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EARTH PHYSICS

MAXIMUM STRESS AT A TECTONIC FAULT PLANE. COULOMB LAW

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Abstract. We give an analytic method to determine the maximum shear stress in the presence of friction. The method employs Lagrange multipliers to find the maxima and minima of a function of a set of variables.

Key words: rock mechanics, friction, faulting surfaces, stress tensor.

1. INTRODUCTION

Lithosphere is the outer port of Earth, consisting of the crust and the upper mantle. The lithosphere is solid but it is also brittle. The lithosphere is approximately 65 to 100 km thick and lies above the layer known as the astenosphere, which consists of softer, less rigid rocky materials.

Earthquake can occur in brittle material. A brittle material is one that breaks abruptly when the applied stress exceeds the strength of the material. It follows that earthquake happen only in the lithosphere. The Earths lithosphere is fractured into plates which are moving relative to each other. The movement of the tectonic plates produces forces that deform Earth's crust. These forces are greatest at the plate boundaries, but can be transmitted throughout any plate.

Earthquakes are now explained by the elastic rebound theory (Reid, 1911) [1]. Stress is applied to rock or to an existing fault over a period of time. This usually happens at a plate boundary where two plates are moving in different directions, or in the same direction with different speeds. During the interseismic stage the stress builds, rock or a locked fault (a fault where the two sides are held together by friction) deform elastically. Immediately prior to rupture, there is the preseismic stage that can be associated with small earthquakes (foreshocks) or other possible precursory effects

As the stress overcomes the rocks strength or the faults friction, and either the rock fractures or the fault slips causing an earthquake. The earthquake itself marks the coseismic phase, during which rapid motion on the fault generates seismic waves. Finally, a postseismic phase occurs following the rupture, and aftershocks and transient slip may occur for up to several years (Fig.1).

The rock or fault rebounds and the process may begin again. The key point of this theory is that the stress is continually building up and the earthquake act to relieve that stress. For a constant rate of stress increase due to plate motion, the greater the time between earthquakes, the greater the stress release when the earthquake occurs (larger magnitude). The elastic rebound idea was a major conceptual breakthrough, because the faulting seen at the surface had previously been regarded as an incidental side-effect of an earthquake, rather than its cause. Recent research uses GPS and InSAR, measuring motion at the fault zone as well as away from the fault .

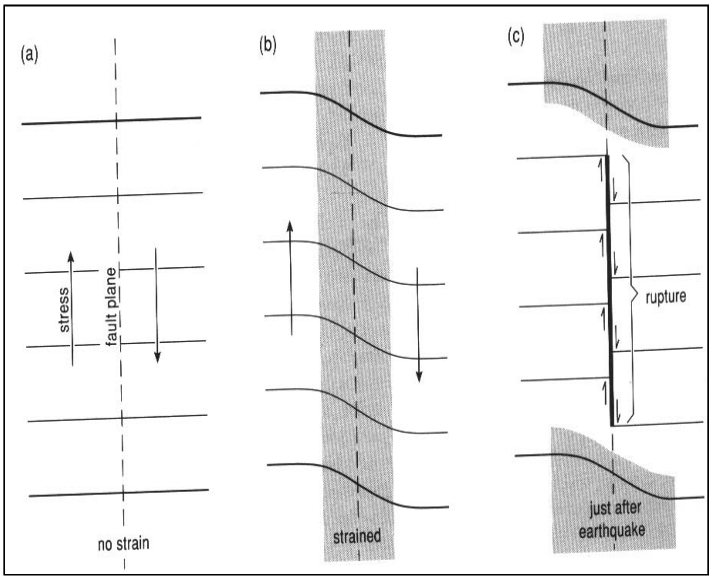

Fig.1 – Schematic of elastic rebound theory; *a*) unstrained rocks (*b*) are distorted by relative movement between the two plates, causing strains within the fault zone that finally become so great that (*c*) the rocks break and rebound to a new unstrained position (after C. R. Allen [3]).

The state of stress at some point in a body of rock is given by the stress tensor $\hat{\sigma}$. If we know the stress tensor $\hat{\sigma}$ at a point in a body we can determine the normal and tangential (shear stress) components, written (σ_n, τ), that act on a plane of any orientation passing through that point.

The Coulomb law of failure describes the shear stress on a failure plane as function of normal stress at the moment of initiation of failure. This law for failure of intact rock may be written in terms of the Coulomb law as follows:

$$S = |\tau| - \mu_i \sigma_n$$

In other words, sufficient shear stress τ is needed to overcome the normal stress σ_n holding surfaces together in frictional contact in order for sliding to occur once the inherent shear strength is overcome. If there is "insufficient" τ (or too much $\mu_i\sigma_n$) the difference $|\tau| - \mu_i\sigma_n$ could be a negative number. This implies that frictional sliding is not possible.

We shall answer to two questions: a) what is the plane of maximum sheer stress, b) what is the plane of maximum difference between τ and $\mu\sigma_n$? We use Lagrange multipliers to find maxima and minima of a function of a set of variable in the one that the variables are subject to additional constrains.

2. THEORY

The stress tensor in Cartesian coordinate system is defined by:

$$\hat{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, \quad (1)$$

where $\sigma_{ij} = \sigma_{ji}$. The first subscript identifies the coordinate direction in which the component acts and second identifies the plane by giving the coordinate direction of its outward normal [3]. We call the stress components σ_{ii} normal stress, while $\sigma_{ij}\,(i \neq j)$ are known as shear stresses. In some texts, the shear stress σ_{ij} is denoted as τ_{ij}. σ_{ii} would be positive if they stretch the material (tension) and negative for compression.The stress components are shown in Fig. 2.

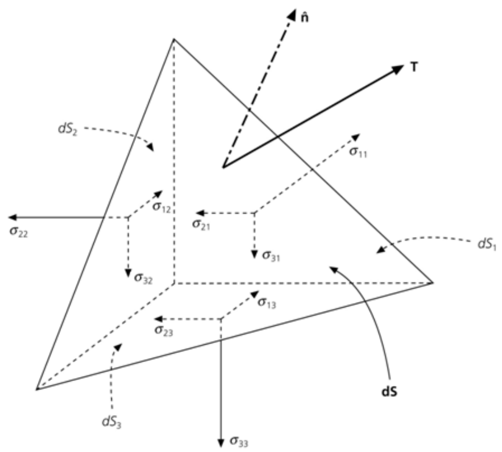

Fig. 2 – Stress components on the faces of a tetrahedron.

There is always a coordinate system where all the off diagonals elements are zero. In this coordinate system the stress tensor can be written as:

$$\hat{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}. \tag{2}$$

These principal stresses are the greatest (σ_1), intermediate (σ_2) and least normal stresses (σ_3). Because normal stresses are predominantly compressive within the Earth in geology, compression is positive and tension negative.

For instance as we descend in the crust the vertical traction on a horizontal plane will increase due to the increasing load of rock overhead [4]. This is known as the lithostatic stress. For a rock density of $2,700\ \mathrm{kg}/\mathrm{m}^3$ and a depth of 1000 m

$$\sigma_v = \rho g h = 26.46\mathrm{MPa},$$

where h is the height of rock column above depth of interest.

Elasticity theory indicates that if you push a block with certain load, it will try to respond by horizontal extension, but adjacent rock is in the way and so a horizontal stress develops. In this case the lateral components are:

$$\sigma_h = \frac{\nu}{1-\nu}\sigma_v,$$

where ν is Poisson's ratio with a typical value about ¼.

The traction $\vec{T}$ across any arbitrary plane of orientation defined by $\vec{n}(n_1, n_2, n_3)$ may be obtained by multiplying the stress tensor by $\vec{n}$, that is:

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \tag{3}$$

Note that $n_1 = \cos\alpha$, $n_2 = \cos\beta$ and $n_3 = \cos\gamma$ are the directional cosines of the unit vector $\vec{n}$.

Generally the direction of $\vec{T}$ and $\vec{n}$ do not coincide. The normal σ_n and tangential τ components on the plane are [4]:

$$\sigma_n = \vec{T} \cdot \vec{n} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2, \tag{4}$$

$$\tau=\sqrt{T^2-\sigma_n^2}=\sqrt{\sigma_1^2n_1^2+\sigma_2^2n_2^2+\sigma_3^2n_3^2-\left(\sigma_1n_1^2+\sigma_2n_2^2+\sigma_3n_3^2\right)^2}=$$
$$=\sqrt{\left(\sigma_1-\sigma_2\right)^2n_1^2n_2^2+\left(\sigma_1-\sigma_3\right)^2n_1^2n_3^2+\left(\sigma_2-\sigma_3\right)^2n_2^2n_3^2}. \quad (5)$$

The simplest theory of faulting states that shear fall occurs on planes of maximum shear stress. We want to find the maximum and minimum values of τ subject of the constraint $n_1^2+n_2^2+n_3^2=1$.

Form the Langrangian:

$$L=\left(\sigma_1^2n_1^2+\sigma_2^2n_2^2+\sigma_3^2n_3^2\right)-\left(\sigma_1n_1^2+\sigma_2n_2^2+\sigma_3n_3^2\right)^2+\lambda\left(n_1^2+n_2^2+n_3^2-1\right), \quad (6)$$

where λ is the Lagrange multiplier. Now for $i=\overline{1,3}$:

$$\frac{\partial L}{\partial n_i}=2\sigma_i^2n_i-4\left(\sigma_1n_1^2+\sigma_2n_2^2+\sigma_3n_3^2\right)\sigma_in_i+2\lambda n_i=0 \quad (7)$$

(no summation over *i*). From (7) either $n_i=0$ ($\tau=0$, a minimum of τ), or

$$-\sigma_1^2+2\left(\sigma_1n_1^2+\sigma_2n_2^2+\sigma_3n_3^2\right)\sigma_i=\lambda. \quad (8)$$

There are three different solutions for $\vec{n}$ and for each of these three solutions, we can find the value of τ at each extremum:

$$\begin{aligned}
&\text{(a)}\quad \vec{n}\left(0,\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2}\right),\quad \lambda=\sigma_2\sigma_3;\tau^2=\left[\frac{1}{2}\left(\sigma_2-\sigma_3\right)\right]^2,\\
&\text{(b)}\quad \vec{n}\left(\pm\frac{\sqrt{2}}{2},0,\pm\frac{\sqrt{2}}{2}\right),\quad \lambda=\sigma_3\sigma_1;\tau^2=\left[\frac{1}{2}\left(\sigma_3-\sigma_1\right)\right]^2,\\
&\text{(c)}\quad \vec{n}\left(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2},0\right),\quad \lambda=\sigma_1\sigma_2;\tau^2=\left[\frac{1}{2}\left(\sigma_1-\sigma_2\right)\right]^2.
\end{aligned} \quad (9)$$

If the principal stresses of σ_1,σ_2 and σ_3 are all different, than one of the three extremes will be greater than the other. If $\sigma_1<\sigma_2<\sigma_3<0$ (all compression) the maximum shear stress has magnitude $\frac{1}{2}\left(\sigma_1-\sigma_3\right)$. We call this "global maximum [5].

The other two are called "local maxima". From the above equations we see that the maximum shear stress (and thus the most likely plane of failure) occurs at $\pm$ 45° to the maximum and minimum stress in a plane parallel to the intermediate stress. This leads to the result that failure always takes place on planes passing through the direction of the intermediate principal stress. The theory of fracture which assumes failure will occur along this plane is known as the Coulomb theory. There are two equivalent directions (planes) in a rock which are equally favorable for failure, called “conjugate fractures”. This result is not always consistent with experiment.

3. THE EFFECT OF FRICTION

It should be noted that when we examine real fractures we find that failure occurs at angles less than $45°$ to the maximum principal stress. Hence the failure criterion derived above has neglected some property of the rock that alters the failure criterion.

Consider first the case of frictional sliding on a simple preexisting plane of rock mass. Amounton’s law states that:

$$f_{max} = \mu_s N. \tag{10}$$

where the constant of proportionality μ_s is the coefficient of friction Dividing (11) by the area of contact we obtain in terms of normal and tangential components of traction:

$$\tau = \mu_s \sigma_n. \tag{11}$$

It may be observed that the shear stress necessary to initiate sliding from a static condition is greater than that required to maintain sliding in the dynamic condition. That is, the coefficient of static friction, μ_s, is greater than the coefficient of dynamic friction μ_d. If there is no preexisting plane of weakness the criterion of shear failure is written as:

$$|\tau| = S_0 + \mu_i \sigma_n, \tag{12}$$

where S_0 is the cohesive shear strength which is the resistance to shear when $\sigma_n = 0$ (the cohesion reflects the strength of the bonds between rock particles), and μ_i is the coefficient of internal friction (not identical with the coefficient of static friction). We use the modulus or magnitude of τ because its direction does not matter. The equation (13) is known as the Coulomb failure criterion [6]. Sometimes the Coulomb criterion is written as

$$\tau = S_0 + \sigma_n \tan \varphi_f, \tag{13}$$

where $\mu_i = \tan \varphi_f$. Here φ_f is the angle of friction.

Byerlee [7] compiled data from a large number of rock-friction experiments and found that maximum friction was nearly independent of rock type. For normal stresses >200MPa the relationship is $|\tau| = 50 + 0.6\sigma_n$ Mpa and for normal stresses less than 200 MPa the relationship is $|\tau| = 0.85\sigma_n$ Mpa.

Fault planes are more likely to develop on planes for which the difference $|\tau| + \mu\ \sigma_n$ is maximized as $\sigma_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$ is negative for compressive principal stresses:

$$\begin{aligned} L = &\sqrt{\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - \left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)^2} + \\ &+ \mu\left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right) + \lambda\left(n_1^2 + n_2^2 + n_3^2 - 1\right). \end{aligned} \tag{14}$$

From $\dfrac{\partial L}{\partial n_i} = 0$ we get 3 equations:

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$$\begin{aligned} &\sigma_1 n_1 \left[\left(n_2^2 + n_3^2 - n_1^2\right)\sigma_1 - 2\sigma_2 n_2^2 - 2\sigma_3 n_3^2\right] + 2\mu\sigma_1 \tau n_1 = -2\lambda n_1 \tau, \\ &\sigma_2 n_2 \left[-2\sigma_1 n_1^2 + \left(n_3 + n_1^2 - n_2^2\right)\sigma_2 - 2\sigma_3 n_3^2\right] + 2\mu\sigma_2 \tau n_2 = -2\lambda n_2 \tau, \\ &\sigma_3 n_3 \left[-2\sigma_1 n_1^2 - 2\sigma_2 n_2^2 + \left(n_1^2 + n_2^2 - n_3^2\right)\sigma_3\right] + 2\mu\sigma_3 \tau n_3 = -2\lambda n_3 \tau. \end{aligned} \tag{15}$$

If $n_2 = 0$ from (5) then $\tau = n_1 n_3 (\sigma_1 - \sigma_3)$, and from the first and third of last equations we can eliminate λ to obtain

$$\frac{1}{\mu} = \frac{2 n_1 n_3}{n_1^2 - n_3^2}, \tag{16}$$

with $n_1 = \cos\alpha$, $n_3 = \cos\gamma = \sin\alpha$

$$\frac{1}{\mu} = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{\sin 2\alpha}{\cos 2\alpha} = \mathrm{tg}2\alpha \tag{17}$$

or

$$\alpha = \frac{1}{2}\mathrm{tg}^{-1}\left(\frac{1}{\mu}\right). \tag{18}$$

The coefficients of friction, μ, for a wide variety of rocks, are comparable, in the range 0.6 – 0.9 [7].

$$\begin{aligned} &\mu = 0, && \alpha = 45°; \\ &\mu = 0.6, && \alpha = 30°; \\ &\mu = 1, && \alpha = 22.5°; \\ &\mu \to \infty, && \alpha = 0°. \end{aligned} \tag{19}$$

4. CONCLUSIONS

Seismologists are interested in calculating when a material will break. In a isotropic medium, planes of all orientations would be equally susceptible to failure, and the optimal failure planes are determined by the orientation of the principal axes of the stress tensor. It might be logical that a material would be most likely to fail along planes for which the absolute value of shear stresses are a maximum. Although this is not exactly true, it gives insight into the relation between fault orientation and regional tectonics.

One of the simplest fault models predicts that faulting occurs on planes 45^0 from the maximum and minimum compressive stresses. The (Mohr-) Coulomb criteria is an empirically derived method for determining when a material will fail (break) under shear [8]. The strength of a rock to resists faulting is derived from two sources:

a. S_0 = "cohesive strength" or "cohesive shear strength".

b. Internal frictional resistance to faulting - once the rock is fractured frictional resistance must be overcome to allow movement (*i.e.*, sliding) along the fracture.

In the case of real fractures we find that failure occurs at angles less than $45°$ to the maximum principal stress. For $\mu = 0.0$, $\alpha = 45°$ and the conjugate failure planes are perpendicular. For $\mu > 0.0$ we find $\alpha < 45°$. In addition, the conjugate failure planes are no longer perpendicular, resulting in two sets of P-T axes per failure [9].

REFERENCES

1. H. F. Reid, *The elastic rebound theory of earthquake*, Bulletin of the Department of Geology, **6**, 19 (1911).
2. C. R. Allen, *The San Andreas Fault*, Eng. Sci. Mag., Calif. Inst. Technol., May 1957, pp. 1–5.
3. L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, London, Pergamon Press, 1959.
4. W. D. Means, *Stress and Strain : Basic Concepts of Continuum Mechanics for Geologists*, Springer, New York, 1976.
5. A.Ben-Menahem and S.J.Singh, *Seismic wave and Sources*, Springer, New York, 1980.
6. Z. Reches, *Faulting of rocks in three-dimensional strain fields. II. Theoretical analysis* Tecttonophysics, **95**, 133–156 (1987).
7. J. D. Byerlee, *Friction of Rocks,* Pure and Applied Geophysics, **116**, 615–626 (1978).
8. W. Hafner, *Stress distribution and faulting*, Geol. Soc. Am. Bull., **62**, 373–398 (1951).
9. T. Lay T. and T. C. Wallace, *Modern global seismology*, Academic Press, NY, 1995.