BAND STRUCTURE OF A BOSE-EINSTEIN CONDENSATE LOADED INTO AN OPTICAL LATTICE

ALEXANDRU I. NICOLIN

"Horia Hulubei" National Institute for Physics and Nuclear Engineering (IFIN-HH),
Department of Theoretical Physics, 407 Atomistilor, Magurele-Bucharest, 077125, Romania
Email: nicolin@theory.nipne.ro

Received November 8, 2010

In this brief study we determine by variational means a loop in the band structure of a Bose-Einstein condensate loaded into an optical lattice.

Key words: Bose-Einstein condensates, band structure, optical lattices.

After the experimental achievement of the Bose-Einstein condensation back in 1995 [1], there has been a surge of papers focused on nonlinear phenomena specific to Bose-Einstein-condensed gases; the range of topics includes the condensate band structure, pattern formation, soliton dynamics and management, dynamics of multi-component condensates, formation and propagation of shock-waves, etc. (see Refs. [2, 3] for the main results). The condensate band structure, in particular, exhibits loops of Bloch states (also known as "swallow tails") [4–6] and stationary states of higher period than that of the underlying lattice [7]. These states are determined using the Gross-Pitaevskii (GP) equation and rely either on heavy numerics or on oversimplified tight-binding models [7]. It is the purpose of this paper to show how to determine a loop in the band structure of the condensate by variational means using an ansatz grafted on the Bloch solution (see Refs. [8–10] for related applications of variational methods to BECs).

Starting from a BEC loaded into an optical lattice of the form $V(x) = V_0 \sin(\pi x/d)$ we write the GP energy functional over two periods of the potential, namely $x \in [d/2, 9d/2]$. To this end, we approximate the full wave function as the sum of the local wave functions in the two wells under scrutiny and the neighboring ones, namely $\psi = \psi_0 + \psi_1 + \psi_2 + \psi_3$, where

$$\begin{aligned}
\psi_0 &= \frac{\sqrt{N}\cos\phi}{\omega_2^{1/2}\pi^{1/4}} \exp\left(-\left(x+d/2\right)^2/2\omega_2^2\right)\exp\left(-ikd/2\right), \\
\psi_1 &= \frac{\sqrt{N}\sin\phi}{\omega_1^{1/2}\pi^{1/4}} \exp\left(-\left(x-3d/2\right)^2/2\omega_1^2\right)\exp\left(i\alpha\right)\exp\left(ik3d/2\right),
\end{aligned} \tag{1}$$

$$\begin{aligned}\psi_2 &= \frac{\sqrt{N}\cos\phi}{\omega_2^{1/2}\pi^{1/4}}\exp\left(-\left(x-7d/2\right)^2/2\omega_2^2\right)\exp\left(ik7d/2\right),\\ \psi_3 &= \frac{\sqrt{N}\sin\phi}{\omega_1^{1/2}\pi^{1/4}}\exp\left(-\left(x-11d/2\right)^2/2\omega_1^2\right)\exp\left(i\alpha\right)\exp\left(ik11d/2\right).\end{aligned} \tag{1}$$

Each local wave function consists of a normalized Gaussian that gives the local density profile within the well, a phase (here incorporated through the phase difference α), and a plane wave of the form $\exp(ikx)$ evaluated at the center of each well. The total number of atoms in the two wells of interest is constant. This approximation holds for small nonlinearities, while at larger nonlinearities the phase in each well acquires a spatial dependence and our ansatz breaks down. Our ansatz captures both Bloch states and non-Bloch states having twice the period of the optical lattice.

With this ansatz we compute the GP energy

$$E[\psi] = \int_{\frac{d}{2}}^{\frac{9d}{2}}\left(\frac{1}{2}\left|\frac{d\psi}{dx}\right|^2 + V(x)\left|\psi\right|^2 + \frac{U}{2}\left|\psi\right|^4\right), \tag{2}$$

taking into account only nearest neighbor interactions. Due to the exponential tail of the local wave functions we can extend the integration domain such that we have

$$\begin{aligned}E[\psi] \approx & \int_{-\infty}^{\infty}dx\left[\left|\frac{d\psi_1}{dx}\right|^2 + \left|\frac{d\psi_2}{dx}\right|^2 + \frac{d\psi_1}{dx}\frac{d\psi_2^*}{dx} + \frac{d\psi_1^*}{dx}\frac{d\psi_2}{dx}\right]\\ &+\frac{1}{2}\int_{-\infty}^{\infty}dx\left[\frac{d\psi_0}{dx}\frac{d\psi_1^*}{dx} + \frac{d\psi_0^*}{dx}\frac{d\psi_1}{dx} + \frac{d\psi_2}{dx}\frac{d\psi_3^*}{dx} + \frac{d\psi_2^*}{dx}\frac{d\psi_3}{dx}\right]\\ &+\frac{1}{2}\int_{-\infty}^{\infty}dxV(x)\left(\psi_0\psi_1^* + \psi_1\psi_0^* + \psi_2\psi_3^* + \psi_3\psi_2^*\right)\\ &+\int_{-\infty}^{\infty}dxV(x)\left(|\psi_1|^2 + |\psi_2|^2 + \psi_1\psi_2^* + \psi_2\psi_1^*\right) + \frac{U}{2}\int_{-\infty}^{\infty}dx\left(|\psi_1|^4 + |\psi_2|^4\right),\end{aligned} \tag{3}$$

which yields

$$\begin{aligned}\frac{E[\psi]}{N} \approx & \frac{2\sqrt{2}\cos 2kd\cos\alpha\sin 2\phi\sqrt{\omega_1\omega_2}\left(\omega_1^2+\omega_2^2-4d^2\right)}{\exp\left(\frac{2d^2}{\omega_1^2+\omega_2^2}\right)\left(\omega_1^2+\omega_2^2\right)^{5/2}} - \frac{\cos(\phi)^2}{\exp\left(\frac{\pi^2\omega_2^2}{4d^2}\right)}\\ &-\frac{\sin(\phi)^2}{\exp\left(\frac{\pi^2\omega_1^2}{4d^2}\right)} + \frac{NU}{2\sqrt{2\pi}}\left(\frac{\sin(\phi)^4}{\omega_1} + \frac{\cos(\phi)^4}{\omega_2}\right)\\ &+\frac{V_0}{2}\left(\frac{\sin(\phi)^2}{\omega_1^2} + \frac{\cos(\phi)^2}{\omega_2^2}\right) + \frac{V_0 2\sqrt{2}\cos 2kd\cos\alpha\sin 2\phi}{\exp\left(\frac{4d^4+\pi^2\omega_1^2\omega_2^2}{2d^2\left(\omega_1^2+\omega_2^2\right)}\right)}\sqrt{\frac{\omega_1\omega_2}{\omega_1^2+\omega_2^2}}\end{aligned} \tag{4}$$

The stationary solutions correspond to $\partial E/\partial q = 0$ where $q = \{\phi, \alpha, \omega_1, \omega_2\}$.

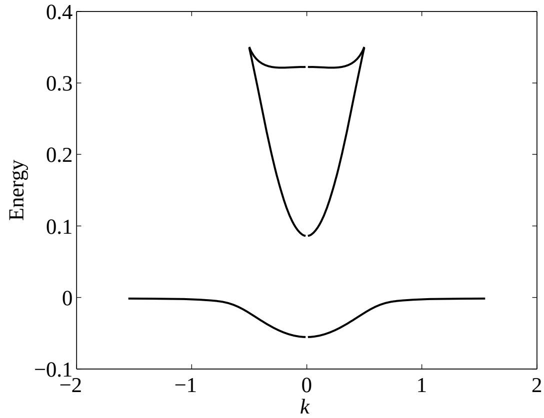

Fig. 1 – The first Brillouin zone ($k \in [-\pi/2d, \pi/2d]$) of the band structure of the condensate for $N = 1$, $U = 1$, $V_0 = 1$, $d = 1$. In the figure we have depicted the lowest Bloch band and the loop at the center of the first Brillouin zone.

Solving numerically these equations we see that for $U > 0$ we have both the regular Bloch solutions (that is $\omega_1 = \omega_2$) and non-Bloch states with a period twice that of the optical lattice (that is $\omega_1 \neq \omega_2$). In Fig. 1 we show a typical loop structure in the middle of Brillouin zone along with the lowest Bloch band; non-Bloch states will be reported elsewhere.The loop reported in this paper using a variational method is consistent with that obtained using the full GP equation [4–6].

Acknowledgements. For this work Alexandru I. Nicolin was supported by CNCSIS-UEFISCDI through the postdoctoral project PD122 contract no. 35/28.07.2010.

REFERENCES

1. C. J. Pethick and H. Smith, *Bose-Einstein condensation in dilute gases* (Cambridge University Press, Cambridge, 2002).
2. B. A. Malomed, *Soliton management in periodic systems* (Springer, Berlin, 2006).
3. P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-Gonzlez, Eds., *Emergent nonlinear phenomena in Bose-Einstein condensates* (Springer, New York, 2007).
4. D. Diakonov, L. M. Jensen, C. J. Pethick, and H. Smith, Phys. Rev. A **66**, 013604 (2002).
5. B. Wu, R. B. Diener, and Q. Niu, Phys. Rev. A **65**, 025601 (2002).
6. M. Machholm, C. J. Pethick, and H. Smith, Phys. Rev. A **67**, 053613 (2003).
7. M. Machholm, A. Nicolin, C. J. Pethick, and H. Smith, Phys. Rev. A **69**, 043604 (2004).
8. A. I. Nicolin and R. Carretero-Gonzlez, Physica A **387**, 6032 (2008).
9. A. I. Nicolin, Romanian Rep. Phys **61**, 641–645 (2009).
10. A. I. Nicolin and M. C. Raportaru, Physica A **389**, 4663 (2010).