equations for the wave amplitudes, which are solved for the both cases. The waves produced by a force localized beneath the surface are stationary waves along the normal to the surface. For a force localized on the body surface two transverse waves are identified, corresponding to the two polarizations (normal and parallel to the propagation plane). Another longitudinal wave appears as an eigen-mode. The surface displacement and the force exerted on the surface are computed in both cases. All these quantities exhibit a characteristic decrease with the distance on the body surface and an oscillatory behaviour. A brief discussion is included regarding some possibilities of extending the present method to treating the effect of the inhomogeneities on the waves propagation.

Key words: semi-infinite elastic body; elastic waves; localized forces; Kirchhoff potentials; normal modes.
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1. INTRODUCTION

The propagation of elastic waves in bodies with special, restricted geometries was originally investigated by Rayleigh [1] and Lamb [2], and underwent various developments during the time [3]. It is sometimes known as Lamb's problem in seismology [4]. The problem exhibits a certain complexity related to difficulties arising mainly from the lack of an adequate treatment of the boundary conditions. Of particular importance is the determination of the waves produced in such elastic bodies by external forces, either localized on the body surface, or within the bulk, or extended over certain spatial volumes. Even more interesting, and more difficult, is the problem of treating the effect of the inhomogeneities, either localized or extended, on the wave propagation infinite elastic bodies. Apart from their practical importance in engineering, such problems are of great relevance for the effect of the seismic waves on the Earth's surface [5–9].

A new method is presented here for studying the wave propagation in isotropic elastic bodies with a finite (or partially finite) structure, based on the Kirchhoff potentials of the wave equation with sources, borrowed from electromagnetism. This method is employed herein to determine the elastic waves produced in a semi-infinite (half-space) body by external forces localized either beneath, or on the body surface. For the force localized beneath the body surface the elastic waves are stationary waves along the direction perpendicular to the body surface. For the force localized on the surface we determine two transverse waves propagating in the body and a longitudinal one which appears as an eigenmode. The surface displacement and the force exerted on the surface are computed in both cases. All these quantities exhibit a characteristic decrease and an oscillatory behavior along the in-plane distance on the body surface. In both cases, the present method leads to coupled integral equations for the wave amplitudes, which are solved. By means of the method presented herein new results are obtained for a point-like force localized under the surface and one of Lamb's problem (force localized on the surface) is generalized. The generalization consists in treating a general distribution of the force acting on the body surface and a general orientation of this force. In addition, in both cases, the effects of a localized pressure are analyzed. Finally, a brief discussion is given regarding the extension of the present method to include the effect of the inhomogeneities on the wave propagation in elastic bodies with finite geometries.

2. GENERAL THEORY

The elastic waves in isotropic bodies are governed by the equation of motion [10]

$$\rho\ddot{\mathbf{u}} = \mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\cdot\mathrm{div}\mathbf{u} + \rho\mathbf{f}, \tag{1}$$

where ρ is the density, u is the displacement field, μ and λ are the Lame coefficients and f is an external force per unit mass (acceleration). By a Fourier transform of the form

$$\mathbf{u}(\mathbf{R},t) = \sum_{\mathbf{K}} \mathrm{d}\omega\mathbf{u}(\mathbf{K},\omega)\,\mathrm{e}^{\mathrm{i}\mathbf{KR}-\mathrm{i}\omega t} \tag{2}$$

and a similar one for the force $\mathbf{f}$, equation (1) becomes

$$\left(-\rho\omega^2 + \mu K^2\right)\mathbf{u} = -(\lambda+\mu)(\mathbf{Ku})\mathbf{K} + \rho\mathbf{f}, \tag{3}$$

where we dropped out the arguments $\mathbf{K},\omega$ for simplicity. Equation (3) can easily be solved. Its solution is given by

$$\mathbf{u} = -\frac{\left(\mathrm{v}_l^2 - \mathrm{v}_t^2\right)\left(\mathbf{Kf}\right)}{\left(\omega - \mathrm{v}_t^2 K^2\right)\left(\omega^2 - \mathrm{v}_l^2 K^2\right)}\mathbf{K} - \frac{\mathbf{f}}{\omega^2 - \mathrm{v}_t^2 K^2}, \tag{4}$$

where

$$\mathrm{v}_t = \sqrt{\frac{\mu}{\rho}}, \quad \mathrm{v}_l = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{5}$$

are the velocities of the transverse and, respectively, longitudinal waves. We can see from equation (4) that for a longitudinal force $\mathbf{f} = f\mathbf{K}/K$ f = fK=K the displacement field is longitudinal and has the eigenfrequencies $\omega = \mathrm{v}_l K$, while for a transverse force, $\mathbf{Kf} = 0$, the field is transverse and has the eigenfrequencies $\omega = \mathrm{v}_t K$. As it is well known, the Lame coefficients can be expressed by the Young modulus E and the Poisson ratio σ,

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)}, \tag{6}$$

and, for reasons of stability, $E > 0$ and $-1 < \sigma < 1/2$ (actually, for usual bodies, $0 < \sigma < 1/2$). In particular, the ratio

$$q = \frac{\mathrm{v}_l^2}{\mathrm{v}_t^2} - 1 = \frac{\lambda}{\mu} + 1 = \frac{1}{1-2\sigma} \tag{7}$$

satisfies the inequality $q > 1/3$ (actually $q > 1$) [10]. In general, the solution of the homogeneous equation (1) (“free waves”) must be added to the particular solution given by equation (4) (“forced waves”).

As it is well known, another, more direct, method can be used for solving equation (1), without resorting to Fourier transforms. The method consists in writing the displacement $\mathbf{u}$ as a sum of two functions, $\mathbf{u} = \mathbf{u}_t + \mathbf{u}_l$, satisfying the conditions $\mathrm{div}\mathbf{u}_t = 0$, as for transverse waves, and $\mathrm{curl}\mathbf{u}_l = 0$ corresponding to longitudinal waves. Then, it is easy to see that functions $\mathbf{u}_{t,l}$ satisfy the wave equations with velocities $\mathrm{v}_{t,l}$, respectively.

We present here a third method, which is used in the present paper, based on Kirchhoff potentials. Indeed, making use of the notations introduced above we write equation (1) as

$$\frac{1}{\mathrm{v}_t^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = q \cdot \mathrm{grad} \cdot \mathrm{div}\mathbf{u} + \frac{\mathbf{f}}{\mathrm{v}_\mathrm{t}^2}, \tag{8}$$

where we can recognize the wave equation with sources $q \cdot \mathrm{grad} \cdot \mathrm{div}\mathbf{u}$ and $\mathbf{f}/\mathrm{v}_\mathrm{t}^2$. As it is well known, its (particular) solution is given by the retarded (Kirchhoff) potential

$$\mathbf{u}\left(\mathbf{R},t\right)=\frac{q}{4\pi}\int \mathrm{d}\mathbf{R}'\,\frac{\mathrm{grad}\cdot\mathrm{div}u\left(\mathbf{R}',t-\left|\mathbf{R}-\mathbf{R}'\right|/\mathrm{v}_t\right)}{\left|\mathbf{R}-\mathbf{R}'\right|}+ \\ +\frac{1}{4\pi\mathrm{v}_t^2}\int \mathrm{d}\mathbf{R}'\,\frac{\mathbf{f}\left(\mathbf{R}',t-\left|\mathbf{R}-\mathbf{R}'\right|/\mathrm{v}_t\right)}{\left|\mathbf{R}-\mathbf{R}'\right|}. \tag{9}$$

Indeed, making use of the Fourier transform given by equation (2) and using also the well-known integral

$$\int \mathrm{d}\mathbf{R}'\,\frac{e^{\mathrm{i}\mathbf{KR}+\mathrm{i}\omega R/\mathrm{v}_t}}{R}=-\frac{4\pi\mathrm{v}_t^2}{\omega^2-\mathrm{v}_t^2K^2} \tag{10}$$

we get easily the solution given by equations (3) and (4). We apply herein this method of Kirchhoff potential, inspired from the theory of electromagnetism, [11] to the elastic waves generated in a semi-infinite body by forces localized either beneath, or on the body surface.

3. SEMI-INFINITE BODY. SURFACE WAVES

We consider a semi-infinite isotropic body extending over the region $z < 0$, with a plane surface at $z = 0$. We may consider also a force **f** acting on this body, either inside it or localized on its surface, and look for solutions of equation (1), *i.e.* for the waves propagating in the body. Of course, we assume that $\mathbf{u};\ \mathbf{f} = 0$ for $z > 0$ (outside the body). This is Lamb's problem [2]. Usually, it is assumed that force **f** is localized, either on the body surface or beneath it, and the problem is approached by making use of a remarkable property of equation (1).

Indeed, as it is well known, equation (1) is another form of the more compact equation written as

$$\rho\ddot{u}_i=\frac{\partial\sigma_{ij}}{\partial x_j}+\rho f_i, \tag{11}$$

where

$$\sigma_{ij}=\lambda u_{ll}\delta_{ij}+2\mu u_{ij} \tag{12}$$

is the stress tensor,

$$u_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{13}$$

is the strain tensor and i, j, l, etc, denote the coordinates axes. We take a small volume encircling a point on the body surface and apply the Gauss theorem for equation (11). It is easy to see that, for a force distributed in the body volume, we get

$$\sigma_{ij} n_j = 0, \tag{14}$$

for any point on the body surface, where n is the unit vector normal to the body surface. Equation (14), which plays the role of a boundary condition, tells that the body surface is free (from any forces). If there exists a force localized on the body surface, say $\mathbf{f}\delta(z)$, then it is easy to see that the same procedure leads to

$$\sigma_{ij} n_j + f_i = 0, \tag{15}$$

which tells again that the total force on the surface is vanishing. In addition, equation (15) tells that the quantity $\sigma_{ij} n_j$ is the surface elastic force (force per unit area) acting on the surface. It is also easy to see that the same procedure can be applied, at least in principle, to a force localized onto a point, a surface, region, etc inside the body. In all these cases we write solutions for the homogeneous equation (11) and impose upon them the continuity and the "jump" boundary condition of the type given by equation (15). This way, the general solution of equation (11) is obtained. This was the method used by Lamb in treating such problems [2]. Apart from less interesting two-dimensional problems, Lamb treated also particular cases of a force localized on the surface of a semi-infinite isotropic body and sketched an integral representation of the solution for a force localized beneath the surface of such a body.

It is worth noting also that the "free-surface" condition given by equation (14) has been used by Rayleigh [1] to identify the damped "free" surface waves $\left(\sim \mathrm{e}^{\kappa z}, \quad \kappa \text{ real}\right)$ propagating on the surface of a semi-infinite body in the absence of any force.

A different procedure is adopted in this paper. A particular solution is obtained by using Kirchhoff potential, as determined by the external force **f** ("forced waves"), and "free waves" of the form $\mathbf{u}_{t,l}$ are added to it, *i.e.* solutions of the homogeneous equation, in order to satisfy the boundary conditions of the type given by equations (14) and (15). We carry out explicit calculations for a point-like force localized beneath the body surface and for a point-like force localized on the body surface. As we shall see, the damped "free" surface waves are not excited by localized external forces, though they may be excited by surface damped forces.

4. FORCE LOCALIZED BENEATH THE SURFACE

We consider a force

$$\mathbf{f}(\mathbf{R}, t) = a^3 \mathbf{f}(t) \delta(\mathbf{R} - \mathbf{R}_0) \tag{16}$$

localized at depth d beneath the plane surface $z = 0$ of a semi-infinite elastic body extending to the region $z < 0$, such as $\mathbf{R}_0 = (0, 0, -d)$. The characteristic length a is, much smaller than the relevant distances, is introduced on one hand for reasons of dimensionality and, on the other, for having a representation of the spatial extension of the "focus" onto which the force acts. In equation (16) the position vector is given by $\mathbf{R} = (x, y, z) = (\mathbf{r}, z)$ and t denotes the time. The propagating spherical waves produced by this point-like force in an infinite body are well known (see for instance Ref. [9]). We derive here the waves produced by such a source in a semi-infinite body. We represent the displacement field as

$$\mathbf{u} = (\mathbf{v}, u_3)\theta(-z), \tag{17}$$

where $\mathbf{v}$ is the in-plane component (parallel to the surface), u_3 is the transverse component (perpendicular to the surface) and $\theta(z) = 0$ for $z < 0$, $\theta(z) = 1$ for $z > 0$ is the step function. We use Fourier transforms of the form

$$\mathbf{v}(\mathbf{r}, z; t) = \sum_{\mathbf{k}} \int \mathrm{d}\omega \mathbf{v}(\mathbf{k}, \omega; z) \mathrm{e}^{\mathrm{i}\mathbf{kr} - \mathrm{i}\omega t}, \tag{18}$$

and a similar one for $u_3(\mathbf{r}, z; t)$, where $\mathbf{k}$ is the in-plane wavevector. Usually, we leave aside the arguments $\mathbf{k}, \omega$, while preserving explicitly the z-dependence of the functions $\mathbf{v}(\mathbf{k}, \omega; z)$ and $u_3(\mathbf{k}, \omega; z)$. The divergence occurring in equation (9) can then be written as

$$\mathrm{div}\mathbf{u} = \left(\mathrm{div}\mathbf{v} + \frac{\partial u_3}{\partial z} \right) \theta(-z) - u_3(0)\delta(z), \tag{19}$$

where we can see the occurrence of specific surface contributions associated with $u_3(0) = u_3(z = 0)$.

We compute $\mathrm{grad} \cdot \mathrm{div}\mathbf{u}$ according to equations (18) and (19) and introduce it, together with the force given by equation (16), in equation (9). The intervening integrals reduce to the known integral [12]

$$\int_{|z|}^{\infty} \mathrm{d}x J_0 \left(k\sqrt{x^2 - z^2} \right) \mathrm{e}^{\mathrm{i}\omega x / c} = \frac{\mathrm{i}}{\kappa} \mathrm{e}^{\mathrm{i}\kappa |z|}, \tag{20}$$

where J_0 is the Bessel function of the first kind and zeroth order and

$$\kappa = \sqrt{\frac{\omega^2}{v_t^2} - k^2}. \tag{21}$$

In addition, we introduce the convenient notations $\mathrm{v}_1 = \mathbf{vk}/k, \mathrm{v}_2 = \mathbf{vk}_\perp/k$ and similar ones for $f_{1,2}$, where $\mathbf{k}_\perp$ is a vector perpendicular to $\mathbf{k}$, $\mathbf{kk}_\perp = 0$, and of the same magnitude k. The force term in equation (9), which we denote by $\mathbf{F}$, can easily be evaluated. Its Fourier transform is given by

$$\mathbf{F} = -\frac{a^3\mathbf{f}}{2\mathrm{v}_t^2\kappa}\sin\kappa\left|z+d\right|, \tag{22}$$

where $\kappa^2 = \omega^2/\mathrm{v}_t^2 - k^2 > 0$. Applying the procedure described above we get straightforwardly from equation (9)

$$\mathrm{v}_2 = F_2 = -\frac{a^3 f_2}{2\mathrm{v}_t^2\kappa}\sin\kappa\left|z+d\right| \tag{23}$$

and the set of coupled integral equations

$$\begin{aligned} \mathrm{v}_1 &= -\frac{\mathrm{i}qk^2}{2\kappa}\int^0 \mathrm{d}z'\mathrm{v}_1(z')\mathrm{e}^{\mathrm{i}\kappa|z-z'|} - \frac{qk}{2\kappa}\frac{\partial}{\partial z}\int^0 \mathrm{d}z'u_3(z')\mathrm{e}^{\mathrm{i}\kappa|z-z'|} + F_1, \\ u_3 &= -\frac{qk}{2\kappa}\frac{\partial}{\partial z}\int^0 \mathrm{d}z'u_1(z')\mathrm{e}^{\mathrm{i}\kappa|z-z'|} + \frac{\mathrm{i}k}{2\kappa}\frac{\partial^2}{\partial z^2}\int^0 \mathrm{d}z'u_3(z')\mathrm{e}^{\mathrm{i}\kappa|z-z'|} + F_3. \end{aligned} \tag{24}$$

In deriving these equations it is worth noting the non-invertibility of the derivatives and the integrals, according to the identity

$$\frac{\partial}{\partial z}\int_0 \mathrm{d}z' f(z')\frac{\partial}{\partial z'}\mathrm{e}^{\mathrm{i}\kappa|z-z'|} = \kappa^2\int_0 \mathrm{d}z' f(z')\mathrm{e}^{\mathrm{i}\kappa|z-z'|} - 2\mathrm{i}\kappa f(z) \tag{25}$$

for any function $f(z)$, $z > 0$; it is due to the discontinuity in the derivative of the function $\mathrm{e}^{\mathrm{i}\kappa|z-z'|}$ for $z = z'$. These equations imply the relationship

$$u_3 = \frac{\mathrm{i}}{k}\frac{\partial \mathrm{v}_1}{\partial z} - \frac{\mathrm{i}}{k}\frac{\partial F_1}{\partial z} - F_3. \tag{26}$$

Introducing $\mathbf{u}_3$ from this equation into the first equation (24) and performing the integrations by parts, we get a single integral equation

$$\begin{aligned} (1+q)\mathrm{v}_1 &= -\frac{\mathrm{i}q\omega^2}{2\mathrm{v}_t^2\kappa}\int^0 \mathrm{d}z'u_1(z')\mathrm{e}^{\mathrm{i}\kappa|z-z'|} + \frac{q}{2}\mathrm{v}_1(0)\mathrm{e}^{-\mathrm{i}\kappa z} + (1-q)F_1 - \\ &-\frac{\mathrm{i}q\kappa}{2}\int^0 \mathrm{d}z'F_1(z')\mathrm{e}^{\mathrm{i}\kappa|z-z'|} + \frac{q}{2}F_1(0)\mathrm{e}^{-\mathrm{i}\kappa z} + \frac{qk}{2\kappa}\frac{\partial}{\partial z}\int^0 \mathrm{d}z'F_3(z')\mathrm{e}^{\mathrm{i}\kappa|z-z'|}. \end{aligned} \tag{27}$$

Taking the second derivative with respect to z in this equation we find

$$\frac{\partial^2 \mathrm{v}_1}{\partial z^2}+\kappa'^2 \mathrm{v}_1=\frac{q}{1+q}\left(\kappa^2 F_1+\mathrm{i}k\frac{\partial F_3}{\partial z}\right), \tag{28}$$

where $\kappa'^2=\omega^2/\mathrm{v}_l^2-k^2$. Now, it is easy to get the solution for v_1. It is given by

$$\mathrm{v}_1=\frac{a^3}{2\omega^2}\left[\kappa f_1\sin\kappa\left|z+d\right|+\mathrm{i}\kappa f_3\,\mathrm{sgn}\left(z+d\right)\cos\kappa\left(z+d\right)\right] \tag{29}$$

and, by equations (22) and (26),

$$u_3=\frac{a^3k}{2\omega^2\kappa}\left[\kappa f_3\sin\kappa\left|z+d\right|+\mathrm{i}kf_1\,\mathrm{sgn}\left(z+d\right)\cos\kappa\left(z+d\right)\right]. \tag{30}$$

We can see that all these solutions $\mathrm{v}_{1,2}$, u_3 are stationary waves along the direction perpendicular to the surface, as generated by the stationary oscillating force given by equation (22). In addition, they are regular functions for $\kappa\to 0$, though v_2 and u_3 may increase indefinitely for $\kappa\to 0$, $\mathrm{v}_2\left(\kappa\to 0\right)$, $u_3\left(\kappa\to 0\right)\sim\left|z+d\right|$; this increase indicates the transition to the damped regime. It is also worth noting the discontinuity occurring at $z=-d$.

It is interesting to note that the wavevectors $\mathbf{k}$ and κ in the localized force given by equation (16) are independent and real variables. Out of them, the equation of the elastic waves selects only those wavevectors which satisfy the condition $\omega^2=\mathrm{v}_t^2\left(k^2+\kappa^2\right)$, and assigns them to the allowed propagating waves. This is the mathematical mechanism through which extended elastic waves are generated by localized forces. Another observation is that, the above waves being stationary, the polarization is meaningless for them, although they are associated with the velocity v_t of the transverse elastic waves. On the other hand, we must notice that the solution of the homogeneous equation (28) is the "free" longitudinal wave propagating with the wavevector κ', *i.e.* with the velocity v_l of the longitudinal waves, and similarly, the solution of the homogeneous wave equation (8) is the "free" transverse wave propagating with the wavevector κ and the velocity v_t.

5. SURFACE DISPLACEMENT

The displacement of the surface z = 0, as caused by the "forced waves" obtained above, can be computed by using the inverse Fourier transforms of $\mathrm{v}_{1,2}\left(\mathbf{K}\right)$ and $u_3\left(\mathbf{K}\right)$ given by equations (23), (29) and (30), where $\mathbf{K}=\left(\mathbf{k},\kappa\right)$. As usually, we leave aside for the moment the argument ω in these expressions. It is worth noting that $\kappa=\sqrt{\omega^2/\mathrm{v}_t^2-k^2}$ is not an independent variable. The Fourier components of the force are given by $\mathbf{f}\left(\mathbf{K}\right)=a^3\mathbf{f}\mathrm{e}^{\mathrm{i}\kappa d}$. We choose an in-plane reference frame with one axis oriented along the in-plane radius **r** (radial axis r) and another perpendicular to the former (tangential axis t). We denote by α the angle between the force vector **f** and radius **r** so that the force vector can be written as $\left(f\cos\alpha, f\sin\alpha, f_3\right)$, where f denotes the in-plane (horizontal) force and f_3 denotes the vertical force. We also denote by φ the angle between the in-plane wavevector **k** and radius **r**, such that $\mathbf{k}=k\left(\cos\varphi,\sin\varphi\right)$ and $\mathbf{k}_{\perp}=k\left(-\sin\varphi,\cos\varphi\right)$. Then, we can compute easily the force projections $f_{1,2,3}$ appearing in equations (23), (29) and (30). They are given by $f=\left(f\cos\left(\alpha-\varphi\right), f\sin\left(\alpha-\varphi\right), f_3\right)$. It is worth noting that on changing $\mathbf{k}\rightarrow-\mathbf{k}$, *i.e.* $\varphi\rightarrow\pi+\varphi$, the quantities $f_{1,2}$ change sign, as they should do; similarly, f_3, being the projection of the force along the wavevector component κ, must change sign under the reversal of the direction of this component, $\kappa\rightarrow-\kappa$. Making use of the above notations and of $\mathbf{v}\left(\mathbf{K}\right)=\mathrm{v}_1\mathbf{k}/k+\mathrm{v}_2\mathbf{k}_{\perp}/k$ we can obtain immediately the radial and tangential components of the displacement, $\mathrm{v}_r\left(\mathbf{K}\right)$ and $\mathrm{v}_t\left(\mathbf{K}\right)$, respectively. However, it is worth noting that for a real displacement the Fourier transforms must satisfy the symmetry relationship $\mathbf{v}^*\left(-\mathbf{K}\right)=\mathbf{v}\left(\mathbf{K}\right)$, and, similarly, $\mathbf{u}_3^*\left(-\mathbf{K}\right)=\mathbf{u}_3\left(\mathbf{K}\right)$. Taking into account the change of sign of the force components $f_{1,2,3}$ under this operation, we can see that a factor $\mathrm{sgn}\left(\pi-\varphi\right)$ must be introduced, in general, wherever relevant, in order to get real displacements. The integrals with respect to angle φ in the Fourier transforms imply the Bessel functions $J_{0,1}$. Some of these integrals are collected in **Appendix**. The surface displacement can be written as

$$
\begin{aligned}
\mathrm{v}_r\left(\mathbf{r}\right) &= \frac{a^3 f}{4\pi\omega^2}\left(I_1-\frac{1}{r}I_2\right)\cos\alpha-\frac{a^3 f}{4\pi\mathrm{v}_{\mathrm{t}}^2 r}I_3\cos\alpha-\frac{a^3 f_3}{4\pi\omega^2}I_4,\\
\mathrm{v}_t\left(\mathbf{r}\right) &= \frac{a^3 f}{4\pi\omega^2 r}I_2\sin\alpha-\frac{a^3 f}{4\pi\mathrm{v}_{\mathrm{t}}^2}\left(I_5-\frac{1}{r}I_3\right)\sin\alpha, \qquad (31)\\
u_3\left(\mathbf{r}\right) &= \frac{a^3 f_3}{4\pi\omega^2}I_6-\frac{a^3 f}{4\pi\omega^2}I_4\cos\alpha,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^{\omega/\mathrm{v}_{\mathrm{t}}}\mathrm{d}k\kappa k\sin\kappa d\cdot J_0\left(kr\right),\quad I_2=\int_0^{\omega/\mathrm{v}_{\mathrm{t}}}\mathrm{d}k\kappa\sin\kappa d\cdot J_1\left(kr\right),\\
I_3 &= \int_0^{\omega/\mathrm{v}_{\mathrm{t}}}\mathrm{d}k\frac{1}{\kappa}\sin\kappa d\cdot J_1\left(kr\right),\quad I_4=\int_0^{\omega/\mathrm{v}_{\mathrm{t}}}\mathrm{d}kk^2\cos\kappa d\cdot J_1\left(kr\right), \qquad (32)\\
I_5 &= \int_0^{\omega/\mathrm{v}_{\mathrm{t}}}\mathrm{d}k\frac{k}{\kappa}\sin\kappa d\cdot J_0\left(kr\right),\quad I_6=\int_0^{\omega/\mathrm{v}_{\mathrm{t}}}\mathrm{d}k\frac{k^3}{\kappa}\sin\kappa d\cdot J_0\left(kr\right).
\end{aligned}
$$

We estimate these integrals in the fast oscillating limit $\omega r / \mathrm{v}_t, \omega d / \mathrm{v}_t \gg 1$. In this case, the main contribution comes from $k \sim 0$ and extends over a range $\Delta k \sim 1/r$ for $r \gg d$ or $\Delta k \sim 1/d$ for $d \gg r$. The leading contributions for $r \gg d$ are given by

$$\mathrm{v}_r(\mathbf{r}) \sim \frac{a^3 f}{\omega \mathrm{v}_t r^2} \cos\alpha, \quad \mathrm{v}_t(\mathbf{r}) \sim \frac{a^3 f}{\omega \mathrm{v}_t r^2} \sin\alpha, \quad u_3(\mathbf{r}) \sim \frac{a^3 f}{\omega^2 r^3} \cos\alpha, \tag{33}$$

where oscillating factors of the form $\sin \omega d / \mathrm{v}_t, \cos \omega d / \mathrm{v}_t$ are left aside. We can see the directional character of the surface displacement (through angle α) and the vertical component (u_3) which is much smaller (by a factor $\omega r / \mathrm{v}_t$) than the horizontal components. We shall see that the directional character as given in equation (33) for the "forced waves" is amended by the contribution of the "free waves". It is also worth noting that the leading contribution to the vertical displacement is caused by the in-plane force *f*, and, in general, the vertical component of the force brings a smaller contribution. This is due to the stationary character of the waves along the vertical direction.

Let us assume now a force derived from a localized pressure *p*. The force components are then given by $f_1 = \mathrm{i}pk/\rho$, $f_2 = 0$ and $f_3 = (\mathrm{i}p\kappa/\rho)\mathrm{e}^{\mathrm{i}\kappa d}$. In computing the Fourier transforms of the surface displacement we must take care again of the general symmetry relations $\mathbf{v}^*(-\mathbf{K}) = \mathbf{v}(\mathbf{K})$ and $\mathbf{u}_3^*(-\mathbf{K}) = \mathbf{u}_3(\mathbf{K})$. We get

$$\begin{aligned}
\mathrm{v}_r(r) &\sim \frac{a^3 p}{4\pi\omega^2\rho} \int_0^{\omega/\mathrm{v_t}} \mathrm{d}k\kappa k^2 \sin\kappa d \left(1 - \cos\kappa d\right) J_1(kr), \\
\mathrm{v}_r(r) &\sim \frac{a^3 p}{2\pi^2\omega^2\rho r} \int_0^{\omega/\mathrm{v_t}} \mathrm{d}k\kappa k \cos^2 \kappa d \sin(\kappa r), \\
\mathrm{v}_r(r) &\sim \frac{a^3 p}{4\pi\omega^2\rho} \int_0^{\omega/\mathrm{v_t}} \mathrm{d}k\kappa k^3 \left(\sin^2 \kappa d + \cos\kappa d\right) J_0(kr).
\end{aligned} \tag{34}$$

In the same limit $\omega r / \mathrm{v}_t \gg \omega d / \mathrm{v}_t \gg 1$ the leading contributions to the above displacements are given by

$$\mathrm{v}_r(r), \mathrm{v}_t(r) \sim \frac{a^3 p}{\omega \mathrm{v}_t \rho r^3}, \quad u_3(r) \sim \frac{a^3 p}{\omega^2 \rho r^4}. \tag{35}$$

We can see that the displacements produced by pressure fall off faster with distance than the corresponding displacements caused by a force (equation (33)).

6. FORCE EXERTED ON THE SURFACE

We are interested now in the force exerted on the surface $z = 0$ by the "forced waves" produced beneath the surface. As shown above, the force exerted by a displacement field **u** per unit area of a surface with unit normal **n** is given (in our notations) by $\rho f_i^s = \sigma_{ij} n_j$, where $\sigma_{ij} = \lambda u_{ll}\delta_{ij} + 2\mu u_{ij}$ is the stress tensor and $u_{ij} = (1/2)\left(\partial u_i / \partial x_j + \partial u_j / \partial x_i\right)$ is the strain tensor. Using the reference frame defined by $\mathbf{k}, \mathbf{k}_\perp$ and z we get

$$\begin{aligned} f_1^s(\mathbf{k},\omega) &= \frac{a^3 \mathrm{v}_t^2 \kappa}{\omega^2}\left(\kappa f_1 \cos\kappa d - \mathrm{i} k f_3 \sin\kappa d\right), \\ f_2^s(\mathbf{k},\omega) &= -a^3 f_2 \cos\kappa d, \\ f_3^s(\mathbf{k},\omega) &= \frac{a^3 \mathrm{v}_t^2 k}{\omega^2}\left(k f_3 \cos\kappa d - \mathrm{i}\kappa f_1 \sin\kappa d\right). \end{aligned} \tag{36}$$

We note that the dilatation vanishes, $\mathrm{v}_{11} + \mathrm{v}_{22} + u_{33} = 0$, in accordance with the fact that these solutions, given by equations (23), (29) and (30), are constructed from transverse waves.

We compute the inverse Fourier transforms of these forces with respect to the wavector **k** according to the procedure described above for the surface displacement. The asymptotic expressions ($\omega r / \mathrm{v}_t \gg \omega d / \mathrm{v}_t \gg 1$) are given by

$$f_r^s(\mathbf{r}) \sim \frac{a^3 f}{r^2}\cos\alpha, \quad f_t^s(\mathbf{r}) \sim \frac{a^3 f}{r^2}\sin\alpha, \quad f_3^s(\mathbf{r}) \sim \frac{a^3 f \mathrm{v}_t}{\omega r^3}\cos\alpha; \tag{37}$$

they are similar with the surface displacements given by equation (33), except for an additional factor ω. In the same manner we can compute the force exerted on the surface by a localized pressure.

7. GENERAL SOLUTION

The general solution of the problem is obtained by adding to the particular solution given by equations (23), (29) and (30) the solutions of the homogeneous equations (8) and (28). These are given by a transverse wave

$$\mathbf{u}_t = \left(A_1, A_2, -\frac{k}{\kappa}A_1\right)\mathrm{e}^{\mathrm{i}\kappa z} \tag{38}$$

and a longitudinal wave

$$\mathbf{u}_l = \left(B, 0, \frac{\kappa'}{k} B \right) \mathrm{e}^{\mathrm{i}\kappa' z}, \tag{39}$$

where the constants $A_{1,2}$, B are determined from the condition of a vanishing total surface force, in accordance with equation (14). On equations (38) and (39) one can check the transversality condition div$\mathbf{u}_t = 0$ and the condition curl$\mathbf{u}_l = 0$ for longitudinal waves. The "free waves" solutions given by equations (38) and (39) are written in the referenec frame defined by $\mathbf{k}, \mathbf{k}_\perp$ and z.

We compute the surface force $\rho f_i^{0s} = \lambda u_{ll}^0 \delta_{i3} + 2\mu u_{i3}^0$, where $\mathbf{u}^0 = \mathbf{u}_t + \mathbf{u}_l$, and impose the condition

$$f_i^{0s} + f_i^s = 0, \tag{40}$$

where f_i^s, given by equations (36), correspond to the surface force generated by the particular solution ("forced waves"). It is worth noting here that equation (40) holds for any point on the surface $z = 0$, *i.e.* it is multiplied in fact by the factor $\mathrm{e}^{i\mathbf{kr}}$ with the same in-plane wavevector **k**. Since ω is the same for both the particular solution and the „free waves", and $\omega^2 = \mathrm{v}_t^2 \left(\kappa^2 + k^2 \right)$, $\omega^2 = \mathrm{v}_l^2 \left(\kappa'^2 + k^2 \right)$ in both cases, it follows that κ, κ' are the same, *i.e.* they are real variables, as corresponding to localized external forces. Consequently, the "free waves" are propagating waves. Damped "free" surface waves (*i.e.* waves with κ, κ' purely imaginary) can be excited by damped external forces [13].

Condition (40) leads to

$$A_2 = \sim \mathrm{i} \frac{a^3 f_2}{\mathrm{v}_t^2 \kappa} \cos \kappa d \tag{41}$$

and the system of equations

$$\begin{aligned} \left(\kappa^2 - k^2 \right) A_1 + 2\kappa\kappa' B &= \frac{a^3 \kappa^2}{\omega^2} \left(k f_3 \sin \kappa d + \mathrm{i}\kappa f_1 \cos \kappa d \right), \\ 2k^2 A_1 - \left(\kappa^2 - k^2 \right) B &= -\frac{a^3 k^2}{\omega^2} \left(\kappa f_1 \sin \kappa d + \mathrm{i} k f_3 \cos \kappa d \right), \end{aligned} \tag{42}$$

whose solution can be written as

$$\begin{aligned} A_1 &= -\frac{4\kappa\kappa' k^2}{\Delta} \mathrm{v}_1 + \frac{2\kappa^3 \left(\kappa^2 - k^2 \right)}{k\Delta} u_3, \\ B &= \frac{4\kappa^3 k}{\Delta} u_3 + \frac{2k^2 \left(\kappa^2 - k^2 \right)}{\Delta} \mathrm{v}_1, \end{aligned} \tag{43}$$

where $\Delta = \left(\kappa^2 - k^2\right)^2 + 4\kappa\kappa' k^2$. Incidentally, we note here that $\Delta = 0$ for $\kappa \to \mathrm{i}\kappa$ and $\kappa' \to \mathrm{i}\kappa'$ gives the dispersion relation $\omega(k)$ for the Rayleigh surface waves. The surface displacements brought about by the "free waves" are given by $\mathrm{v}_1^0 = A_1 + B, \quad \mathrm{v}_2^0 = A_2$ and $u_3^0 = -\frac{k}{\kappa}A_1 + \frac{\kappa'}{k}B$. We compute their inverse Fourier transforms by the same procedure as that described in **Section 5**. Under the same conditions as those employed in this **Section** we get the asymptotic behaviour

$$\begin{aligned}
\mathrm{v}_\mathrm{r}^\mathrm{tot}(\mathbf{r}) &\sim \frac{a^3 f}{\omega \mathrm{v}_t r^2}\cos\alpha, \quad \frac{a^3 f}{\omega \mathrm{v}_t r^2}\sin\alpha, \\
\mathrm{v}_\mathrm{r}^\mathrm{tot}(\mathbf{r}) &\sim \frac{a^3 f}{\omega \mathrm{v}_t r^2}\sin\alpha, \quad \frac{a^3 f}{\omega \mathrm{v}_t r^2}\cos\alpha, \\
u_3^\mathrm{tot}(\mathbf{r}) &\sim \left(1 - \mathrm{v}_\mathrm{t} / \mathrm{v}_\mathrm{l}\right)\frac{a^3 f}{\omega r^3}\cos\alpha, \quad \frac{a^3 \mathrm{v}_\mathrm{t} f}{\omega^2 \mathrm{v}_l r^3}\sin\alpha
\end{aligned} \tag{44}$$

for the total displacement $\mathbf{u}^\mathrm{tot} = \mathbf{u}^0 + \mathbf{u}$. We can see that the "free waves" do not change the *r*-dependence, but introduce an additional directional character. In addition, the vertical displacement is affected by factors depending on $\mathrm{v}_\mathrm{t} / \mathrm{v}_\mathrm{l}$. A similar conclusion (except for the directional character) holds for a force derived from pressure.

8. FORCE LOCALIZED ON THE SURFACE

We consider a semi-infinite isotropic elastic body extending over the region $z > 0$ and assume a localized force

$$\mathbf{f}(\mathbf{R}, t) = a\sum_{\mathbf{k}}\int \mathrm{d}\omega \mathbf{f}(\mathbf{k}, \omega)\mathrm{e}^{\mathrm{i}\mathbf{kr} - \mathrm{i}\omega t}\delta(z) \tag{45}$$

acting on the body plane surface $z = 0$, where a is a characteristic length, $\mathbf{R} = (\mathbf{r}, z)$ and $\mathbf{k}$ is the in-plane wavevector. This is a generalization of one of Lamb's problems [2].

We represent the displacement field as

$$\mathbf{u} = (\mathbf{v}, u_3)\theta(z), \tag{46}$$

where $\mathbf{v}$ is the in-plane component (parallel to the surface), u_3 is the transverse component (perpendicular to the surface) and $\theta(z) = 0$ for $z < 0$, $\theta(z) = 1$ for $z > 0$ is the step function. The divergence occurring in equation (9) can then be written as

$$\mathrm{div}\mathbf{u} = \left(\mathrm{div}\mathbf{v} + \frac{\partial u_3}{\partial z}\right)\theta(z) + u_3(0)\delta(z), \tag{47}$$

where we can see the occurrence of specific surface contributions associated with $u_3(0) = u_3(z=0)$. As before, we use a Fourier transform of the form

$$\mathbf{v}(\mathbf{r}, z; t) = \sum_{\mathbf{k}} \int \mathrm{d}\omega \mathbf{v}(\mathbf{k}, \omega; z) \mathrm{e}^{\mathrm{i}\mathbf{kr} - \mathrm{i}\omega t}, \tag{48}$$

and a similar one for $u_3(\mathbf{r}, z; t)$. Usually, we leave aside the arguments $\mathbf{k}, \omega$, while preserving explicitly the z-dependence of the functions $\mathbf{v}(\mathbf{k}, \omega; z)$ and $u_3(\mathbf{k}, \omega; z)$. We compute grad·div**u** according to equations (47) and (48) and introduce it, together with the force given by equation (45), in equation (9). As for a force localized beneath the surface, the intervening integrals reduce to the known integral [12]

$$\int_{|z|}^{\infty} \mathrm{d}x J_0\left(k\sqrt{x^2 - z^2}\right) \mathrm{e}^{\mathrm{i}\omega x / c} = \frac{\mathrm{i}}{\kappa} \mathrm{e}^{\mathrm{i}\kappa |z|}, \tag{49}$$

where

$$\kappa = \sqrt{\frac{\omega^2}{\mathrm{v}_t^2} - k^2}. \tag{50}$$

We use also the same convenient notations $\mathrm{v}_1 = \mathbf{vk}/k$, $\mathrm{v}_2 = \mathbf{vk}_\perp / k$, and similar ones for $f_{1,2}$, where $\mathbf{k}_\perp$ is a vector perpendicular to $\mathbf{k}$, $\mathbf{kk}_\perp = 0$, and of the same magnitude k. Then, equation (9) reduces to

$$\mathrm{v}_2 = \frac{\mathrm{i}af_2}{2\mathrm{v}_t^2 \kappa} \mathrm{e}^{\mathrm{i}\kappa z} \tag{51}$$

and to a set of two coupled integral equations which read

$$\mathrm{v}_1 = -\frac{\mathrm{i}qk^2}{2\kappa} \int_0 \mathrm{d}z' \mathrm{v}_1(z') \mathrm{e}^{\mathrm{i}\kappa|z-z'|} - \frac{qk}{2\kappa} \frac{\partial}{\partial z} \int_0 \mathrm{d}z' u_3(z') \mathrm{e}^{\mathrm{i}\kappa|z-z'|} + \\ + \frac{\mathrm{i}af_1}{2\mathrm{v}_t^2 \kappa} \mathrm{e}^{\mathrm{i}\kappa z} \tag{52}$$

and

$$u_3 = -\frac{qk}{2\kappa} \frac{\partial}{\partial z} \int_0 \mathrm{d}z' \mathrm{v}_1(z') \mathrm{e}^{\mathrm{i}\kappa|z-z'|} + \frac{\mathrm{i}q}{2\kappa} \frac{\partial^2}{\partial z^2} \int_0 \mathrm{d}z' u_3(z') \mathrm{e}^{\mathrm{i}\kappa|z-z'|} + \\ + \frac{\mathrm{i}af_3}{2\mathrm{v}_t^2 \kappa} \mathrm{e}^{\mathrm{i}\kappa z}. \tag{53}$$

We note here again that in deriving these equations it is worth observing the non-invertibility of the derivatives and the integrals, according to the identity

$$\frac{\partial}{\partial z}\int_0 dz' f(z')\frac{\partial}{\partial z'}e^{i\kappa|z-z'|} = \kappa^2\int_0 dz' f(z')e^{i\kappa|z-z'|} - 2i\kappa f(z) \tag{54}$$

for any function $f(z)$, $z > 0$; it is due to the discontinuity in the derivative of the function $e^{i\kappa|z-z'|}$ for $z = z'$. From equations (52) and (53) we get easily

$$u_3 = -\frac{i}{k}\frac{\partial v_1}{\partial z} - \frac{ia(\kappa f_1 - kf_3)}{2v_t^2\kappa k}e^{i\kappa z}. \tag{55}$$

Equation (51) gives the transverse wave v_2 (for κ real) propagating with the velocity v_t, according to equation (50). Its polarization is normal to the plane of propagation (the plane determined by the vectors **k** and κ). This wave is usually known as the *s*-wave in the theory of electromagnetism (from the German word "senkrecht" which means "perpendicular"). From equation (51) we can see that the *s*-wave becomes singular for κ = 0.

We pass now to the system of coupled equations (52) and (53), and the relationship given by equation (55). We introduce u_3 from equation (55) into equation (52) and get

$$\begin{aligned}(1+q)v_1 &= -\frac{iq\omega^2}{2v_t^2\kappa}\int_0 dz' v_1(z')e^{i\kappa|z-z'|} + \\ &+\frac{iaq}{4v_t^2\kappa^2}(\kappa f_1 - kf_3)\frac{\partial}{\partial z}\int_0 dz' e^{i\kappa z'}e^{i\kappa|z-z'|} + \frac{1}{2}\left[\frac{iaf_1}{v_t^2\kappa} + qv_1(0)\right]e^{i\kappa z}.\end{aligned} \tag{56}$$

This equation can easily be solved by taking the second derivative with respect to z and using the non-invertibility equation (54). We get

$$\frac{\partial^2 v_1}{\partial z^2} + \kappa'^2 v_1 = -\frac{iaq}{2v_t^2(1+q)}(\kappa f_1 - kf_3)e^{i\kappa z}, \tag{57}$$

where

$$\kappa' = \sqrt{\frac{\omega^2}{v_l^2} - k^2}\ . \tag{58}$$

For a longitudinal force $\kappa f_1 - k f_3 = 0$ we obtain from equation (57) longitudinal waves propagating with wavevector κ' (for κ' real) and with the velocity v_l. For a general force, equation (57) has the particular solution

$$\mathrm{v}_1 = \frac{\mathrm{i}a}{2\omega^2}\left(\kappa f_1 - k f_3\right)\mathrm{e}^{\mathrm{i}\kappa z} \tag{59}$$

and

$$u_3 = \frac{\mathrm{i}ak}{2\omega^2\kappa}\left(\kappa f_1 - k f_3\right)\mathrm{e}^{\mathrm{i}\kappa z}. \tag{60}$$

We can see that v_1 and u_3 given above correspond to a transverse wave, $k\mathrm{v}_1 + \kappa u_3 = 0$, whose polarization lies in the plane of propagation. This is called the *p*-wave, where *p* stands for "parallel". We can see also that v_1 is a regular function, while u_3 may exhibit the same singularity as v_2 does for $\kappa = 0$.

9. SURFACE DISPLACEMENT CAUSED BY A FORCE LOCALIZED ON THE SURFACE

The displacement of the surface $z = 0$ can be computed by using the inverse Fourier transforms of $\mathrm{v}_{1,2}(\mathbf{K})$ and $u_3(\mathbf{K})$ given by equations (51), (59) and (60), where $\mathbf{K} = (\mathbf{k}, \kappa)$. As usually, we leave aside for the moment the argument ω in these expressions. It is worth noting that $\kappa = \sqrt{\omega^2 / \mathrm{v}_\mathrm{t}^2 - k^2}$ is not an independent variable. First, we consider a δ-type force localized on the surface, $\mathbf{f}(\mathbf{R}) = ab^2\mathbf{f}\,\delta(\mathbf{r})\delta(z)$, where $\mathbf{f}$ is a constant vector and b is a characteristic localization length on the surface. Again, this is another generalization of one of Lamb's problems [2]. The Fourier components $\mathbf{f}(\mathbf{K}) = ab^2\mathbf{f}$ of this force do not depend on $\mathbf{K}$ (but they may have an ω-dependence). As before, we choose an in-plane reference frame with one axis oriented along the in-plane radius $\mathbf{r}$ (radial axis r) and another perpendicular to the former (tangential axis t). We denote by α the angle between the force vector $\mathbf{f}$ and radius $\mathbf{r}$. Then, the force vector can be written as $f = (f\cos\alpha, f\sin\alpha, f_\mathrm{v})$, where f denotes the in-plane (horizontal) force and f_v denotes the vertical force. Similarly, we denote by φ the angle between the in-plane wavevector $\mathbf{k}$ and radius $\mathbf{r}$, such that $\mathbf{k} = k(\cos\varphi, \sin\varphi)$ and $\mathbf{k}_\perp = k(-\sin\varphi, \cos\varphi)$. Then, the force projections $f_{1,2,3}$ entering equations (51), (59) and (60) can be written as

$$f_1 = b^2 f\cos(\alpha - \varphi), \quad f_2 = b^2 f\sin(\alpha - \varphi), \quad f_3 = b^2 f_\mathrm{v}. \tag{61}$$

It is worth noting that on changing $\mathbf{k} \to -\mathbf{k}$, *i.e.* $\varphi \to \pi + \varphi$, the quantities $f_{1,2}$ change sign, as they should do; similarly, f_3, being the projection of the force along the wavevector component κ, must change sign under the reversal of the direction of this component, $\kappa \to -\kappa$. Making use of the above notations and of $\mathbf{v}(\mathbf{K}) = \mathrm{v}_2 \mathbf{k}/k + \mathrm{v}_2 \mathbf{k}_\perp / k$ we can obtain immediately the radial and tangential components of the displacement, $\mathrm{v}_r(\mathbf{K})$ and $\mathrm{v}_t(\mathbf{K})$, respectively. However, it is worth noting that for a real displacement the Fourier transforms must satisfy the symmetry relationship $\mathbf{v}^*(-\mathbf{K}) = \mathbf{v}(\mathbf{K})$, and, similarly, $\mathbf{u}_3^*(-\mathbf{K}) = \mathbf{u}_3(\mathbf{K})$. Taking into account the change of sign of the force components $f_{1,2,3}$ under this operation, we can see that a factor sgn(π – φ) must be introduced wherever relevant. The Fourier components of the displacement can be written as

$$\begin{aligned}
\mathrm{v}_r(\mathbf{k}) &= \left[\frac{\mathrm{i}ab^2}{2\omega^2} \kappa f \cos(\alpha - \varphi) \cos\varphi - \frac{\mathrm{i}ab^2 f}{2\mathrm{v}_t^2 \kappa} \sin(\alpha - \varphi) \sin\varphi \right] \mathrm{sgn}(\pi - \varphi) - \\
&\quad - \frac{\mathrm{i}ab^2}{2\omega^2} k f_\mathrm{v} \cos\varphi, \\
\mathrm{v}_t(\mathbf{k}) &= \left[\frac{\mathrm{i}ab^2}{2\omega^2} \kappa f \cos(\alpha - \varphi) \sin\varphi + \frac{\mathrm{i}ab^2 f}{2\mathrm{v}_t^2 \kappa} \sin(\alpha - \varphi) \cos\varphi \right] \mathrm{sgn}(\pi - \varphi) - \\
&\quad - \frac{\mathrm{i}ab^2}{2\omega^2} k f_\mathrm{v} \sin\varphi, \\
u_3(\mathbf{k}) &= \frac{\mathrm{i}ab^2}{2\omega^2} k f \cos(\alpha - \varphi) + \frac{\mathrm{i}ab^2 k^2}{2\omega^2 \kappa} f_\mathrm{v} \,\mathrm{sgn}(\pi - \varphi)
\end{aligned} \tag{62}$$

(for $z = 0$). Now we can take the inverse Fourier transforms of these quantities. It is easy to see that the integrals over angle φ which contain factors $\sin^2\varphi$ and $\cos^2\varphi$ are vanishing. For the radial component we are left with

$$\begin{aligned}
\mathrm{v}_t(\mathbf{r}) &= \frac{\mathrm{i}ab^2 f}{2(2\pi)^2 \omega^2} \sin\varphi \int_0^{\omega/\mathrm{v}_t} \mathrm{d}k \frac{k^3}{\kappa} \int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{sgn}(\pi - \varphi) \sin\varphi \cos\varphi\, \mathrm{e}^{\mathrm{i}kr\cos\varphi} - \\
&\quad - \frac{\mathrm{i}ab^2 f_\mathrm{v}}{2(2\pi)^2 \omega^2} \int_0^{\omega/\mathrm{v}_t} \mathrm{d}k\, k^2 \int_0^{2\pi} \mathrm{d}\varphi \cos\varphi\, \mathrm{e}^{\mathrm{i}kr\cos\varphi}.
\end{aligned} \tag{63}$$

The integrals in equation (63) can be performed straightforwardly, by making use of the properties of the Bessel functions [12, 14]. We get

$$\mathrm{v}_r(\mathbf{r}) = -\frac{ab^2}{4\pi \mathrm{v}_t^2 r} (f \sin\alpha + f_\mathrm{v}) \left[J_0\left(\frac{\omega r}{\mathrm{v}_t}\right) - \frac{2\mathrm{v}_t}{\omega r} J_1\left(\frac{\omega r}{\mathrm{v}_t}\right) \right]. \tag{64}$$

In the limit $\omega r / \mathrm{v}_t \gg 1$ we get

$$\mathrm{v}_r(\mathbf{r}) \sim_{\omega r/\mathrm{v}_t \gg 1} -\frac{ab^2}{\omega^{1/2}}\left(f\sin\alpha + f_{\mathrm{v}}\right)\frac{1}{\left(\mathrm{v}_t r\right)^{2/3}}\cos\left(\frac{\omega r}{\mathrm{v}_t}-\frac{\pi}{4}\right). \tag{65}$$

We can see that the radial component of the surface displacement attains its maximum value along a direction perpendicular to the direction of the force $(\alpha=\pi/2)$, as expected for a transverse wave generated by such a localized force. It has a characteristic oscillatory behaviour with the in-plane distance and goes like $r^{-3/2}$ for long distances. The temporal Fourier transform of the spectrum given by equation (65) for f and f_{v} independent of ω (related to Fresnel integrals) exhibits a characteristic oscillatory wave front of the form $\sim\left(r-\mathrm{v}_t t\right)^{-1/2}$, as expected. Such qualitative characteristics of the solution to this problem are similar with those indicated long time ago by Lamb [2] (See also Ref. [3]).

Similar calculations can be done for the tangential component $\mathrm{v}_t(\mathbf{r})$) and the vertical component $u_3(\mathbf{r})$. The result for $\mathrm{v}_t(\mathbf{r})$ can be obtained from equations (64) and (65) by putting formally $f_{\mathrm{v}}=0$ and replacing sinα by cosα. The vertical component can be obtained from equations (64) and (65) by replacing sinα by 1 and putting $f_{\mathrm{v}}=0$.

Next, we consider an in-plane localized pressure $pb^2\delta(\mathbf{r})$. The Fourier components of the force are given by $f_1=\left(-\mathrm{i}b^2 p/\rho\right)k,\ f_2=f_3=0$ and the Fourier components of the displacement are

$$\mathbf{v}_1(\mathbf{k})=\frac{ab^2 p}{2\rho\omega^2}\kappa k\left(\cos\varphi,\,\sin\varphi\right)\mathrm{sgn}\left(\pi-\varphi\right),\quad u_3(\mathbf{k})=-\frac{ab^2 p}{2\rho\omega^2}k^2. \tag{66}$$

The inverse Fourier transforms of these displacements gives $\mathrm{v}_r(\mathbf{r})=0$ and

$$\begin{aligned}\mathrm{v}_t(r)&=\frac{ab^2 p\omega}{16\pi\rho\mathrm{v}_t^3 r}\left[J_1\left(\frac{\omega r}{\mathrm{v}_t}\right)+J_3\left(\frac{\omega r}{\mathrm{v}_t}\right)\right],\\ u_3(r)&=-\frac{ab^2 p\omega}{4\pi\rho\mathrm{v}_t^3 r}\left[J_1\left(\frac{\omega r}{\mathrm{v}_t}\right)-\frac{2\mathrm{v}_t}{\omega r}J_2\left(\frac{\omega r}{\mathrm{v}_t}\right)\right],\end{aligned} \tag{67}$$

where $J_{2,3}$ are Bessel functions of the first kind and second and, repectively, third order. The leading term $\left(\sim r^{-3/2}\right)$ in v_t is vanishing in the limit $\omega r/\mathrm{v}_t \gg 1$, while u_3 behaves like

$$u_3(r)\sim_{\omega r/v_t\gg 1}-\frac{ab^2p}{\rho v_t}\frac{\omega^{1/2}}{(v_r r)^{3/2}}\cos\left(\frac{\omega r}{v_t}-\frac{3\pi}{4}\right). \tag{68}$$

The vertical component of the surface displacement has a wave front of the form $\sim (r-v_t t)^{-3/2}$.

10. GENERAL SOLUTION FOR A FORCE LOCALIZED ON THE SURFACE

The force exerted on the surface by the "forced waves" is given by $\rho f_i^s=\sigma_{i3}=\lambda u_{ll\delta_{i3}}+2\mu u_{i3}$, where the particular solution given by equations (51), (59) and (60) is used for computing the strain tensor. Its components are given by

$$f_1^s=-\frac{a v_t^2\left(\kappa^2-k^2\right)}{2\omega^2\kappa}\left(\kappa f_1-k f_3\right),$$
$$f_2^s=-\frac{af_2}{2}, \tag{69}$$
$$f_3^s=-\frac{a v_t^2 k}{\omega^2}\left(\kappa f_1-k f_3\right),$$

where f_i are given by equation (61). We can estimate this force following the same procedure described in the preceding *Section* for the displacement. The results are similar with the corresponding displacements. For instance, the asymptotic expression for the radial component of such a force is given by

$$f_r^s(\mathbf{r})\sim_{\omega r/v_t\gg 1}-\frac{ab^2}{r}(\omega/v_t r)^{1/2} f\cos\alpha\cos\left(\frac{\omega r}{v_t}-\frac{3\pi}{4}\right), \tag{70}$$

which is similar with equation (65).

The contribution f_i^{0s} of the "free waves", as given by equations (38) and (39), must be added to this force, in order to satisfy the "free-surface" boundary condition given by equation (15). For the in-plane Fourier transforms this condition reads

$$f_i^{0s}+f_i^s+af_i=0. \tag{71}$$

The solutions of this system of equations are

$$A_1 = -\frac{\mathrm{i}af_1\kappa\left(\kappa^2 - k^2\right)}{2\Delta \mathrm{v}_t^2} + \frac{\mathrm{i}a\kappa}{2\omega^2 k}\left(kf_1 - \kappa f_3\right) + \frac{\mathrm{i}a\left(\kappa^2 - k^2\right)^3}{2\Delta\omega^2 k} f_3,$$

$$A_2 = -\frac{\mathrm{i}af_2}{2\mathrm{v}_t^2\kappa}, \tag{72}$$

$$B = -\frac{\mathrm{i}af_1\kappa k^2}{\Delta \mathrm{v}_t^2} + \frac{\mathrm{i}ak\left(\kappa^2 - k^2\right)}{\Delta\omega^2}\left[2\kappa kf_1 + \left(\kappa^2 - k^2\right)f_3\right],$$

where $\Delta = \left(\kappa^2 - k^2\right)^2 + 4\kappa\kappa' k^2$. The surface displacement caused by the free waves is given by

$$\mathrm{v}_1^0 = A_1 + B, \quad \mathrm{v}_2^0 = A_2, \quad u_3^0 = -\frac{k}{\kappa}A_1 + \frac{\kappa'}{k}B. \tag{73}$$

In the asymptotic limit $\omega r / \mathrm{v}_t \gg 1$ the main contribution to the in-plane Fourier transforms of the quantities given by equation (73) in the reference frame defined by the radial axis *r*, tangential axis *t* and the vertical axis *z* is brought by $k \sim 0$. Within this approximation we find

$$\mathrm{v}_3^0\left(r\right) \sim \frac{ab^2 f_\mathrm{v}}{\mathrm{v}_t^2 r} J_0\left(\frac{\omega r}{\mathrm{v}_t}\right) \sim \frac{ab^2 f_\mathrm{v}}{\omega^{1/2}\left(\mathrm{v}_t r\right)^{3/2}} \cos\left(\frac{\omega r}{\mathrm{v}_t} - \frac{\pi}{4}\right) \tag{74}$$

for the radial component, a similar expresion for the tangential component and

$$u_3^0\left(r\right) \sim \frac{ab^2 f}{\mathrm{v}_l \mathrm{v}_t r} J_0\left(\frac{\omega r}{\mathrm{v}_t}\right)\cos\alpha \sim \frac{ab^2 f}{\mathrm{v}_l\left(\mathrm{v}_t\omega\right)^{1/2} r^{3/2}}\cos\alpha\cos\left(\frac{\omega r}{\mathrm{v}_t} - \frac{\pi}{4}\right) \tag{75}$$

for the vertical component of the displacement. As we can see, the "free waves" do not modify the asymptotic *r*-dependence of the displacement caused by the "forced waves" (equation (65)), but introduce an additional angle dependence and amplitude factors related to the velocity of the longitudinal waves. A similar conclusion holds for the displacement caused by a point-like pressure localized on the surface.

11. TEMPORAL DEPENDENCE

The asymptotic surface displacements for a force localized beneath the surface contains frequency factors of the form $1/\omega$, $1/\omega^2$ etc. In addition, for $d \ll r$, they have also oscillating factors of the form $\sin\left(\omega d / \mathrm{v}_t\right))$, $\cos\left(\omega d / \mathrm{v}_t\right)$

(not included in equations (44)). In the opposite case $r \ll d$ these oscillating factors are of the form $\sin\left(\omega r / \mathrm{v}_t\right)$, $\cos\left(\omega r / \mathrm{v}_t\right)$, so we may take a general behaviour of the type $\sin\left(\omega R / \mathrm{v}_t\right)$, $\cos\left(\omega R / \mathrm{v}_t\right)$, where *R* is a length related to the distance from the source to the point on the surface. In addition, the free waves may bring also contributions propagating with velocity v_l along the distance *R*, especially for small values of the in-plane radius *r*.

Consequently, for the time dependence of the surface displacement we have to estimate, for instance, integrals of the form

$$I = \int_0^{\Delta\omega} \mathrm{d}\omega \frac{\cos\omega\tau}{\omega}, \tag{76}$$

where $\tau \sim t - R / \mathrm{v}$, where v denotes a generic velocity and $\Delta\omega$ is a range of frequencies. It is easy to see that for small τ the integral in equation (76) is approximately given by $I \sim \ln\left(\Delta\omega\tau\right)$. It tells that the front waves has an abrupt rise for $\tau = 0$, as expected.

For the surface displacements caused by a force localized on the surface the characteristic frequency factor is $\sim \omega^{-1/2}$ and we have to estimate integrals of the form

$$I = \int_0^{\Delta\omega} \mathrm{d}\omega \frac{\cos\omega\tau}{\sqrt{\omega}}, \tag{77}$$

By a change of variable $\omega\tau = z^2$ this integral can be reduced to a Fresnel integral. The Fresnel integral is given by [11]

$$\int_0^{\infty} \mathrm{d}z \mathrm{e}^{\mathrm{i}z^2} = \sqrt{\frac{\pi}{2}} \frac{1+\mathrm{i}}{2}. \tag{78}$$

We can see that the wave front goes like $\tau^{-1/2} = \left(t - r / \mathrm{v}\right)^{-1/2}$.

12. CONCLUSIONS

In conclusion, we may say that we have introduced herein a new method of studying the propagation of the elastic waves in isotropic bodies, based on the Kirchhoff potentials for wave equation with sources, borrowed from the theory of electromagnetism. The method implies coupled integral equations for the waves amplitudes, which we solved. Making use of this method we have determined the waves produced in an isotropic elastic semi-infinite body by an external force localized either on the body surface or beneath the surface at some distance d. In the latter case the waves are stationary along the direction perpendicular to the body surface. We have also computed the surface displacement produced by these forces as well as the force exerted on the surface as caused by a force localized beneath. We have estimated these quantities in the fast oscillating regime ($\omega r / v_t \gg \omega d / v_t \gg 1$, where ω denotes the frequency and v_t is the velocity of the transverse waves). These quantities exhibit a characteristic decrease along the in-plane distance on the body surface and a characteristic oscillatory behaviour. By making use of this method we have generalized one of Lamb's problem (force localized on the surface of the body) and obtained new results for a point-like force localized beneath the body surface. Various other results can be obtained by means of this method, for various other geometries and force distributions.

The present approach can be extended to determine the waves propagating in elastic bodies with special, finite geometries, either as eigenmodes or caused by some external forces (both localized or extended). More interesting, we can extend the present approach to include the effect of various inhomogeneities placed in elastic bodies, as caused by local variations in the body density or elastic constants.

Indeed, suppose for instance that a small irregularity $\delta\rho$ occurs in the density ρ in equation (1). The corresponding term $\delta\rho\ddot{u}$ can be transferred into the *rhs* of equation (8) and can be treated as a "wave source". It will bring an additional contribution to the "potential" given by equation (9), which allows one to compute the changes brought by this inhomogeneity both in the eigenmodes and the elastic response of the body. Of particular importance is the case when this inhomogeneity is placed on the body surface. Obviously, a similar treatment can be applied to inhomogeneities occurring in the elastic coefficients λ and μ, both on the body surface or in the bulk. Some results in this direction will be reported in a forthcoming publication.

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APPENDIX

A few integrals

We give here a few integrals occurring in the calculations described in the main text:

$$\int_0^{2\pi} \mathrm{d}\varphi \cos\varphi\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = 2i\pi J_1(z), \quad \int_0^{2\pi} \mathrm{d}\varphi \sin\varphi\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = 0,$$

$$\int_0^{2\pi} \mathrm{d}\varphi \cos^2\varphi\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = 2i\pi J_0(z) - \frac{2\pi}{z} J_1(z), \quad \int_0^{2\pi} \mathrm{d}\varphi \sin\varphi \cos\varphi\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = 0,$$

$$\int_0^{2\pi} \mathrm{d}\varphi \sin^2\varphi\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = \frac{2\pi}{z} J_1(z), \quad \int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{sgn}(\pi - \varphi)\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = 0,$$

$$\int_0^{2\pi} \mathrm{d}\varphi\ \mathrm{sgn}(\pi - \varphi) \cos\varphi\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = 0, \quad \int_0^{2\pi} \mathrm{d}\varphi\ \mathrm{sgn}(\pi - \varphi) \sin\varphi\ \mathrm{e}^{\mathrm{i}z\cos\varphi} = 4\frac{\sin z}{z},$$

$$\int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{sgn}(\pi - \varphi) \cos^2\varphi\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = 0, \quad \int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{sgn}(\pi - \varphi) \sin^2\varphi\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = 0,$$

$$\int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{sgn}(\pi - \varphi) \sin\varphi \cos\varphi\, \mathrm{e}^{\mathrm{i}z\cos\varphi} = -4\mathrm{i}\frac{\partial}{\partial z}\frac{\sin z}{z}.$$

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