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TOPOLOGICAL DISSIPATIVE NONLINEAR MODES IN TWO- AND THREE-DIMENSIONAL GINZBURG-LANDAU MODELS WITH TRAPPING POTENTIALS

D. MIHALACHE[1,2]

[1]"Horia Hulubei" National Institute for Physics and Nuclear Engineering (IFIN-HH), 407 Atomistilor, Magurele-Bucharest, RO-077125, Romania, E-mail: Dumitru.Mihalache@nipne.ro
[2]Academy of Romanian Scientists, 54 Splaiul Independentei, Bucharest 050094, Romania

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Abstract. A brief overview of recent theoretical results in the area of two- and three-dimensional dissipative topological solitons described by the complex cubic-quintic Ginzburg-Landau partial differential equation with two kinds of external potentials is given.

Key words: Dissipative solitons, Ginzburg-Landau equation, vortex solitons.

1. INTRODUCTION

Two-dimensional (2D) spatial optical solitons are spatially confined light beams propagating in nonlinear media, i.e. in optical media whose refractive indices depend on light intensity. They are localized (self-guided) in two transverse (spatial) dimensions, therefore they constitute nondiffracting electromagnetic wave-packets. Their spatiotemporal counterparts are nondiffracting and nondispersing wave-packets, which form in certain nonlinear optical media under special conditions. Thus the three-dimensional (3D) spatiotemporal optical solitons are localized (self-guided) in the two transverse (spatial) dimensions and in the direction of propagation due to the balance of anomalous group-velocity dispersion (GVD) of the medium in which they form and nonlinear self-phase modulation. Therefore, a spatiotemporal optical soliton is a fully 3D localized physical object in both space and time coordinates; for a relevant literature of the theoretical and experimental advances in the area of multidimensional optical solitons in the past two decades see Refs. [1-22]. It is commonly believed that the two- and three-dimensional optical solitons could be used as information carriers in future all-optical information processing systems due to their remarkable potential for massive parallelism (in space) and pipelining (in time) [1-3].

Both 2D and 3D optical solitons propagating in self-focusing cubic (Kerr-like) optical media are unstable because of the occurrence of collapse in the governing nonlinear Schrödinger model [4]. However, in the past decade, a lot of possibilities to "arrest" the wave collapse which is unavoidable in cubic self-focusing media were put forward. We mention here only a few of these approaches: (i) the use of weaker nonlinearities: either saturable nonlinearities [9] or quadratic nonlinearities [10-11], (ii) the combination of (a) quadratic and cubic nonlinearities, and of (b) self-focusing cubic and self-defocusing quintic nonlinearities, (iii) the use of off-resonance two-level systems [12], and (iv) the use of tandem layered structures [13]. The unique features of both 2D and 3D optical solitons in a lot of relevant physical settings have been investigated both theoretically and experimentally in the last two decades [1-22].

The localized optical vortices, i.e., *vortex solitons*, have drawn much attention within the past years as physical objects of fundamental interest, and also due to their potential applications to all-optical information processing, as well as to the guiding and trapping of atoms. In the core of an optical vortex the complex electromagnetic field is equal to zero, however the circulation C of the gradient of the phase of the complex field on an arbitrary closed contour around the vortex core is a multiple of 2π, i.e., $C = 2\pi S$, where the integer S is the *topological number* of the vortex ("spin"). Thus the phase dislocations carried by the wavefront of a light beam are associated with a zero-intensity point (a vortex core); the phase is twisted around such points where the light intensity vanishes, creating an optical vortex.

It is worthy to mention that unique properties are also featured by vortex clusters, such as rotation similar to the vortex motion in superfluids. The complex dynamics of two- and three-dimensional soliton clusters in optical media with competing nonlinearities has been studied too [22]. Various complex patterns based on both fundamental (nonspinning) solitons and vortices were theoretically investigated in optics and in the usual BEC models governed by the Gross-Pitaevskii equation with both local and nonlocal nonlinearity. Stable nondissipative spatiotemporal spinning solitons (vortex tori) with the topological charge S=1, described by the three-dimensional nonlinear Schrödinger equation with focusing cubic and defocusing quintic nonlinearities were found to exist for sufficiently large energies [14]. This result also holds for the case of competing quadratic and self-defocusing cubic nonlinearities [14]. A general conclusion of these theoretical studies is that stable spinning solitons are possible as a result of competition between focusing and defocusing optical nonlinearities.

The presence of gain and loss (due to optical amplifiers and saturable absorbers) in optically nonlinear media would influence the unique properties of 2D and 3D optical solitons in conservative (dissipativeless) systems. Thus *dissipative optical solitons* are possible in several physical settings, including laser cavities [23-32]. Recently we performed a comprehensive stability analysis of three-dimensional dissipative solitons with intrinsic vorticity S governed by the complex Ginzburg-Landau equation with cubic and quintic terms in its dissipative and conservative parts [31]. It was found that a necessary stability condition for all vortex solitons, but not for the fundamental ones ($S = 0$), is the presence of nonzero diffusivity in the transverse plane. The fundamental solitons are stable in all cases when they exist, while the vortex solitons are stable only in a part of their existence domain. However, the spectral filtering (i.e., the temporal-domain diffusivity) is not necessary for the stability of any species of dissipative solitons. Stability domains were found for three-dimensional vortex solitons (alias vortex tori) with "spin" (topological charge) $S = 1$, 2, and 3, suggesting that spinning solitons with any vorticity S can be stable in certain portions of their existence domains [31].

Recently [33] we studied the existence and stability domains of vortex solitons in the Ginzburg-Landau model of a two-dimensional lasing medium with a transverse grating (a 2D harmonic potential in the transverse spatial coordinates x and y). We introduced a two-dimensional model of a laser cavity based on the complex Ginzburg-Landau equation with the cubic-quintic nonlinearity and a lattice (harmonic) potential accounting for the transverse grating. We found that *localized vortices*, built as sets of four peaks pinned to the periodic potential, may be stable without the unphysical diffusion term, which was necessary for the stabilization in previously studied Ginzburg-Landau models for dissipative optical solitons. The vortices were chiefly considered in the onsite form, but the stabilization of offsite vortices and quadrupoles was shown too [33]. Stability regions for the rhombic (onsite) vortices and fundamental ($S = 0$) solitons were identified in the parameter space of the cubic-quintic Ginzburg-Landau model, and scenarios of the evolution of unstable vortices were also described, see Ref. [33]. The main result of this study was the necessity of a minimum strength of the lattice potential in order to stabilize the vortices. A stability border was also identified in the case of the self-focusing quintic term in the underlying model, which suggested a possibility of the supercritical collapse. Notice that beyond this border, the stationary vortex turns into a vortical breather, which is subsequently replaced by a dipolar breather and eventually by a single-peak breather; see Ref. [33].

In Refs. [34] we investigated in detail families of spatiotemporal dissipative solitons in a model of 3D laser cavities including a combination of gain, saturable absorption, and transverse grating. The governing model was based on the complex Ginzburg-Landau equation with the cubic-quintic nonlinearity and a 2D external grating potential. The shapes of both fundamental (vorticityless) and vortex solitons were found by adequate numerical techniques as robust attractors in this dissipative model and their stability against strong random perturbations was tested by direct numerical simulations. The vorticityless solitons were found to be completely stable in this model. The 3D vortex solitons, built as rhombus-shaped complexes of four fundamental solitons, may be split by perturbations into their constituents separating in the temporal direction, see Ref. [34]. Nevertheless, we have found that a sufficiently strong grating makes the 3D vortex solitons practically stable physical objects.

Recently we have analyzed in detail the existence and stability characteristics of topological modes in 2D Ginzburg-Landau models with external trapping potentials [35]. It is well known that complex Ginzburg-Landau models of laser media with cubic-quintic nonlinearities do not contain an effective diffusion term, which makes all vortex solitons unstable in these models. However, as it was said above, in a recent publication [33] it has been demonstrated that the addition of a two-dimensional periodic potential, which may be induced by a transverse grating in the laser cavity, to the cubic-quintic complex Ginzburg-Landau model is able to support stable compound (four-peak) vortices, but the most fundamental "crater-shaped" vortices, alias vortex rings, which are essentially squeezed into a single cell of the external potential, have not been found before in a stable form. In Ref. [35] we reported on families of stable compact crater-shaped vortices with vorticity $S = 1$ in the complex Ginzburg-Landau model with external potentials of two different types: (a) an axisymmetric parabolic trapping potential, and (b) a periodic harmonic potential. In both situations, we identified stability regions for the crater-shaped vortices and for the fundamental (vorticityless) solitons with $S = 0$. Those crater-shaped vortices which are unstable in the axisymmetric (parabolic) potential break up into robust dipoles. All the vortices with vorticity number $S = 2$ are unstable, splitting into stable tripoles. Stability regions for the dipoles and tripoles were identified, too in Ref. [35]. Moreover, it was shown that the harmonic periodic potential cannot stabilize compact crater-shaped vortices with S larger than 2 either; instead, families of stable compact square-shaped quadrupoles were found; see Ref. [35] for a comprehensive study of this issue.

2. TWO- AND THREE-DIMENSIONAL TOPOLOGICAL DISSIPATIVE MODES IN GINZBURG-LANDAU MODELS WITH EXTERNAL POTENTIALS

The most general three-dimensional complex cubic-quintic Ginzburg-Landau partial differential equation is

$$\mathrm{i}E_z + (1/2-\beta)(E_{xx}+E_{yy}) + (1/2)E_{tt} + [\mathrm{i}\delta + (1-\mathrm{i}\varepsilon)|E|^2 - (\nu - \mathrm{i}\mu)|E|^4]E = 0, \quad (1)$$

where $E(x, y, z, t)$ is the complex field amplitude, (x, y) are the normalized transverse coordinates, and t is the reduced temporal variable [34]. The coefficients which are scaled to be (1/2 – β) and 1 account, respectively, for the diffraction in the transverse plane and self-focusing Kerr nonlinearity, the coefficient ν takes into account the quintic nonlinearity, that may compete with the cubic term, and the coefficient (1/2) in front of the double time derivative in Eq. (1) is the GVD. In this work I consider only anomalous GVD. In the dissipative part of the dynamic equation, the real constants δ, ε and μ represent, respectively, the linear loss, cubic gain, and quintic loss, which are the basic ingredients of the cubic-quintic Ginzburg-Landau equation.

The physical interpretation of all terms in Eq. (1) is straightforward, except for the diffusion term proportional with the parameter β. This term arises in some models of large-aspect-ratio laser cavities, close to the lasing threshold. Note that the diffusion term is rather artificial in the applications to optics. Nevertheless, $\beta > 0$ is a necessary condition for the stability of dissipative vortex solitons, while the fundamental ($S = 0$) solitons may be stable at $\beta = 0$, see Refs. 29 and 31. Therefore, a challenging problem is to develop a physically relevant modification of the 2D and 3D complex Ginzburg-Landau models, without the diffusivity ($\beta = 0$), that can support stable vortex solitons. Recently [33], it has been demonstrated that this problem can be resolved by adding a transverse periodic potential to Eq. (1), which casts the three-dimensional complex cubic-quintic Ginzburg-Landau partial differential equation into the following form:

$$\mathrm{i}E_z + (1/2)(E_{xx}+E_{yy}) + (1/2)E_{tt} + [\mathrm{i}\delta + (1-\mathrm{i}\varepsilon)|E|^2 - (\nu - \mathrm{i}\mu)|E|^4]E + V(x,y)E = 0. \quad (2)$$

Here the external potential $V(x, y)$ might be (i) the harmonic 2D grating potential $V(x, y) = p[\cos(2x) + \cos(2y)]$, where the parameter p is the strength of the external trapping potential whose period is scaled to be π, and (ii) the axisymmetric trapping potential (the parabolic potential) $V(x, y) = -(\Omega^2/2)\, r^2$, where the parameter Ω is the trapping frequency.

In the 2D case, Eq. (2) simplifies to the following complex cubic-quintic Ginzburg-Landau equation with the 2D trapping potential $V = V(x, y)$:

$$\mathrm{i}E_z + (1/2)(E_{xx} + E_{yy}) + [\mathrm{i}\delta + (1 - \mathrm{i}\varepsilon)|E|^2 - (\nu - \mathrm{i}\mu)|E|^4]E + V(x, y)E = 0, \quad (3)$$

where the potential $V(x, y)$ is either the harmonic 2D grating potential or the axisymmetric trapping potential (the parabolic potential).

In Fig. 1 we show an illustrative plot of the 2D periodic external potential with strength $p = 1$. Stationary 3D topological solitons were generated as attractors by direct simulations of Eq. (2). Thus found objects are single-peaked fundamental solitons (with $S = 0$), and rhombus-shaped vortical solitons (alias onsite vortices), built as compound objects, consisting of four separate peaks of the local field intensity, set at local potential minima of the lattice potential, with an empty site in the middle, see Fig. 2 below. The topological charge (intrinsic vorticity) of the complex patterns is provided by the phase shift $\pi/2$ between adjacent peaks, which corresponds to the total phase circulation of 2π around the core of the pattern, as it should be for topological dissipative nonlinear modes with vorticity ("spin") $S = 1$.

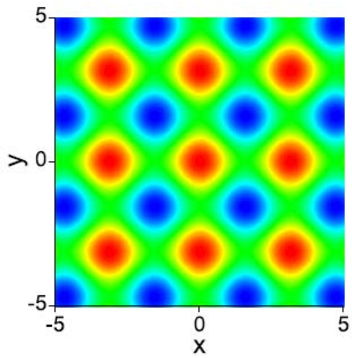

Fig. 1 – The amplitude distribution of the external two-dimensional harmonic potential for $p = 1$.

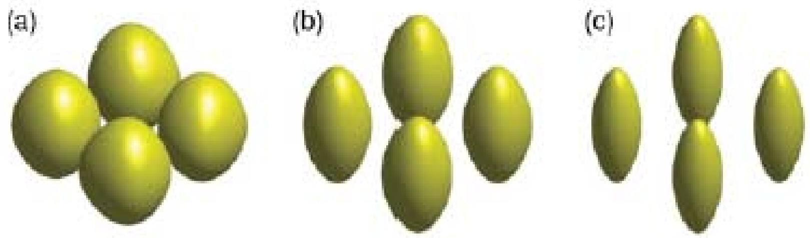

Fig. 2 – Isosurface plots of total intensity showing typical quasistable rhombic vortex solitons: a) $p = 0.25$, $\varepsilon = 1.9$; b) $p = 1$, $\varepsilon = 1.7$, and c) $p = 4$, $\varepsilon = 1.8$, see Ref. 34.

The families of 3D fundamental solitons are completely stable, i.e., the solitons restore their stationary shape after the addition of random (white noise) perturbations. The stability of fundamental solitons in the present 3D model is not surprising, as they are stable in both the 2D [29] and 3D [31] versions of Eq. (2) without the lattice (periodic) potential (i.e. for $p = 0$), unlike localized vortices, which cannot be stable in either case. The temporal trajectories of the four constituents of unstable $S = 1$ rhombic vortex solitons whose stationary field distributions are shown in Fig. 2 are plotted in Fig. 3 for two representative values of the parameters p and ε. Fig. 3 clearly illustrates *quasistabilization* of the vortex solitons with the increase of the grating's strength p. We see from Fig. 3 that the splitting distance for $p = 1$ is $z \approx 100$, which is about ten soliton's dispersion lengths, with the temporal width of each constituent being $T \sim 2$.

Next we present the output of numerical simulations of fundamental solitons, topological solitons with "spin" $S = 1$, and other complex structures such as dipoles and tripoles which exist and are stable in the parabolic axisymmetric external potential $V(x, y) = -(\Omega^2/2)\, r^2$, for different values of the trapping frequency Ω. In Fig. 4 we show the recovery of a perturbed stable fundamental soliton ($S = 0$) for a representative set of parameters: ε = 1.8 and Ω = 2, whereas in Fig. 5 we display the recovery of a perturbed stable vortex soliton with $S = 1$ for the same set of parameters. The recovery of a perturbed stable dipole soliton for ε = 1.9 and Ω = 1 is shown in Fig. 6. The generation of a robust rotating tripole in the axisymmetric parabolic potential from an input cluster formed by three Gaussians with phase differences 2π/3 between them is displayed in Fig. 7 for ε = 1.7 and Ω = 0.5.

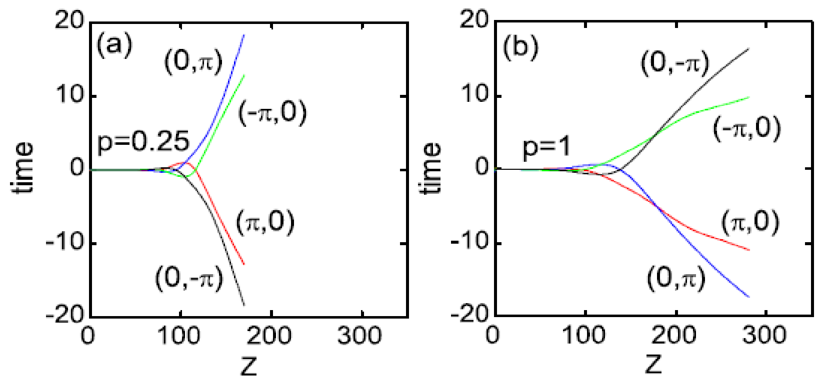

Fig. 3 – Temporal trajectories of the four constituents of quasistable $S = 1$ rhombic vortices showing their temporal separation during propagation: a) $p = 0.25$, ε = 1.9; b) $p = 1$, ε = 1.7.

In Ref. 35 we also applied the generalized variational approximation, which was developed in Ref. [30] for a class of complex Ginzburg-Landau equations, as an extension of the well-known variational method for conservative nonlinear-wave systems. The existence limits of both fundamental and crater-shaped spinning solitons with $S = 1$ were found to be in excellent agreement with the corresponding values obtained from direct numerical simulations, see Ref. 35 for details of this comprehensive study.

We have found that higher-order vortex solitons, with topological charge $S \geq 2$, are found to be completely unstable. If vortices with $S = 1$ are unstable, they spontaneously split into *stable dipoles*, whereas those with $S = 2$ split into *stable tripoles*, see Ref. 35.

In the following we present a few representative examples of the output of extensive numerical simulations of 2D topological dissipative nonlinear modes which form in the 2D harmonic potential (transverse grating potential) $V(x, y) = p$ $[\cos(2x) + \cos(2y)]$, for representative values of the strength parameter p. Thus in Fig. 8 we show the amplitude and phase structure of stable compact vortices with $S = 1$ for $\beta = 0.1$ and $\varepsilon = 2.8$. The external potential strength is $p = 2$ (Fig. 8a, b and $p = 5$ Fig. 8 c, d). The evolution of the power P of a stable compact vortex with $S = 1$ for the parameters $\beta = 0$, $\varepsilon = 2$, and $p = 2$ is shown in Fig. 9a and the evolution of an unstable compact vortex for the same set of parameters but for a higher value of the cubic gain $\varepsilon = 2.5$ is displayed in Fig. 9b.

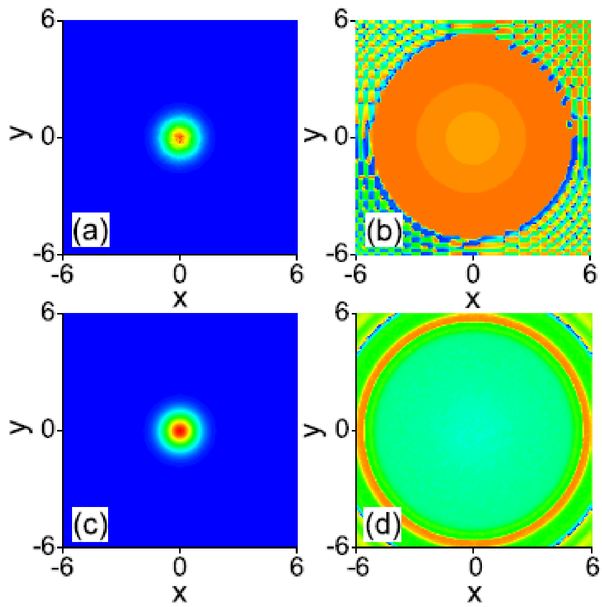

Fig. 4 – The recovery of a perturbed stable fundamental soliton ($S = 0$) for $\varepsilon = 1.8$ and $\Omega = 2$: a) and (b) – perturbed input amplitude and phase distributions; (c) and (d) – self-cleaned output amplitude and phase distributions (at $z = 200$).

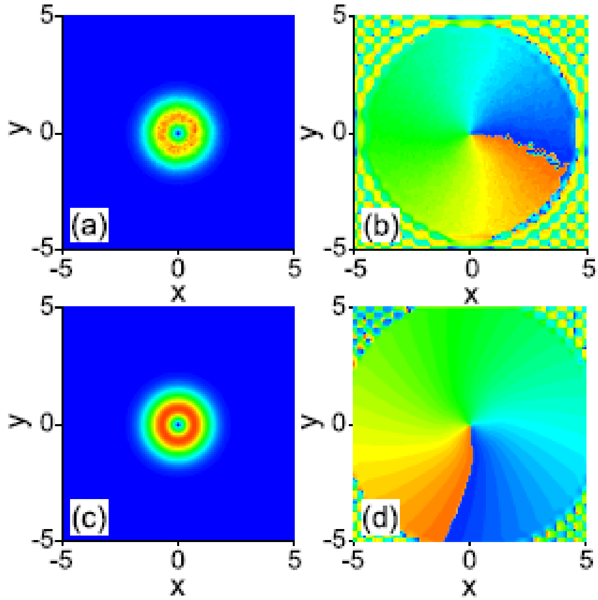

Fig. 5 – The recovery of a perturbed stable vortex soliton with $S = 1$ for ε = 1.8 and Ω = 2: a) and b) – perturbed input amplitude and phase distributions; c) and d) – self-cleaned output amplitude and phase distributions (at $z = 200$).

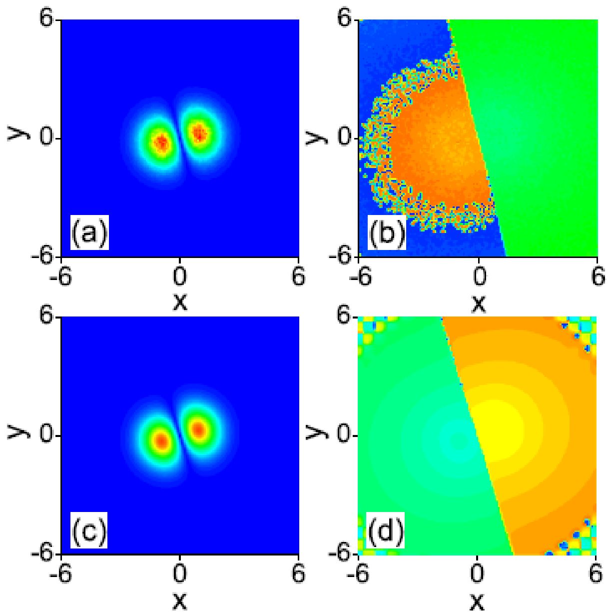

Fig. 6 – The recovery of a perturbed stable dipole soliton for ε = 1.9 and Ω = 1: a) and b) – perturbed input amplitude and phase distributions; c) and d): self-cleaned output amplitude and phase distributions (at $z = 200$), see Ref. 35.

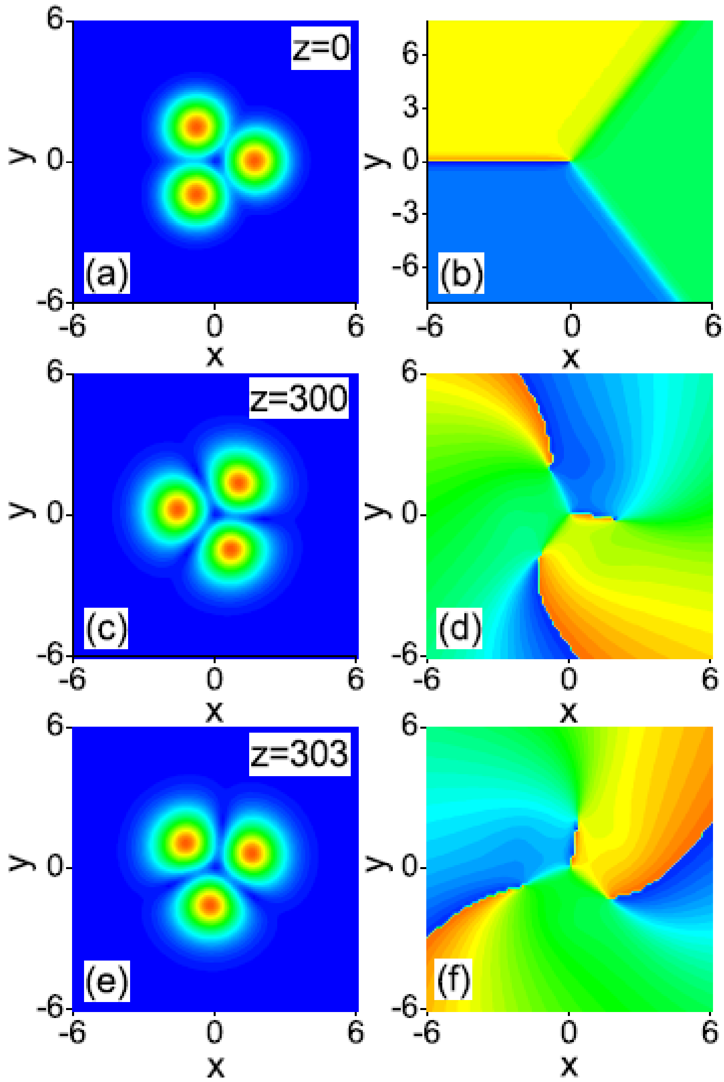

Fig. 7 – The generation of a robust rotating tripole in the axisymmetric parabolic potential from an input cluster formed by three Gaussians with phase differences $2\pi/3$ between them. Left panels: the input field (a), and the established field amplitude at $z = 300$ (c) and at $z = 303$ (e). Right panels: the phase of the input field (b), and the phases of the established pattern at $z = 300$ (d) and at $z = 303$ (f). Here the parameters are $\varepsilon = 1.7$ and $\Omega = 0.5$, see Ref. 35.

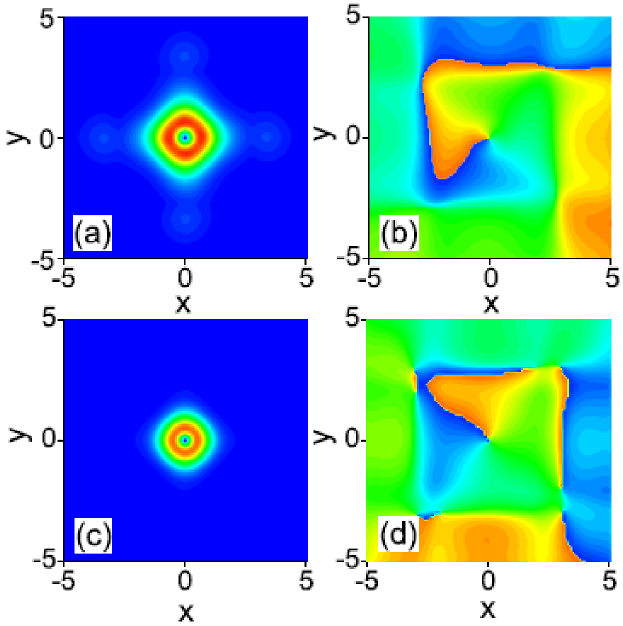

Fig. 8 – The amplitude and phase structure of stable compact vortices with $S = 1$ for $\beta = 0.1$ and $\varepsilon = 2.8$. The external potential strength is: a) and b) – $p = 2$; c) and d) – $p = 5$.

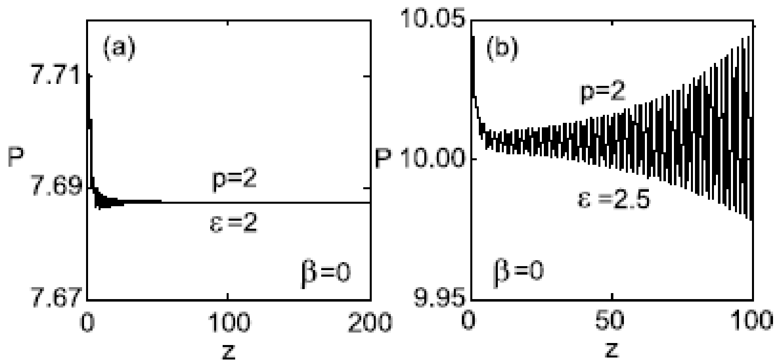

Fig. 9 – Evolution of the power P of a stable compact vortex with $S = 1$ for the parameters $\beta = 0$, $\varepsilon = 2$, and $p = 2$ (a) and of an unstable compact vortex for the same parameters but for a higher value of the cubic gain $\varepsilon = 2.5$ (b).

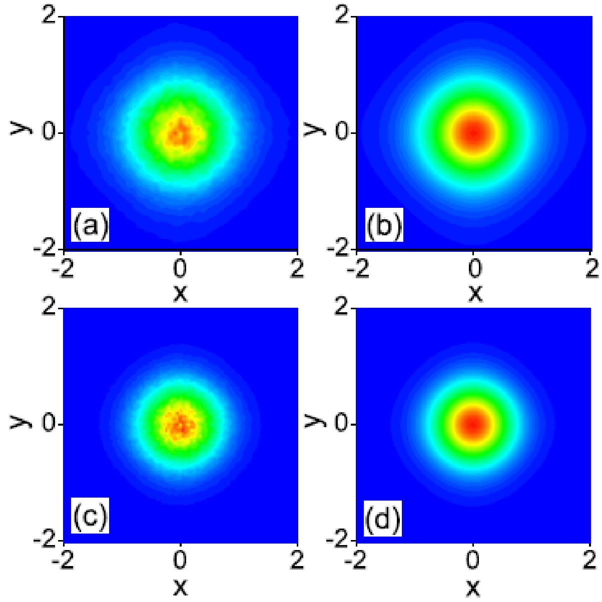

Fig. 10 – Self-cleaning process of perturbed stable fundamental solitons for two sets of parameters: a) and b) $\beta = 0$, $\varepsilon = 2$, $p = 2$; c) and d) $\beta = 0.1$, $\varepsilon = 3$, $p = 5$. The random noise in the inputs plotted in panels a) and c) was at the 10% level and the self-cleaned solitons plotted in panels b) and d) are shown at the propagation distance $z = 300$.

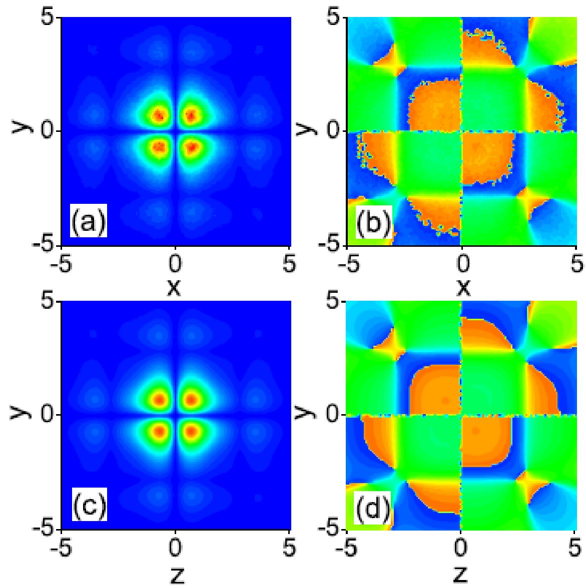

Fig. 11 – Top: the amplitude a), and phase b) of the perturbed input compact quadrupole for the parameters $\beta = 0$, $\varepsilon = 2.2$, and $p = 2$. Bottom: the amplitude c), and phase d) of the self-cleaned compact quadrupole at $z = 1000$, see Ref. 35.

In Fig. 10 we show the self-cleaning process of perturbed stable fundamental solitons ($S = 0$) for two representative sets of parameters: $\beta = 0$, $\varepsilon = 2$, $p = 2$ (Fig. 10a and b), and $\beta = 0.1$, $\varepsilon = 3$, $p = 5$ (Fig. 10c and d). The random noise in the inputs plotted in panels a) and c) of Fig. 10 was at the 10% level. Stable crater-shaped vortices with vorticities $S = 2$ have not been found in direct simulations of Eq. (3) with the periodic (harmonic) potential; instead, families of robust compact square-shaped quadrupoles, into which unstable vortices with $S = 2$ are spontaneously transformed, were found at different values of the strength p of the periodic potential. In Fig. 11 we show the amplitude (Fig. 11a and phase (Fig. 11b) of the perturbed input compact quadrupole for the parameters $\beta = 0$, $\varepsilon = 2.2$, and $p = 2$. The amplitude and phase of the self-cleaned compact quadrupole at $z = 1000$ are plotted in Fig. 11 c–d.

For the direct numerical simulations of the nonlinear partial differential equation (3), the Crank-Nicolson algorithm was used for the numerical simulations, with transverse and longitudinal step sizes $\Delta x = \Delta y = 0.1$ and $\Delta z = 0.005$ for the grating strength $p = 1$. For larger values of p, it was necessary to use smaller step sizes: $\Delta x = \Delta y = 0.08$, $\Delta z = 0.004$ for $p = 2$, and $\Delta x = \Delta y = 0.06$, $\Delta z = 0.003$ for $p = 5$; see Ref. 35 for more details of these extensive numerical simulations.

3. CONCLUSIONS

In this work, I gave an overview of recent studies of families of both fundamental (vorticityless) and vortical dissipative solitons in the framework of both two-dimensional and three-dimensional complex Ginzburg-Landau models with the cubic-quintic nonlinearity and with two kinds of external potentials: (i) an axisymmetric (parabolic) potential, and (ii) a periodic potential in the transverse plane. In the three-dimensional setting an anomalous group-velocity dispersion in the temporal direction was considered. In this case, only the fundamental solitons are completely stable against strong random (white noise) perturbations, while the vortices, built as rhombus-shaped complexes of four fundamental solitons with appropriate phase shifts between them, are quasistable physical objects; they may be split by white noise perturbations into four stable vorticityless three-dimensional solitons separating in the temporal direction. In the two-dimensional physical setting, in addition to compact crater-shaped vortices (alias vortex rings) which exist and are stable in both kinds of external potentials we have found families of stable topological dissipative nonlinear modes such as vortex dipoles, vortex tripoles (in the case of axisymmetric trapping potentials) and vortex quadrupoles (in the case of periodic grating potentials).

An important open problem suggested by these studies is to find stable compact solitons with embedded vorticity in the three-dimensional (spatiotemporal) version of the complex Ginzburg-Landau equation with the periodic (harmonic) two-dimensional potential. This issue is extremely interesting because this physical model does not support completely stable compound vortices formed by four fundamental (vorticityless) solitons [34].

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