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INVESTIGATIONS OF COUPLING OF $Z = (z - t)$ TYPE PLANE FRONTED WAVES AND ELECTROMAGNETIC WAVES WITH MASSLESS SCALAR PLANE WAVES AND MASSIVE SCALAR WAVES IN PERES SPACE-TIME

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Abstract. In this paper, we deduce the existence of massless scalar plane waves coupled to plane fronted waves and electromagnetic waves and non-existence of massive scalar waves coupled to plane fronted gravitational waves and electromagnetic waves in Peres space-time.

Key words: Ricci tensor, massless scalar field, massive scalar field, energy momentum tensor, electromagnetic waves.

1. INTRODUCTION

The plane gravitational waves g_{ij} are mathematically exposed by H. Takeno [1] in general relativity. He has studied $(z - t)$ and (t/z)-type plane gravitational waves and obtained the line element for both waves as

$$ds^2 = -A dx^2 - 2D dx dy - B dy^2 - \left(C - E\right) dz^2 - 2E dz dt + \left(C + E\right) dt^2, \quad (1.1)$$

and

$$ds^2 = -A dx^2 - 2D dx dy - B dy^2 - Z^2 \left(C - E\right) dz^2 - 2ZE dz dt + \left(C + E\right) dt^2, \quad (1.2)$$

where A, B, C, D and E are function of Z.

Peres [2] deduced the metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - 2 f(x, y, Z)(dz - dt)^2, \quad (1.3)$$

in which $g_{ij} = g_{ij}(f)$ and $f = f(x, y, Z)$, $Z = z - t$.

The necessary and sufficient condition that space-time (1.3) satisfies field equation

$$R_{ij} = 0 \text{ and } \Delta f = 0 \ \ \Delta\partial_{11} + \partial_{22}, \ \ \partial_{11} = \frac{\partial^2}{\partial x \partial x}, \ \ \partial_{22} = \frac{\partial^2}{\partial y \partial y}.$$

Takeno (1961) denote (0) – system, the original co-ordinate system in which metric is (1.3) and 0- system is a co-ordinate system in which the metric is of the form as (1.3) and shows co-existence of gravitational waves with electromagnetic waves.

For space-time (1.3), he has obtained the non-vanishing components of F_{ij} as

$$\begin{aligned} F_{14} = F_{31} &= \sigma, \\ -F_{23} = F_{24} &= -\rho, \end{aligned} \tag{1.4}$$

where σ and ρ are functions of $Z = (z - t)$.

Deshmukh and Karade (2004) have shown the existence of massless scalar waves coupled with gravitational and electromagnetic waves. They further introduced massive scalar field in place of massless scalar field and shows the non-existence of massive scalar waves coupled with gravitational and electromagnetic waves for metric (1.1) and (1.2) respectively. In this paper, we want to investigate the same for plane fronted waves in Peres space-time whose metric is (1.3), in which $Z = z - t$.

For our investigations we use Einstein's equations

$$R_{ij} = -8\pi\left(T_{ij} - \frac{1}{2} g_{ij} T\right), \ \ i, j = 1, ..., 4, \tag{1.6}$$

with the metric (1.3).

2. EXISTENCE THEOREM

THEOREM. *The existence of massless Scalar waves coupled to plane fronted gravitational waves and electromagnetic waves is assured by the four zero eigen values of the energy momentum tensor of the combined distribution of scalar and electromagnetic fields.*

Proof: Here we consider following two cases.

Case – I. *Scalar field is coupled to gravitational field*

The scalar field is described by the tensor

$$T_{ij} = \frac{1}{4\pi}\left(V_i V_j - \frac{1}{2} g_{ij} V_s V^s \right), \qquad s = 1, \ldots, 4\,, \tag{2.1}$$

where V is scalar function of Z and

$$V_i = \frac{\partial V}{\partial x^i}, \qquad x^i = (x, y, z, t).$$

From (2.1),

$$T = T_i^i = \frac{1}{4\pi}\left(V_i V_j - \frac{1}{2} g_i^i V_s V^s \right) = \frac{1}{4\pi}\left(V_i V^i - 2 V_s V^s \right) = \frac{1}{4\pi}\left(-V_s V^s \right). \tag{2.2}$$

Now

$$\begin{aligned}
V_1 &= \frac{\partial V}{\partial x^1} = \frac{\partial V}{\partial x} = \frac{\partial V}{\partial Z} \cdot \frac{\partial Z}{\partial x} = \overline{V} \cdot 0 = 0,\\
V_2 &= \frac{\partial V}{\partial x^2} = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial Z} \cdot \frac{\partial Z}{\partial y} = \overline{V} \cdot 0 = 0,\\
V_3 &= \frac{\partial V}{\partial x^3} = \frac{\partial V}{\partial z} = \frac{\partial V}{\partial Z} \cdot \frac{\partial Z}{\partial z} = \overline{V} \cdot (1) = \overline{V},\\
V_4 &= \frac{\partial V}{\partial x^4} = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial Z} \cdot \frac{\partial Z}{\partial t} = \overline{V} \cdot (-1) = -\overline{V},
\end{aligned} \tag{2.3}$$

where bar (-) over letter means derivative with respect to Z. So,

$$\begin{aligned}
V_s V^s = V_s g^{sp} V_p = \overline{V}\left[\overline{V}(-1 - 2f) + \overline{V}(2f) - \overline{V}(2f) - \overline{V}(1 + 2f) \right] = \\
\overline{V}\left(-\overline{V} + 2f\overline{V} - \overline{V} - 2f\overline{V} \right) = 0.
\end{aligned} \tag{2.4}$$

We find that, for (1.3): $g^{33} = -(1 + 2f),\ g^{34} = 2f,\ g^{44} = (1 - 2f).$

Expression (2.2) implies $T = 0$ and then

$$T_{ij} = \frac{1}{4\pi} V_i V_j, \tag{2.5}$$

and

$$R_{ij} = (-8\pi)\frac{1}{4\pi} V_i V_j = (-2) V_i V_j. \tag{2.6}$$

The non-vanishing V_3, V_4 assume the form

$$V_3 V_3 = \overline{VV}, \quad V_4 V_3 = \overline{VV}, \quad V_4 V_4 = \overline{VV}. \tag{2.7}$$

For the line element (1.2), we have (Takeno 1961, 55.5)

$$R_{33} = -R_{34} = R_{44}. \tag{2.8}$$

Equation (2.6) with (2.7) yield,

$$R_{33} = -2\overline{VV}, \quad R_{34} = -2\overline{VV}, \quad R_{44} = -2\overline{VV}. \tag{2.9}$$

Above equations are compatible with (2.8) and they imply the co-existence of scalar waves with gravitational waves.

Now we relate the existence of waves with eigen values of matter tensor. The eigen values of T_{ij} in this case are given by

$$\left|T_{ij} - \lambda g_{ij}\right| = 0,$$

$$\begin{vmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & T_{33} + \lambda(1+2f) & T_{34} - 2\lambda f \\ 0 & 0 & T_{43} - 2\lambda f & T_{44} - \lambda(1-2f) \end{vmatrix} = 0. \tag{2.10}$$

From (2.5) we have

$$\begin{aligned} T_{33} &= \frac{1}{4\pi} V_3 V_3 = \frac{\overline{V}^2}{4\pi}, \\ T_{43} &= \frac{1}{4\pi} V_3 V_4 = \frac{\overline{V}^2}{4\pi}, \\ T_{44} &= \frac{1}{4\pi} V_4 V_4 = \frac{\overline{V}^2}{4\pi}. \end{aligned} \tag{2.11}$$

Solving (2.10) we get

$$\lambda^2 \left\{ \left[T_{33} + \lambda(1+2f)\right]\left[T_{44} - \lambda(1-2f)\right] - \left[T_{34} - 2\lambda f\right]^2 \right\} = 0,$$

$$\lambda^2 \left\{ T_{33}T_{44} - T_{33}\lambda(1-2f) + T_{33}\lambda(1+2f) - \lambda^2(1-4f^2) - T_{34}^2 - 4\lambda f T_{34} - 4\lambda^2 f^2 \right\} = 0.$$

$$\lambda^4 = 0.$$

In this case all four eigen values are zero and hence principal directions are not unique. Thus the co-existence of plane fronted gravitational waves with scalar waves is related to the eigen values of T_{ij} which are all zero.

Case II. *Scalar field coupled to gravitational and electromagnetic field*

The matter tensor in this case is

$$T_{ij} = E_{ij} \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2} g_{ij} V_s V^s\right], \tag{2.12}$$

where

$$E_{ij} = \frac{1}{4\pi}\left[\frac{1}{4}g_{ij}F_{sp}F^{sp} - F_{is}F_j^s\right], \quad p = 1,\ 2,\ 3,\ 4.$$

Here F_{is} is an electromagnetic Maxwell skew tensor given by

$$F_{is} = K_{s;i} - K_{s;i} = K_{s;i} - K_{i;s}, \quad K_i = K_i(Z).$$

Semicolon (l) means covariant derivative with respect to g_{ij} and the comma (,) stands for partial derivatives with respect to x^i.

The gravitational field is given by (1.2). As $V_sV^s = 0,$ by (2.5) one obtains

$$T_{ij} = E_{ij} + \frac{1}{4\pi}\left[V_iV_j\right], \quad R_{ij} = (-8\pi)\left[E_{ij} + \frac{1}{4\pi}\left(V_iV_j - \frac{1}{2}g_{ij}T\right)\right]. \tag{2.13}$$

Now

$$T = T_i^i = E_i^i + \frac{1}{4\pi}V_iV^i = E_i^i = \frac{1}{4\pi}\left(\frac{1}{4}g_{ij}F_{sp}F^{sp} - F_{is}F^{is}\right) = \frac{1}{4\pi}\left(\frac{1}{4}\cdot 4F_{sp}F^{sp} - F_{is}F^{is}\right) = 0.$$

Then

$$R_{ij} = (-8\pi)\left(E_{ij} + \frac{1}{4\pi}V_iV_j\right). \tag{2.14}$$

Using (2.12) and T

$$R_{ij} = -2\left(4\pi E_{ij} + V_iV_j\right).$$

Since electromagnetic waves coexist with gravitational waves we have (Takeno 1961, 56.2)

$$E_{33} = -E_{34} = E_{44}. \tag{2.15}$$

Equations (2.14) becomes

$$\begin{aligned} R_{33} &= -2\left(4\pi E_{33} + V_3V_3\right) = -2\left(4\pi\left(\rho^2 + \sigma^2\right) + \overline{V}^2\right), \\ R_{34} &= -2\left(4\pi E_{34} + V_3V_4\right) = 2\left(4\pi\left(\rho^2 + \sigma^2\right) + \overline{V}^2\right), \\ R_{44} &= -2\left(4\pi E_{44} + V_4V_4\right) = 2\left(4\pi\left(\rho^2 + \sigma^2\right) + \overline{V}^2\right). \end{aligned} \tag{2.16}$$

Equation (2.15) with (2.16) imply

$$R_{33} = -R_{34} = R_{44}. \tag{2.17}$$

Equation (2.17) is compatible with (2.8) and hence follows the existence of scalar waves coupled to plane fronted gravitational and electromagnetic waves.

The eigen value equation $\left|T_{ij} - \lambda g_{ij}\right| = 0$ gives

$$\begin{vmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & T_{33} - \lambda g_{33} & T_{34} - \lambda g_{34} \\ 0 & 0 & T_{43} - \lambda g_{34} & T_{44} - \lambda g_{44} \end{vmatrix} = 0,$$

where

$$T_{33} + \lambda g_{33} = T_{33} + \lambda\left(1 + 2f\right) = E_{33} + \frac{1}{4\pi}\overline{V}^2 + \lambda\left(1 + 2f\right),$$

$$T_{43} + \lambda g_{34} = T_{43} - \lambda\left(2f\right) = E_{43} - \frac{1}{4\pi}\overline{V}^2 - 2\lambda f,$$

and

$$T_{44} + \lambda g_{44} = T_{44} - \lambda\left(1 - 2f\right) = E_{44} + \frac{1}{4\pi}\overline{V}^2 - \lambda\left(1 - 2f\right),$$

Solving above determinantal equation we have

$$\lambda^2 \left\{ \frac{1}{16\pi^2}\left[4\pi E_{33} + \overline{V}^2 + 4\pi\lambda\left(1 + 2f\right)\right]\left[4\pi E_{44} + \overline{V}^2 - 4\pi\lambda\left(1 - 2f\right)\right] - \right.$$

$$\frac{1}{16\pi^2}\left[4\pi E_{34} - \overline{V}^2 - 8\pi\lambda f\right]\left[4\pi E_{34} - \overline{V}^2 - 8\pi\lambda f\right] = 0,$$

$$\lambda^4 = 0.$$

In this case also all eigen values are zero and hence principal directions are not unique. Thus the co-existence of gravitational waves with scalar and electromagnetic waves is related to the eigen values of T_{ij} which are all zero.

3. NON-EXISTENCE THEOREM

THEOREM. *Non-existence of massive Scalar waves coupled to plane fronted gravitational and electromagnetic waves is guaranteed by the four nonzero* $\frac{1}{8\pi}m^2V^2$ *eigen values of the energy momentum tensor for the combined distribution of massive scalar field and electromagnetic field*.

Proof : We consider following two cases.

Case I. *Massive scalar field to gravitational field*
The matter tensor for massive scalar field is

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2} g_{ij}\left(V_s V^s - m^2 V^2\right)\right], \quad s = 1,\ 4. \tag{3.1}$$

where $V_i = \frac{\partial V}{\partial x^i}$, V is a scalar function of $z - t$.

So,

$$T = T_i^i = \frac{1}{4\pi}\left[V_i V^i - \frac{1}{2}\partial_i^i\left(V_s V^s - m^2 V^2\right)\right] = \frac{1}{4\pi}\left[V_i V^i - 2\left(V_s V^s - m^2 V^2\right)\right] =$$

$$\frac{1}{4\pi}\left(-V_s V^s + 2m^2V^2\right) = \frac{1}{4\pi}\left(2m^2V^2\right), \quad \text{as} \quad V_s V^s = 0, \quad \text{by}(1.4).$$

So,

$$R_{ij} = -8\pi\left\{\left[\frac{1}{4\pi}\left(V_i V_j - \frac{1}{2} g_{ij}\left(V_s V^s - m^2 V^2\right)\right)\right] - \frac{1}{2} g_{ij} \frac{1}{4\pi} 2m^2V^2\right\} = -2\left(V_i V_j - \frac{1}{2} g_{ij} m^2 V^2\right), \tag{3.2}$$

and then we deduce

$$\begin{aligned} R_{33} &= -2\left(\overline{V}^2 + \frac{m^2V^2}{2} + f\, m^2V^2\right), \\ R_{34} &= -2\left(-\overline{V}^2 + f\, m^2V^2\right), \\ R_{44} &= -2\left(\overline{V}^2 - \frac{m^2V^2}{2} + f\, m^2V^2\right). \end{aligned} \tag{3.3}$$

where $V_3 = \overline{V},\ V_3 = \overline{V},\ g_{33} = -\left(1 + 2f\right),\ g_{34} = 2f,\ g_{44} = 1 - 2f$. The (-) over a letter means derivative with respect to Z.

But for (1.3) we have (Takeno 1961, 55.4)

$$R_{33} = R_{34} = R_{44}. \tag{3.4}$$

The equation (2.3) are compatible with (2.4) only when $m^2 = 0$, expressing the fact that massive scalar waves does not exist with gravitational waves. It is analogous result of Ray and Rao (1972) with respect to axially symmetric scalar field.

The eigen values of T_{ij} are given by

$$\begin{vmatrix} T_{11}-\lambda g_{11} & 0 & 0 & 0 \\ 0 & T_{22}-\lambda g_{22} & 0 & 0 \\ 0 & 0 & T_{33}-\lambda g_{33} & T_{34}-\lambda g_{34} \\ 0 & 0 & T_{43}-\lambda g_{34} & T_{44}-\lambda g_{44} \end{vmatrix}=0, \tag{3.5}$$

where

$$T_{11}-\lambda g_{11}=\lambda-\frac{m^2V^2}{8\pi},$$

$$T_{22}-\lambda g_{22}=\lambda-\frac{m^2V^2}{8\pi},$$

$$T_{33}-\lambda g_{33}=\frac{1}{4\pi}\left[\overline{V}^2-\frac{1}{2}m^2V^2+4\pi\lambda+f\left(8\pi\lambda-m^2V^2\right)\right],$$

$$T_{34}-\lambda g_{34}=\frac{1}{4\pi}\left[-\overline{V}^2-f\left(8\pi\lambda-m^2V^2\right)\right],$$

$$T_{44}-\lambda g_{44}=\frac{1}{4\pi}\left[\overline{V}^2+\frac{1}{2}m^2V^2-4\pi\lambda+f\left(8\pi\lambda-m^2V^2\right)\right].$$

On simplifying (3.4),

$$\left(\lambda-\frac{m^2V^2}{8\pi}\right)^2\frac{1}{16\pi^2}\left\{\left[\overline{V}^2-\frac{1}{2}m^2V^2+4\pi\lambda+f\left(8\pi\lambda-m^2V^2\right)\right]\right.$$

$$\left.\left[\overline{V}^2+\frac{1}{2}m^2V^2-4\pi\lambda+f\left(8\pi\lambda-m^2V^2\right)\right]-\left[\overline{V}^2-f\left(8\pi\lambda-m^2V^2\right)\right]=0,\right.$$

$$\lambda=\frac{1}{8\pi}m^2V^2.$$

In this case we obtained four equal non-zero eigen values equal to $\left(\lambda=\frac{1}{8\pi}m^2v^2\right)$ and hence principal directions are not unique. Thus the non-existence of gravitational waves with massive scalar waves is related to non-zero eigen values of T_{ij}.

Case II. *Massive Scalar field coupled to gravitational and electromagnetic field*

In this case, we consider the energy momentum tensor,

$$T_{ij}=\frac{1}{4\pi}\left(V_iV_j-\frac{1}{2}g_{ij}\left(V_sV^s-m^2V^2\right)\right)+E_{ij}.$$

Here

$$E_{ij} = \frac{1}{4\pi}\left(g_{ij} F_{sp} F^{sp} - F_{is} F_j^s\right) \qquad P = 1,\ 4.$$

Then Einstein's equations (1.5) become

$$R_{ij} = -2\left(4\pi E_{ij} + V_i V_j - \frac{1}{2} g_{ij} m^2 V^2\right),$$

which can be written as

$$\begin{aligned} R_{33} &= -2\left(4\pi E_{33} + \overline{V}^2 + \frac{1}{2} m^2 V^2 + m^2 V^2 f\right), \\ R_{34} &= -2\left(4\pi E_{34} - \overline{V}^2 - m^2 V^2 f\right), \\ R_{44} &= -2\left(4\pi E_{44} + \overline{V}^2 - \frac{1}{2} m^2 V^2 + m^2 V^2 f\right). \end{aligned} \tag{3.6}$$

Equation (3.6) are not compatible with (1.5) and hence follows the non-existence of massive scalar waves coupled to the gravitational and electromagnetic waves.

The eigen values of the energy-momentum tensor are given by

$$\left|T_{ij} - \lambda g_{ij}\right| = 0,$$

$$\begin{vmatrix} T_{11} - \lambda g_{11} & 0 & 0 & 0 \\ 0 & T_{22} - \lambda g_{22} & 0 & 0 \\ 0 & 0 & T_{33} - \lambda g_{33} & T_{34} - \lambda g_{34} \\ 0 & 0 & T_{43} - \lambda g_{34} & T_{44} - \lambda g_{44} \end{vmatrix} = 0, \tag{3.7}$$

where

$$T_{11} - \lambda g_{11} = \left(\lambda - \frac{m^2 V^2}{8\pi}\right),$$

$$T_{22} - \lambda g_{22} = \left(\lambda - \frac{m^2 V^2}{8\pi}\right),$$

$$T_{33} - \lambda g_{33} = \frac{1}{4\pi}\left[\overline{V}^2 - \frac{m^2 V^2 (1 + 2f)}{2} + 4\pi\lambda(1 + 2f) + 8\pi E_{33}\right],$$

$$T_{34} - \lambda g_{34} = \frac{1}{4\pi}\left[-\overline{V}^2 + f m^2 V^2 + 4\pi E_{34} - 8\pi\lambda f\right],$$

$$T_{44}-\lambda g_{44}=\frac{1}{4\pi}\left[\overline{V}^2+\frac{m^2V^2(1-2f)}{2}-4\pi\lambda(1-2f)+4\pi E_{44}\right].$$

Simplifying (3.7),

$$\frac{1}{16\pi^2}\left(\lambda-\frac{m^2V^2}{8\pi}\right)\left\{\frac{-m^4V^4}{4}+4\pi\lambda m^2V^2-16\lambda^2\pi^2\right\}=0$$

$$\lambda-\frac{1}{8\pi}m^2V^2.$$

In this case also we obtained four equal non-zero eigen values. Hence it can relate non-existence of massive scalar waves coupled to the gravitational and electromagnetic waves with non-zero eigen values T_{ij}.

4. CONCLUSIONS

It is interesting that for Peres space-time (1.3):

a) the existence of massless scalar waves coupled to plane fronted gravitational waves and electromagnetic waves is assured by four zero eigen-values ($\lambda^4 = 0$) of energy momentum tensor of combined distribution of scalar and electromagnetic field;

b) the non-existence of massive scalar waves coupled to plane fronted gravitational and electromagnetic waves is guaranteed by the four non-zero $\left(\lambda-\frac{1}{8\pi}m^2V^2\right)$ eigen values of energy momentum tensor for combined distribution of massive scalar field and electromagnetic field;

c) furthermore these results are analogous to [3] for Takeno's space-times (1.1) and (1.2) also to Ray and Rao (1972) [4] with respect to axially symmetric massive scalar field.

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