ON FRACTIONAL HAMILTONIAN SYSTEMS POSSESSING FIRST-CLASS CONSTRAINTS WITHIN CAPUTO DERIVATIVES

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The fractional constrained systems possessing only first class constraints are analyzed within Caputo fractional derivatives. It was proved that the fractional Hamilton-Jacobi like equations appear naturally in the process of finding the full canonical transformations. An illustrative example is analyzed.

Key words: fractional Lagrangian, Caputo derivative, fractional Euler-Lagrange equations, first class constraints.

1. INTRODUCTION

The fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order [1–4] and it was extensively used during the last decades in various fields of science and engineering (see for example [5–16] and the references therein). The interest in fractional calculus has been growing during the last decade. This type of calculus has some specific properties. Namely, the form of the fractional Leibniz rule together with the fractional chain rule [17] have totally different forms that the classical ones. In addition of this, the fractional Taylor series (see for example Refs. [18, 19] and the references therein) has a different form reflecting the non-locality of the fractional operators.

The fractional variational principles consists one new and important part of the field of the fractional calculus and they were applied successfully in control theory [20–22] as well as in physics [23–35]. However, for the case of the constrained systems [36, 37] the investigation of the fractional Lagrangian and Hamiltonian formulations are still at the beginning of their development. By using the natural generalization of the classical variational principles, we obtain the fractional Euler-Lagrange equations which differ from the classical one except when the order of the fractional derivative is integer.

An important issue was to obtain the fractional Euler-Lagrange equations for a given fractional Lagrangian and to construct the corresponding Hamiltonian. The non-locality of the fractional Lagrangian is embedded in the definition of the fractional derivatives and it is the main obstacle in finding an appropriate fractional quantization scheme. On the other hand, by replacing the classical derivatives by the fractional ones and following the generalization of the classical variational principles, we obtain the fractional Euler-Lagrange equations which involve both the left and the right derivatives. The immediate consequence of this step is that only in some specific cases we can solve the equations explicitly. For these reasons some other new techniques can be used to describe the fractional dynamics of a given system. An open problem is to investigate the fractional algebra of primary constrains for some specific examples.

The main aim of this paper is to investigate the fractional counterpart of classical systems possessing only first class constraints within Caputo derivatives. The plan of this paper is as follows: in Section 2, we analyze the fractional constrained systems having only the first class constrains. The algebra of constraints is obtained. Section 3 an illustrative is analyzed. Finally, Section 4 is devoted to our conclusion.

2. FRACTIONAL SYSTEMS WITH PRIMARY FIRST CLASS CONSTRAINTS

The use of Caputo derivatives in Physics is of gaining importance because of the specific properties. The fractional variational principles were formulated in terms of Caputo fractional derivatives but many thinks remained unsolved in this area. The left Caputo fractional derivative is given by

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (-\tau + t)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \qquad (1) $$

and the right Caputo fractional derivative has the following form

$$ {}_t^C D_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau - t)^{n-\alpha-1} f^{(n)}(\tau) d\tau \qquad (2) $$

where the order α fulfills $n-1 \leq \alpha < n$ and Γ represents the Euler's Gamma function. The previous definitions implies that the Caputo derivative of a constant becomes zero. We observe that if α becomes an integer, we obtain the usual definitions, namely,

$$ {}_a^C D_t^\alpha f(t) = \left(\frac{d}{dt}\right)^\alpha f(t), \tag{3} $$

$$ {}_t^C D_b^\alpha f(t) = \left(-\frac{d}{dt}\right)^\alpha f(t),\ \alpha = 1,2,.... \tag{4} $$

In addition of this we have the following property.

$$ \int_a^b f(t)[{}_a D_t^\alpha g(t)]dt = \int_a^b g(t)[{}_t D_b^\alpha f(t)]dt. \tag{5} $$

Formula (5) is valid under the assumption that $f(t) \in {}_t I_b^\alpha(L_p)$, $g(t) \in {}_a I_t^\alpha(L_p)$, $\frac{1}{p} + \frac{1}{q} \leq 1+\alpha$.
The fractional dynamics of the constrained systems remains an interesting topics in the area of fractional calculus. We recall that a classical system has primary first class constraints [37] $\Phi_i\ i = 1,...,p$ if

$$ \{\Phi_i, \Phi_j\} = C_{ij}^k \Phi_k. \tag{6} $$

One particular case of constrained systems is when the Hamiltonian is a linear combination of first class constraints only. By making use of (6) we observe that a new subset of constraints can be constructed in such a way that their Poisson brackets are zero. On the other hand we know that the transformation $(p_i, q_i) \to (P_i, Q_i)$ is canonical if and only if

$$ [P_i, P_k] = 0 = [Q_i, Q_k]_{q,k} \ \text{ and } \ [Q_i, P_k]_{q,p} = \delta_{ik}. \tag{7} $$

As a result we can think that a subset of involutive constraints

$$ \{\Phi_i^*, \Phi_j^*\} = 0, \tag{8} $$

can be taken as defining only half of a canonical transformation to a new set of canonical coordinates.

In order to complete the remaining momenta transformations and all q^* transformation we can try to find the generating function S. This is not an easy task, but it can be done using the technique proposed by [38].

3. ILLUSTRATIVE EXAMPLE

In order to apply the above suggested method for the fractional constrained system we are going to investigate an example. The starting point is the classical Lagrangian given by

$$ L = \frac{(x\dot{x} + y\dot{y} + z\dot{z})^2}{2r^2} - V(r), \tag{9} $$

where $r^2 = x^2 + y^2 + z^2$. The next step is to replace the classical derivative by the fractional ones. Therefore, the fractional generalization of (9) becomes

$$L_f = \frac{(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)^2}{2r^2} - V(r). \tag{10}$$

The fractional Euler-Lagrange equations corresponding to (9) are given below

$$\begin{aligned} &-\frac{x(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)^2}{(x^2+y^2+z^2)^2} + \frac{{}_a^C D_t^\alpha x(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)}{(x^2+y^2+z^2)} \\ &-\frac{2x\partial V}{\partial x} + {}_t^C D_b^\alpha \frac{x(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)}{(x^2+y^2+z^2)} = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &-\frac{y(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)^2}{(x^2+y^2+z^2)^2} + \frac{{}_a^C D_t^\alpha y(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)}{(x^2+y^2+z^2)} \\ &-\frac{2y\partial V}{\partial y} + {}_t^C D_b^\alpha \frac{y(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)}{(x^2+y^2+z^2)} = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &-\frac{z(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)^2}{(x^2+y^2+z^2)^2} + \frac{{}_a^C D_t^\alpha z(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)}{(x^2+y^2+z^2)} \\ &-\frac{2z\partial V}{\partial z} + {}_t^C D_b^\alpha \frac{z(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)}{(x^2+y^2+z^2)} = 0. \end{aligned} \tag{13}$$

The fractional canonical momenta defined as

$$p_x = \frac{\partial L_f}{\partial\,{}_a^C D_t^\alpha x}, \quad p_y = \frac{\partial L_f}{\partial\,{}_a^C D_t^\alpha y}, \quad p_z = \frac{\partial L_f}{\partial\,{}_a^C D_t^\alpha z}, \tag{14}$$

have the following forms

$$P_x = x\frac{(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)}{r^2}, \tag{15}$$

$$P_y = y\frac{(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)}{r^2}, \tag{16}$$

and

$$P_z = z\frac{(x\,{}_a^C D_t^\alpha x + y\,{}_a^C D_t^\alpha y + z\,{}_a^C D_t^\alpha z)}{r^2}, \tag{17}$$

respectively. By using (15, 16, 17) we obtain the following constraints among the fractional momenta

$$yP_z - zP_y = 0, \quad zP_x - xP_z = 0, \quad xP_y - yP_x = 0. \tag{18}$$

Making the following notations

$$\Phi_1 = yP_z - zP_y, \quad \Phi_2 = zP_x - xP_z = 0, \quad \Phi_3 = xP_y - yP_x = 0, \tag{19}$$

we conclude that the following algebra is obtained

$$\{\Phi_1, \Phi_2\} = \Phi_3, \quad \{\Phi_2, \Phi_3\} = \Phi_1, \quad \{\Phi_3, \Phi_1\} = \Phi_2. \tag{20}$$

As a result, we observed that not all constraints are independent, for example if we choose Φ_2 and Φ_1 to be independent we obtain

$$P_y = \frac{y}{x} P_x, \quad P_z = \frac{z}{x} P_x. \tag{21}$$

The new obtained constraints

$$\Phi_2^{'} = P_y - \frac{y}{x} P_x, \quad \Phi_3^{'} = P_z - \frac{z}{x} P_x \tag{22}$$

give the same constraints surface as Φ_2 and Φ_1. On the surface of constraints the above constraints commute. By using (22) and taking into account that the above mentioned constraints defined a half canonical transformation we obtain the following fractional partial differential equations

$$P_y^* = {}^{C}D_{-,y}^{\alpha} - \frac{y}{x}\, {}^{C}D_{-,x}^{\alpha}, \tag{23}$$

$$P_z^* = {}^{C}D_{-,z}^{\alpha} - \frac{z}{x}\, {}^{C}D_{-,x}^{\alpha}, \tag{24}$$

where ${}^{C}D_{-,x}^{\alpha}$, ${}^{C}D_{-,y}^{\alpha}$, ${}^{C}D_{-,z}^{\alpha}$ denotes the fractional partial derivatives.

The generalized fractional Hamilton-Jacobi equation associated with this example are given below

$$P_y^* = {}^{C}D_{-,y}^{\alpha} S - \frac{y}{x}\, {}^{C}D_{-,x}^{\alpha} S, P_z^* = {}^{C}D_{-,yz}^{\alpha} S - \frac{yz}{x}\, {}^{C}D_{-,x}^{\alpha} S. \tag{25}$$

4. CONCLUSION

In this paper we have analyzed constrained system only with first class constraints. The fractional generalized Hamilton-Jacobi like equations were obtained naturally in the process of finding the remaining part of the fractional canonical transformations. The algebra of constraints was obtained and the Euler-Lagrange equations were written for a fractional mechanical possessing only first primary constraints. When $\alpha \to 1$ the classical results are reobtained.

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