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MATHEMATICAL PHYSICS

SPIRAL SOLITONS IN TWO-DIMENSIONAL COMPLEX CUBIC-QUINTIC GINZBURG-LANDAU MODELS

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Abstract. A brief overview of studies in the area of two-dimensional dissipative spiral solitons (localized vortices) described by the complex cubic-quintic Ginzburg-Landau partial differential equation is given.

Key words: dissipative solitons, Ginzburg-Landau equation, localized vortices, spiral solitons.

1. INTRODUCTION

Nonlinear localized structures, alias solitons, are ubiquitous in nature. They appear in diverse physical and biological systems, such as shallow water waves in hydrodynamics, DNA excitations in biological systems, matter waves in Bose-Einstein condensates, and electromagnetics wavepackets in nonlinear optics and photonics [1-2]. Two-dimensional (2D) spatial *optical solitons*, which are spatially confined light beams propagating in nonlinear media, and their three-dimensional (3D) spatiotemporal counterparts, which are nondiffracting and nondispersing wavepackets, which form in certain nonlinear optical media under special conditions [3-23] could be used as information carriers in future all-optical signal processing and logic devices due to their remarkable potential for massive parallelism (in space) and pipelining (in time), see *e.g.* Ref. [1].

Optical vortex beams are fundamental physical objects with phase singularities in their wavefronts which are associated with phase dislocations carried by diffracting optical beams [24]. In self-defocusing nonlinear optical media, the vortex core with a phase dislocation becomes self-trapped and an optical *vortex soliton* on a nonvanishing background wave is formed [25]. On the other hand, in self-focusing nonlinear optical media, such optical vortices are ring-like beams which carry phase singularities; the wave intensity is vanishing in the core of such an optical vortex. The localized optical vortices [26-29], *i.e.*, vortex solitons, have drawn much attention within the past years as physical objects of fundamental interest, and also due to their potential applications to all-optical information processing, as well as to the guiding and trapping of atoms; for a review of this active research area, see Ref. 29.

Thus in the core of an optical vortex the complex electromagnetic field is equal to zero, however the circulation C of the gradient of the phase of the complex field on an arbitrary closed contour around the vortex core is a multiple of 2π, *i.e.*, $C = 2\pi S$, where S is an integer number known as the *vorticity* or the *topological number* of the soliton. It is worthy to notice that the phase dislocations carried by the wavefront of a light beam are associated with a zero-intensity point (a vortex core). The phase is twisted around such points where the light intensity vanishes, thus creating an optical vortex.

Unique properties are also featured by vortex clusters, such as rotation similar to the vortex motion in superfluid flows. The complex dynamics of two- and three-dimensional soliton clusters in optical media with competing nonlinearities has been studied, too [23]. Various complex localized patterns based on both fundamental (nonspinning) solitons and vortices were theoretically investigated in optics and in the usual BEC models governed by the Gross-Pitaevskii equation with both local and nonlocal nonlinearities.

Stable nondissipative spatiotemporal spinning solitons (vortex tori) with the topological charge $S = 1$, described by the 3D nonlinear Schrödinger equation with focusing cubic and defocusing quintic nonlinearities were found to exist for sufficiently large energies of the soliton [15]. This result also holds for the case of competing quadratic and self-defocusing cubic nonlinearities [15]. The main result of these comprehensive theoretical studies is that stable spinning solitons in conservative (dissipativeless) systems are possible as a result of competition between focusing and defocusing optical nonlinearities.

In conservative physical systems the key mechanism that defines the optical soliton paradigm is the exact balance between the spreading and the concentration of optical wavepackets. Thus in this case we get soliton families (one- or multiparameter families of stationary solutions of the nonlinear dynamical equations) as a result of the exact balance of nonlinear modulation and linear transport effects such as diffraction, dispersion, and diffusion. In dissipative optical systems, which are defined as lossy media fed with external energy we get fixed point solutions of the nonlinear governing equations as a result of the exact balance between nonlinear modulation, linear transport effects, (non)linear gain, and (non)linear loss effects. Thus *dissipative optical solitons* (temporal, spatial or spatiotemporal ones) can form in the presence of *gain and loss* due to optical amplifiers and saturable absorbers in optically nonlinear media, including laser cavities [30-44].

Recently we performed a comprehensive stability analysis of three-dimensional dissipative solitons with intrinsic vorticity S governed by the complex Ginzburg-Landau equation with cubic and quintic terms in its dissipative and conservative parts [40]. It was found that a necessary stability condition for all vortex solitons, but not for the fundamental ones ($S = 0$), is the presence of nonzero diffusivity in the transverse (x,y) plane (a nonzero spatial-domain diffusivity). The fundamental solitons are stable in their whole existence domain, while the vortex solitons are stable only in a part of their existence domain. However, the spectral filtering (*i.e.*, the temporal-domain diffusivity) is not a necessary ingredient for the stability of any species of dissipative solitons. Stability domains were found for three-dimensional vortex solitons (alias vortex tori) with "spin" (topological charge) $S = 1$, 2, and 3, suggesting that spinning solitons with any vorticity S can be stable in certain portions of their existence domains [40].

We also studied the existence and stability domains of vortex solitons in the Ginzburg-Landau model of a two-dimensional lasing medium with a transverse grating (a 2D harmonic potential in the transverse spatial coordinates) [42]. We introduced a two-dimensional model of a laser cavity based on the complex Ginzburg-Landau equation with the cubic-quintic nonlinearity and a lattice (harmonic) potential accounting for the transverse grating. We found that *localized vortices*, built as sets of four peaks pinned to the periodic potential, may be stable without the rather unphysical diffusion term, which was necessary for the stabilization in previously studied Ginzburg-Landau models for dissipative optical solitons. The vortices were chiefly considered in the onsite form, but the stabilization of offsite vortices and quadrupoles was shown too [42]. Stability regions for the rhombic (onsite) vortices and fundamental ($S = 0$) solitons were identified in the parameter space of the cubic-quintic Ginzburg-Landau model, and scenarios of the evolution of unstable vortices were also described, see Ref. [42] for more details. The main result of this study was the necessity of a minimum strength of the lattice potential in order to stabilize the vortices. A stability border was also identified in the case of the self-focusing quintic term in the underlying model, which suggested a possibility of the supercritical collapse. Notice that beyond this border, the stationary vortex turns into a vortical breather, which is subsequently replaced by a dipolar breather and eventually by a single-peak breather; see Ref. [42].

In Refs. [43] we investigated families of *spatiotemporal dissipative solitons* in a model of 3D laser cavities including a combination of gain, saturable absorption, and transverse grating. The dynamical model was based on the complex Ginzburg-Landau equation with the cubic-quintic nonlinearity and a 2D external grating potential. The shapes of both fundamental (vorticityless) and vortex solitons were found by adequate numerical techniques as robust attractors in this dissipative dynamical model and their stability against strong random perturbations was tested by direct numerical simulations. The vorticityless solitons were found to be completely stable in this Ginzburg-Landau model. The 3D vortex solitons, built as rhombus-shaped complexes of four fundamental solitons, may be split by perturbations into their constituents separating in the temporal direction, see Ref. [43]. Nevertheless, we have found that a sufficiently strong 2D transverse grating makes the 3D vortex solitons practically stable physical objects.

Recently we have analyzed in detail the existence and stability characteristics of topological modes in 2D Ginzburg-Landau models with external trapping potentials [44]. It is well known that complex Ginzburg-Landau models of laser media with cubic-quintic nonlinearities do not contain an effective diffusion term, which makes all vortex solitons unstable in these models. However, as it was said above, in a recent publication [42] it has been demonstrated that the addition of a two-dimensional periodic potential, which may be induced by a transverse grating in the laser cavity, to the cubic-quintic complex Ginzburg-Landau model is able to support stable compound (four-peak) vortices, but the most fundamental "crater-shaped" vortices, alias vortex rings, which are essentially squeezed into a single cell of the external potential, have not been found before in a stable form.

Subsequently, in Ref. [44] we reported on families of stable compact crater-shaped vortices with vorticity $S = 1$ in the complex Ginzburg-Landau model model with external potentials of two different types: (a) an axisymmetric parabolic trapping potential, and (b) a periodic harmonic potential. In both situations, we identified stability regions for the crater-shaped vortices and for the fundamental (vorticityless) solitons with $S = 0$. Those crater-shaped vortices which are unstable in the axisymmetric (parabolic) potential break up into robust *dipoles*. All the vortices with vorticity number $S = 2$ are unstable, splitting into stable *tripoles*. Stability regions for the dipoles and tripoles were identified, too in Ref. [44]. Moreover, it was shown that the harmonic periodic potential cannot stabilize compact crater-shaped vortices with S larger than 2 either; instead, families of stable compact square-shaped *quadrupoles* were found; see Ref. [44] for a comprehensive study of this issue.

In the present work I give a brief overview of theoretical results in the area of two-dimensional dissipative spiral solitons (localized vortices) described by the complex cubic-quintic Ginzburg-Landau model. It is worthy to notice that a recent experimental work on bistable and addresable localized vortices in semiconductor lasers [38] confirmed the earlier theoretical predictions [34–37] on the existence of stable localized vortices which would be able to carry an orbital angular momentum.

2. TWO-DIMENSIONAL DISSIPATIVE SPIRAL SOLITONS IN GINZBURG-LANDAU MODELS

The most general two-dimensional complex cubic-quintic Ginzburg-Landau partial differential equation is [34–35]:

$$iE_z + (1/2 - \beta)(E_{xx} + E_{yy}) + [i\delta + (1 - i\varepsilon)|E|^2 - (\nu - i\mu)|E|^4]E = 0, \qquad (1)$$

where $E(x,y,z)$ is the complex amplitude of the electromagnetic field and (x, y) are the transverse spatial coordinates. The coefficients which are scaled to be $(1/2 - \beta)$ and 1 account, respectively, for the diffraction in the transverse plane and self-focusing Kerr nonlinearity, the coefficient ν takes into account the quintic nonlinearity, that may compete with the cubic term. In the dissipative part of the dynamic equation, the real constants δ, ε and μ represent, respectively, the linear loss, cubic gain, and quintic loss, which are the basic ingredients of the cubic-quintic Ginzburg-Landau partial differential equation.

The physical interpretation of all terms in Eq. (1) is straightforward, except for the diffusion term proportional with the parameter β. This term arises in some models of large-aspect-ratio laser cavities, close to the lasing threshold. The condition $\beta > 0$ is a necessary one for finding stable 2D vortex (spiraling) solitons in the cubic-quintic Ginzburg-Landau models, while the fundamental (nonspiraling) solitons (with topological number $S = 0$) may be stable at zero diffusivity ($\beta = 0$), see Refs. 40 and 42.

In the framework of the complex cubic-quintic Ginzburg-Landau equation (1), we performed a systematic analysis of 2D axisymmetric doughnut-shaped localized wavepackets with the inner phase field in the form of a rotating spiral [34]. We gave a qualitative argument which suggests that, on the contrary to the known fundamental azimuthal instability of spinning doughnut-shaped solitons in the cubic-quintic nonlinear Schrödinger equation, their Ginzburg-Landau counterparts may be stable [34]. This theoretical prediction was confirmed by extensive direct numerical calculations, and, in a more rigorous way, by calculating the growth rate of the dominant perturbation eigenmode. It was shown that stable spiral solitons with the values of the topological number S = 0, 1, and 2 can be easily generated from a variety of initial waveforms having the same values of intrinsic vorticity S. In a large domain of the parameter space, it was found that all the stable solitons coexist, each one being a strong attractor inside its own class of localized 2D waveforms distinguished by their vorticity. In a smaller region of the parameter space, stable solitons with S = 1 and 2 coexist, while the one with S = 0 is absent. Stable nonspiraling and spiraling breather solitons, demonstrating persistent quasiperiodic internal vibrations, were found too; see Ref. [34] for more details.

An important characteristic of spiral solitons is their power. The power has been found to take very different values for the coexisting solitons with different values of the vorticity S. This is illustrated by Fig. 1a, where the power is shown as a function of the nonlinear quintic loss parameter μ. In Fig. 1b we display the existence domains for both spiraling and nonspiraling solitons in the parameter plane (ε,μ). In the blue (dark gray) region, there exist both spiraling and nonspiraling stable solitons, whereas in the lower white strip only spiraling solitons have been found to form. In the red (medium gray) region, dissipative solitons do not form at all. Lastly, in the upper white region, initial wavepackets have been found to expand indefinitely, generating the above-mentioned fronts.

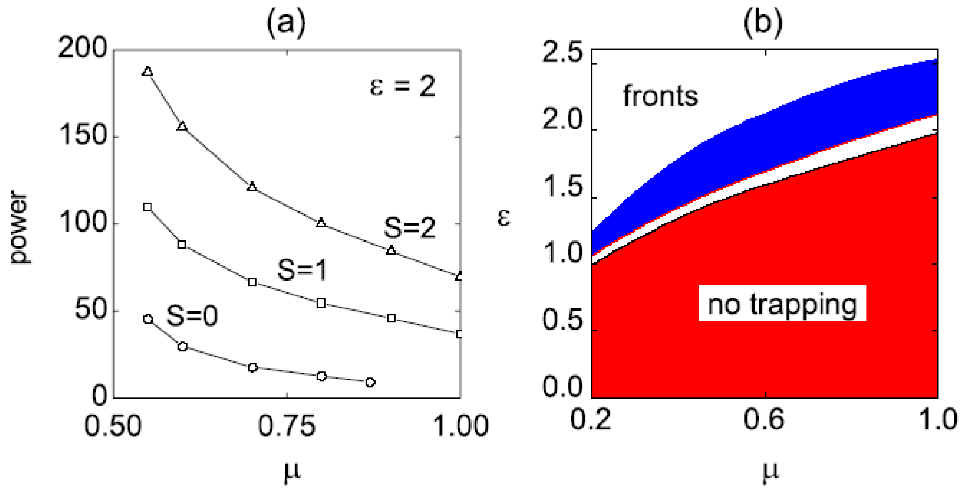

Fig. 1 – a) Power *versus* quintic loss for a fixed cubic gain. The circles, squares, and triangles correspond to, respectively, nonspiraling solitons ($S = 0$), spiraling solitons with $S = 1$, and spiraling ones with $S = 2$; b) The existence domain in the parameter plane (ε,μ) for both spiraling and nonspiraling stable solitons; see Ref. [34].

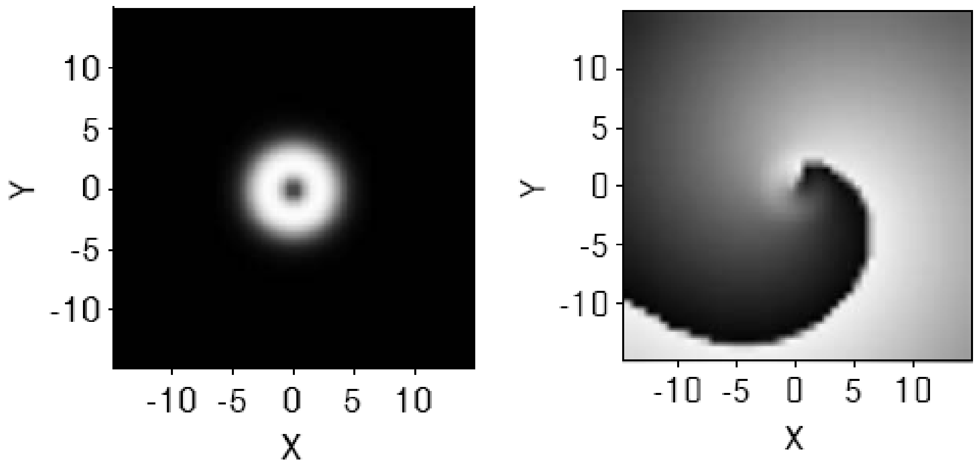

Fig. 2 – Typical shape of stable spiral soliton with vorticity number $S = 1$: a) amplitude distribution; b) phase distribution, see Ref. [34].

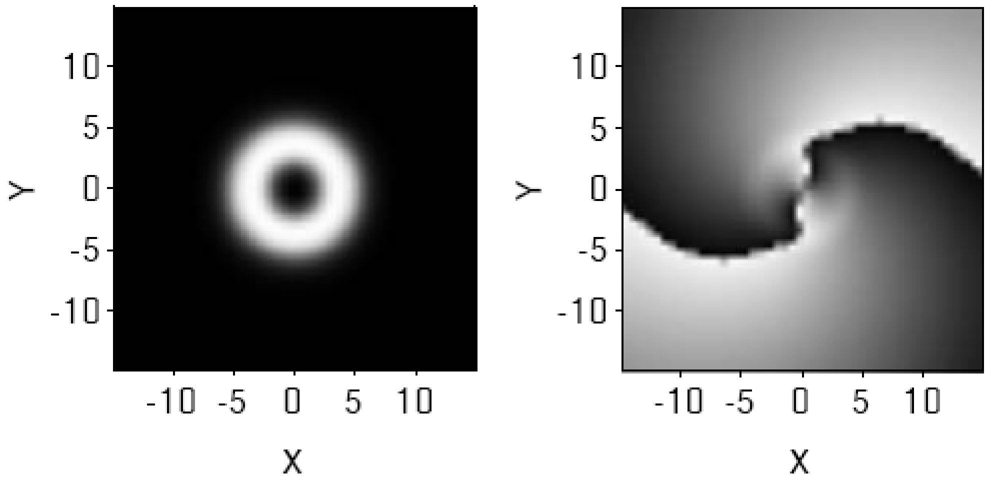

Fig. 3 – Typical shape of stable spiral soliton with vorticity number $S = 2$: a) amplitude distribution; b) phase distribution, see Ref. [34].

In order to assess the range in the space of the initial configurations for which the stable solitons are attractors, we have performed a large number of direct numerical simulations with initial waveforms in the form of Gaussians with intrinsic vorticity S, see Ref. [34]. In all the cases in which the existence of stable spiral solitons was known, the initial Gaussians rapidly developed into them, keeping the initial value of the intrinsic vorticity. To illustrate this issue, in Fig. 2 we show typical shape (amplitude and phase distributions) of a stable spiral soliton with vorticity number $S = 1$ for the parameters $\beta = 0.5$, $\delta = 0.5$, $\nu = 0.1$, $\mu = 1$, and $\varepsilon = 2.5$. In Fig. 3 we show typical shape (amplitude and phase distributions) of a stable spiral soliton with vorticity number $S = 2$ for the same model parameters as in Fig. 2. Next we present three novel varieties of spiraling and nonspiraling axisymmetric solitons in the complex cubic–quintic Ginzburg–Landau equation, see Ref. [35] for a comprehensive study of these 2D localized dissipative structures. Thus we found irregularly "erupting" 2D wavepackets and two different types of very broad stationary ones found near a border between ordinary dissipative solitons and expanding fronts. The region of existence of each waveform was identified numerically. Also we searched in Ref. [35] for irregularly pulsating but robust (nondecaying) axisymmetric both spiraling, and nonspiraling soliton solutions to Eq. (1) of the *erupting type*, similar to those found in the one-dimensional cubic-quintic Ginzburg-Landau equation [45]. It is worthy to notice that they may be regarded as patterns belonging to a class of waveforms with chaotic intrinsic dynamics, which are known in one-dimensional nonlinear systems of the reaction – diffusion type. Also, we presented two new types of stationary broad axisymmetric localized patterns found in the framework of the same model, which we termed *composite* and *flat-top* two-dimensional solitons, similar to their one-dimensional counterparts investigated in detail in Refs. [46].

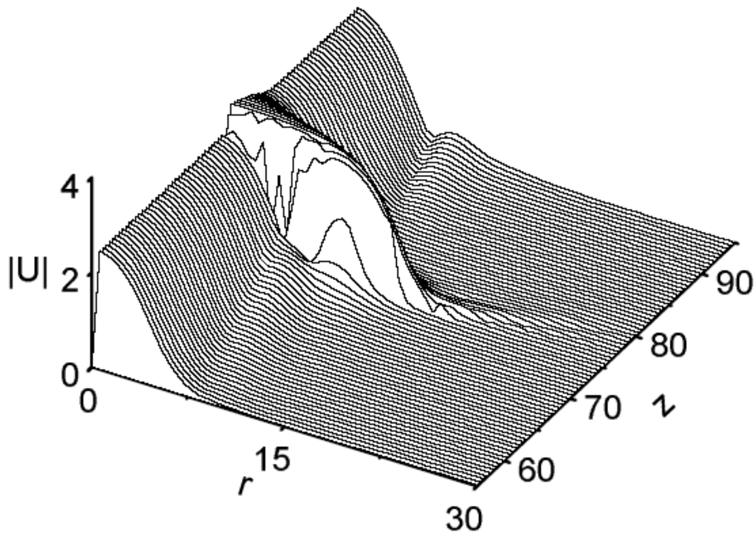

Fig. 4 – Quasi-periodic evolution of an erupting spiral soliton with vorticity number $S = 1$, including one "eruption"; see Ref. [35].

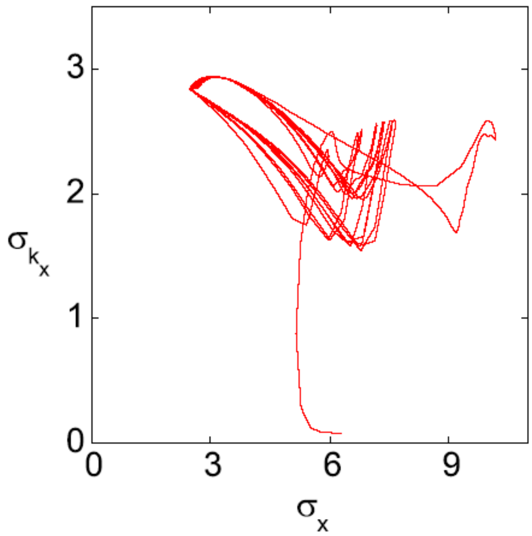

Fig. 5 – The trajectory, in the plane (σ_x,σ_{kx}), of the erupting soliton shown in Fig. 4 illustrating its chaotic intrinsic dynamics, see Ref. [35].

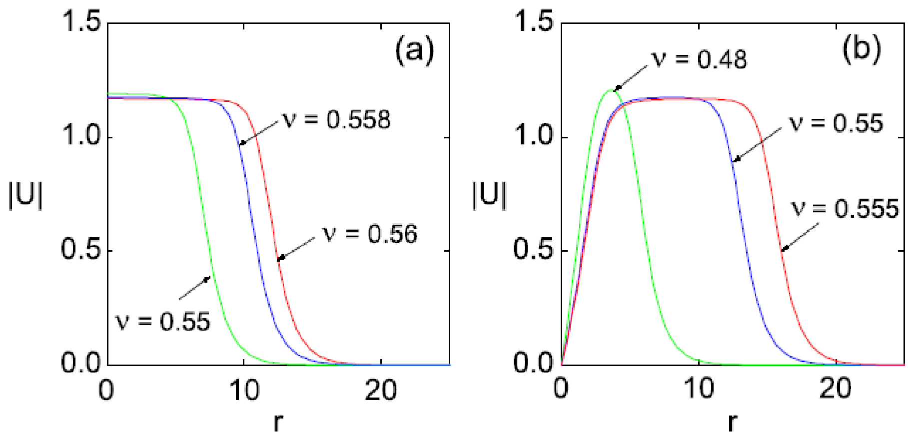

Fig. 6 – Radial field profiles of: a) nonspiraling ($S = 0$); b) spiraling ($S = 1$), flat-top solitons at slightly different values of the quintic self-focusing parameter ν, while the other parameters are fixed: $\beta = 0.03$, $\delta = 0.05$, $\mu = 0.2$, and $\varepsilon = 0.3$. Note that the spiraling soliton pertaining to $\nu = 0.48$ in panel (b) is still an ordinary soliton, rather than a flat-top one; see Ref. [35].

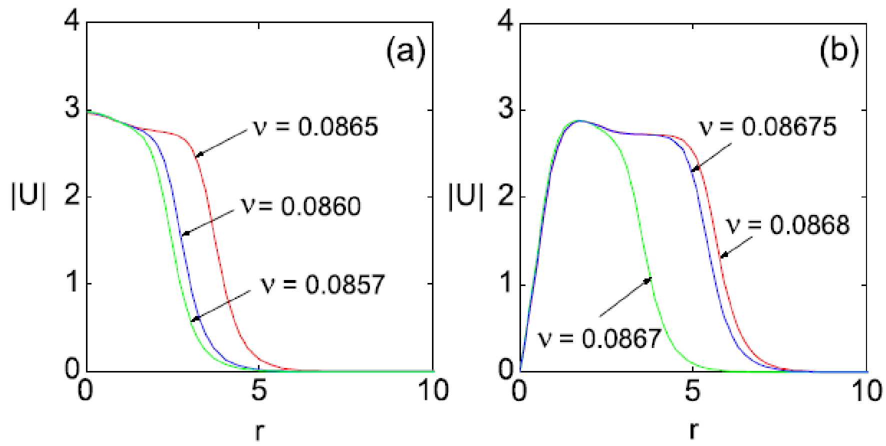

Fig. 7 – Radial field profiles of: a) nonspiraling ($S = 0$); b) spiraling ($S = 1$), composite solitons at slightly different values of the quintic self-focusing parameter ν, while the other parameters are fixed: $\beta = 0.08$, $\delta = 0.1$, $\mu = 0.1$, and $\varepsilon = 0.75$; see Ref. [35].

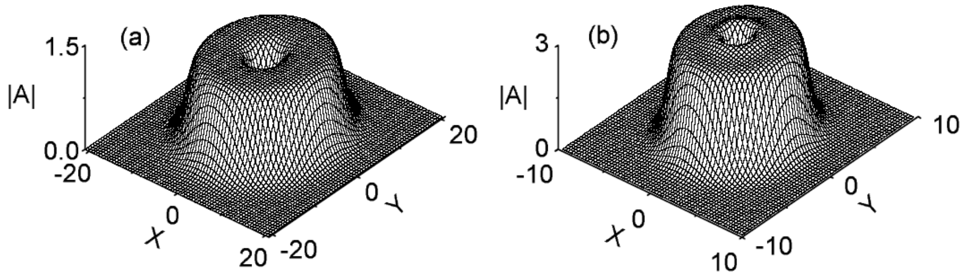

Fig. 8 – Surface plots of the field distribution of the spiraling ($S = 1$) flat-top (a) and composite (b) solitons corresponding to the radial profiles displayed, respectively, in (a) Fig. 6 for ν = 0.55 and (b) Fig. 7 for ν = 0.0868, see Ref. [35].

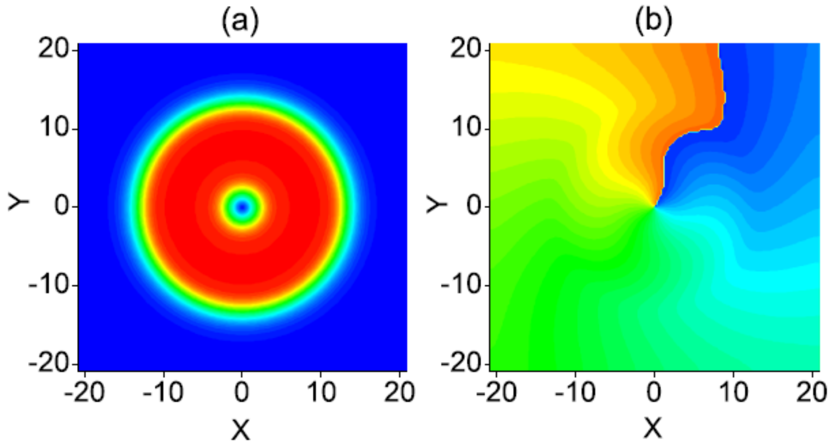

Fig. 9 – Amplitude and phase distributions of the flat-top spiral soliton created by an input Gaussian beam with the intrinsic vorticity $S = 1$; see Ref. [35].

As in the case of one-dimensional solitons [45], the eruptions happen irregularly (*i.e.*, chaotically), although each quasi-period contains only one eruption, see Fig. 4 for a typical quasi-periodic evolution of an erupting spiral soliton with vorticity number $S = 1$. In order to demonstrate the irregularity of the intrinsic dynamics of spiral erupting soliton, in Fig. 5 we illustrate a generic dynamical trajectory in the plane of the soliton's average width in the x-direction, σ_x, and the soliton's mean width in the k_x -direction of the Fourier space.

In Fig. 6 we plot radial field profiles of both nonspiraling ($S = 0$), and spiraling ($S = 1$), *flat-top solitons* at slightly different values of the quintic self-focusing parameter ν, while the other parameters are fixed. The model parameters are: $\beta = 0.03$, $\delta = 0.05$, $\mu = 0.2$, and $\varepsilon = 0.3$. It is worthy to notice that the spiraling soliton pertaining to $\nu = 0.48$ in panel 6b is still an ordinary soliton, rather than a flat-top one; see Ref. 35. Radial field profiles of both nonspiraling ($S = 0$), and spiraling (S = 1) *composite solitons* at slightly different values of the quintic self-focusing parameter ν, while the other parameters are fixed to the values β = 0.08, δ = 0.1, μ = 0.1 and ε = 0.75 are shown in Fig. 7; see Ref. 35. In Fig. 8 we display surface plots of the field distribution of the spiraling (S = 1) flat-top (Fig. 8a) and composite (Fig. 8b) solitons corresponding to the radial profiles displayed, respectively, in Fig. 6 for ν = 0.55 and in Fig. 7 for ν = 0.0868, see Ref. 35 for more details of this study. Finally, in Fig. 9 we show both the amplitude and phase distributions of the flat-top spiral soliton created by an input Gaussian beam with the intrinsic vorticity S = 1. Here the parameter ν = 0.55 and the other parameters of the Ginzburg-Landau model are β = 0.03, δ = 0.05, μ = 0.2, and ε = 0.3 [35].

3. CONCLUSIONS

In this work, I gave an overview of theoretical studies of families of both fundamental (nonspiraling) and spiraling dissipative solitons in the framework of two-dimensional complex Ginzburg-Landau models with the cubic-quintic nonlinearity. New species of nonstationary "erupting" two-dimensional solitons whose distinctive feature are irregularly repeating limited bursts were also found [35]. Nonspiraling (zero-vorticity) and spiraling (with the vorticity S = 1) solitons of this type may coexist with each other, but they do not coexist with the ordinary stationary stable solitons having the same spin. The *spiraling erupting solitons* were shown to be unstable against azimuthal perturbations. Additionally, it was found that the transition from ordinary stationary solitons to expanding circular fronts does not take place by a jump, but, instead, it goes *via* very broad stationary solitons (both nonspiraling and spiraling), which may be of two different types, depending on values of the parameters of the Ginzburg-Landau model: *flat-top solitons,* which are stable, or unstable *composite solitons*, that, unlike the "flat-top" ones, have some weakly pronounced intrinsic structure.

Recently, the existence of localized vortices was demonstrated experimentally [38]; thus it was shown experimentally that localized emission states in coupled broad-area semiconductor lasers can carry a finite orbital angular momentum. The resulting structures therefore possess the chirality of optical vortices together with the unique features of localized structures in dissipative systems, namely, the coexistence with a low intensity homogeneous emission and the mutual independence [38]. Also, in this seminal experimental study of localized vortices in broad-area semiconductor lasers, the possibility to generate states with higher topological charge (higher values of the "spin" S) if the perturbation itself possesses adequate vorticity was suggested, too [38].

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