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ON THE TRAJECTORIES OF THE SEISMIC RAYS

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Abstract. The traveling times of the seismic rays are measured experimentally as functions of the radial angle at the Earth's centre. These measurements are used to derive information about the velocities of the seismic waves inside the Earth, making use of the Herglotz-Wiechert equation. It is well known that the velocity of the seismic waves increases with depth. We assume a simple linear model of in-depth velocities profile and derive analytical forms for the rays' trajectories, both for a spherical model of Earth and a medium with a plane surface. The trajectories' equations enable us to compute the traveling times, the penetration depth and the distance covered by rays on the surface. The corrections to the traveling times arising from the depth of the seismic focus are also computed. The comparison of our analytical results with well-known experimental data of traveling times is satisfactory, which may validate the velocities model.

Key words: traveling times, seismic waves, inhomogeneous Earth.

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It is well known that the traveling times of the seismic waves can provide relevant information about the velocity of these waves at any point inside the Earth, and, thereby, data about the Earth's structure [1]-[3]. Typically, the traveling times of the seismic rays are measured as functions of the angular distance (radial angle), and the velocities profile is extracted from these data by using the well known Herglotz-Wiechert formula; basically, this is a numerical inversion technique of Abel-type integral equations. We present here explicit analytic formulae for the propagation of the seismic rays in an isotropic spherical model of the Earth, as well for an elastic medium with a plane surface, by assuming that the velocity increases linearly with the penetration depth. We give also here the corrections to the traveling times arising from the in-depth location of the seismic source. The comparison with the experimental data produces a satisfactory agreement, which may help in validating such a model. We start by recalling the essential physics behind the rays approximation (or geometrical optics approximation) [4]-[6].

If the wavelength is much shorter than the lengths of interest a wave may be approximated locally by plane waves. The phase Φ of such a wave $e^{i\Phi}$ is then a large number, and we can define wave surfaces $\Phi = const$. The local wavevector $\mathbf{k} = \mathrm{grad}\,\Phi$ gives the direction of the wave propagation, so we have local plane waves which propagates as rays. This is the geometrical optics approximation for waves, or rays approximation. The phase Φ is called eikonal. Obviously, such local plane waves propagate between two points fixed in space such as $\Delta\Phi = \int \mathbf{k} d\mathbf{r}$ be minimal, *i.e.* $\delta\Delta\Phi = 0$. Since, for a monochromatic wave with frequency ω we can write $k = \omega/v$, where v is the wave velocity, the minimum of the eikonal means the minimum of the time spent on the trajectory. This is Fermat's principle of the least time. Similarly, since the ratio of velocities in two media is the refraction coefficient n in optics, Fermat's principle means also the minimum of the optical path nl, *i.e.* the minimum of the geometrical path l multiplied by the refraction coefficient. It is also easy to see that for a wave packet the velocity here means the group velocity, and, in this form, Fermat's principle is known as the Maupertuis' principle for a particle. This is the basis of the Hamilton-Jacobi analogy between waves and particles.

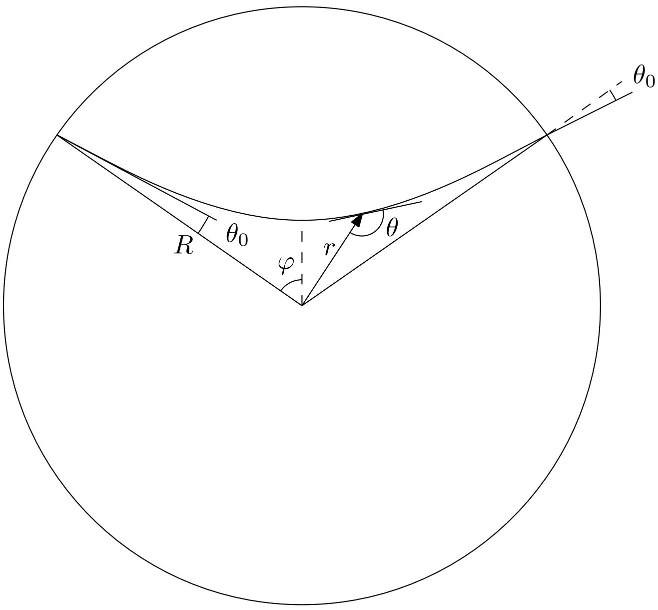

Fig. 1 – Ray through a stratified model of the Earth.

The eikonal at some point $\mathbf{r}$ in space can be written as $\Phi = \mathbf{kr} = kr\cos\theta$, where θ is the angle between the wavevector $\mathbf{k}$ and position vector $\mathbf{r}$. At that point, the variation of the eikonal is given by the variation of the angle θ, *i.e.* $\delta\Phi = -kr\sin\theta\cdot\delta\theta$.

If the change in the eikonal is going to be minimal, its virtual change $\delta\Phi$ must be a constant multiplied by the virtual variation $\delta\theta$. We get $kr\sin\theta = const$, which is Snellius' refraction law; indeed, with $k = \omega/v = \omega n/c$, where c is the velocity of the light in vacuum, we get $n\sin\theta = const$, where we can recognize the Snellius' law.

Let us assume an isotropic spherical model of the Earth, with radius R, as shown in Fig. 1. We prefer to use the angle θ made by the wavevector $\mathbf{k}$ and $-\mathbf{r}$, where $\mathbf{r}$ is the position vector. The refraction law derived above gives immediately

$$\frac{r}{v}\sin\theta = C \tag{1}$$

for a seismic ray, where C is a constant. Let us assume that the ray starts from the Earth's surface with angle θ_0 and velocity v_0. We assume a dependence of velocity with radius of the form

$$v = v_0 + a(R - r), \tag{2}$$

where $a > 0$ is a constant. Then, by equation (1), it is easy to get

$$r = R\sin\theta_0 \frac{1 + \frac{aR}{v_0}}{\sin\theta + \frac{aR}{v_0}\sin\theta_0}. \tag{3}$$

We can see that the ray reaches the Earth's surface at some other point under the same emerging angle θ_0, and the trajectory has a minimum radius for $\theta = \pi/2$. Making use of the notations in Fig. 2 we have

$$\cot\theta_1 = \tan(\varphi - \theta) = \frac{dx}{dy} = \frac{dr\sin\varphi + r\cos\varphi d\varphi}{dr\cos\varphi - r\sin\varphi d\varphi}, \tag{4}$$

whence we get

$$\frac{d\varphi}{d\theta} = -\frac{r'(\theta)}{r(\theta)}\tan\theta. \tag{5}$$

Therefore, the angle φ is given by

$$\varphi = \int_{\pi/2}^{\pi-\theta_0} d\theta \frac{\sin\theta}{\sin\theta + \frac{aR}{v_0}\sin\theta_0}; \tag{6}$$

usually the parameter aR/v_0 is much larger than unity, so the above integral can be approximated by

$$\varphi \simeq \frac{v_0}{aR}\cot\theta_0. \tag{7}$$

From measurements at two different locations we can get, by equation (7), the two parameters v_0 and a of the velocities model.

Let us compute the time T needed by the ray to go along its trajectory, *i.e.* to sweep the angle $\phi = 2\varphi$ (traveling time). From

$$v = \frac{dl}{dt} = \frac{\sqrt{dr^2 + r^2 d\varphi^2}}{dt} = \frac{|r'|}{\cos\theta}\frac{d\theta}{dt}, \tag{8}$$

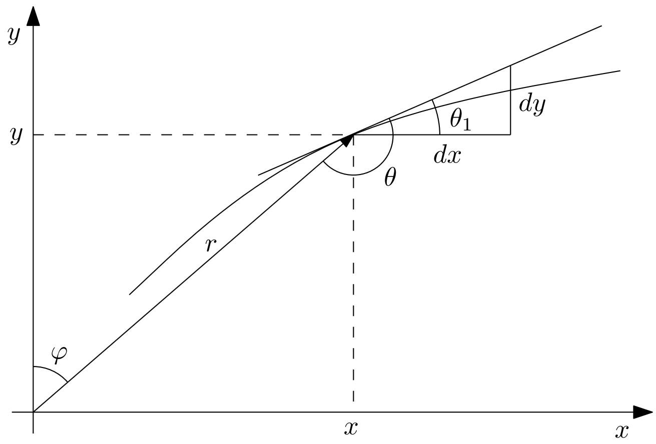

Fig. 2 – Polar coordinates and the derivative of the function.

where we used equation (5), we get easily

$$dt = \frac{R\sin\theta_0}{v_0}\frac{|r'|}{r\sin\theta\cos\theta}d\theta\,, \tag{9}$$

and

$$T = \frac{2R\sin\theta_0}{v_0}\int_{\pi/2}^{\pi-\theta_0} d\theta \frac{1}{\sin\theta\left(\sin\theta + \frac{aR}{v_0}\sin\theta_0\right)}. \tag{10}$$

Again, we take $aR/v_0 \gg 1$ and approximate this integral by

$$T = \frac{2}{a}\ln\cot\frac{\theta_0}{2}. \tag{11}$$

From equations (7) and (11) we get immediately

$$T = \frac{2}{a}\ln\left[\sqrt{\left(\frac{aR}{2v_0}\right)\phi^2 + 1} + \frac{aR}{2v_0}\phi\right]. \tag{12}$$

These traveling times as functions of the angle ϕ are measured experimentally, so, by making use of equation (12), we may derive the parameters a and v_0 and test the validity of the velocities model given by equation (2). Usually, the velocity is determined by the well-known Herglotz-Wiechert equation, which implies solving Abel-type integral equations and numerical computation.

We can consider also the ray propagation in a medium with a plane surface, as shown in Fig. 3. We write

$$\rho^2 = r^2 + R^2 - 2rR\cos\varphi \tag{13}$$

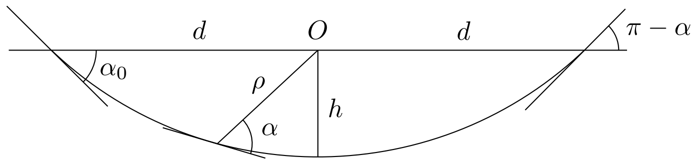

Fig. 3 – Ray through a stratified half-space.

for ρ^2, with the rest of notations given in Fig. 1, and take the limit $R \to \infty$. From equation (6) we get

$$\varphi \simeq -\frac{v_0}{aR\sin\theta_0}\cos\theta\,, \tag{14}$$

such that

$$\cos\varphi \simeq 1-\frac{v_0^2}{2a^2R^2\sin^2\theta_0}\cos^2\theta\,. \tag{15}$$

Introducing this $\cos\varphi$ in equation (13) and making use of r given by equation (3) in the limit $R \to \infty$ we get easily

$$\rho^2=\frac{v_0^2}{a^2}\left(1-\frac{\sin\theta}{\sin\theta_0}\right)^2+\frac{v_0^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta_0}; \tag{16}$$

here, the angle θ is the angle between the vertical axis and the tangent at the curve. It is more convenient to use the incidence angle α, as shown in Fig. 3, $\alpha=\pi/2-\theta$. We get

$$\rho^2=\frac{v_0^2}{a^2}\left(1-\frac{\cos\alpha}{\cos\alpha_0}\right)^2+\frac{v_0^2}{a^2}\frac{\sin^2\alpha}{\cos^2\alpha_0}. \tag{17}$$

The distance d in Fig. 3 is given by $\alpha=\alpha_0$,

$$d=\frac{v_0}{a}\tan\alpha_0, \tag{18}$$

and the depth of the trajectory is given by $\alpha=0$,

$$h=\frac{v_0}{a}\frac{1-\cos\alpha_0}{\cos\alpha_0}. \tag{19}$$

The traveling time is given by $T=(2/a)\ln\tan\alpha_0=(2/a)\ln(ad/v_0)$ (for $ad/v_0>1$).

Let us assume that the source of the seismic rays is located at depth H below the surface, $H \ll R$. Then, the traveling time will record a slight decrease given by

$$\delta T=\frac{2}{a\sin\theta_0}\delta\theta. \tag{20}$$

On the other hand, the radius changes by

$$H = \delta r|_{\theta_0} = \frac{R}{1+\frac{aR}{v_0}} \cot\theta_0 \delta\theta, \tag{21}$$

so we get a change

$$\delta T = \frac{2H}{aR\cos\theta_0}\left(1+\frac{aR}{v_0}\right) \simeq \frac{2H}{v_0\cos\theta_0} \tag{22}$$

in the traveling time. It corresponds to a change $\delta\varphi = 2H/R\sin 2\theta_0$ in the radial angle. Equation (22) gives the corrections to the traveling times caused by the depth of the seismic focus. Making use of equation (7) we can write again the correction as

$$\delta T = \frac{2H}{v_0}\sqrt{1+\frac{v_0^2}{a^2R^2\varphi^2}}, \tag{23}$$

which includes explicitly the radial distance $R\varphi = d/2$.

Various other models of velocities profiles can be investigated along the same lines as those described herein, with the aim of improving the characterization of the propagation of the seismic rays in realistic Earth models. Further investigations in this direction will be reported in forthcoming publications.

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