Romanian Reports in Physics, Vol. 63, No. 3, P. 609–623, 2011

THEORETICAL PHYSICS

HOMOTOPY PERTURBATION METHOD FOR SOLVING A SYSTEM OF SCHRÖDINGER-KORTEWEG-DE VRIES EQUATIONS

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Received November 25, 2010

Abstract. Numerical methods used to find exact solution for the nonlinear differential equations. During the past decades Iterative methods has attracted attention of researcher for solving fractional differential equations. In the present paper, the homotopy perturbation method has been successively used to obtain approximate analytical solutions of the fractional coupled Schrödinger-Korteweg-de Vries and coupled system of diffusion-reaction equation equations. We consider fractional derivative in the Caputo sense. We have illustrated by examples the ability of proposed algorithm for solving fractional system of nonlinear equation.

Key words: Caputo fractional derivative, fractional Schrödinger-Korteweg-de Vries, homotopy perturbation method.

1. INTRODUCTION

Fractional differential systems have recently been proved to be useful in physics, engineering and control processing in various fields of sciences such as viscoelasticity, diffusion, control, relaxation processes. The properties of the fractional derivative and integrals make this kind of calculus a good candidate to describe the phenomena possessing the memory effect. Fractional image method and fractional multipoles have been generalized using fractional derivatives [1–11]. Therefore, various methods have been developed to solve linear/nonlinear fractional differential equations (FDE) [1–10, 12–15].

Nonlinear problems are important for engineers, physicists and mathematicians namely because most physical systems are inherently nonlinear in nature. However, the nonlinear equations are difficult to solve and lead to interesting phenomena, *e.g.* chaos. The investigation of the exact solutions of nonlinear evolution equations play an important role in the study of nonlinear physical phenomena. There are many approaches for seeking exact solutions of nonlinear equations, such as, Hirota's method, Bäcklund and Darboux transformation, Painlevé expansions. Recently, many alternative method used for solving both nonlinear and linear differential equations in terms of convergent power series. Homotopy Perturbation Method(HPM) has been introduced by He which possesses a great potential in solving different kinds of differential and functional equations. Both linear and nonlinear and systems of such types are amenable to the method. In the nonlinear case for differential equation and partial differential equation. The method has the advantage of dealing directly with the problem. That is, the equations are solved without transforming them also avoids linearization, discretization or any unrealistic assumption and provides an efficient numerical solution. In dealing with nonlinear equations the nonlinearity terms is replaced by a series. Then it is an easy algorithm for computing the solution. As a result, it yields a very rapidly convergent series solution, and usually a few iterations lead to very accurate approximation of the exact solution [16–20, 22].

The Nonlinear Schrödinger equation of fractional order has been solved by Rida *et al.* [21]. Chowdhury and Hashim have employed HPM for solving Klein-Gordon equations [22].

In the present paper we employ HPM to solve nonlinear system of Schrödinger-Korteweg-de Vries and diffusion-reaction equations of fractional order. A comparison with the Adomian decomposition (ADM) shows solutions are the same but this method is easier than ADM since for nonlinear terms we have to find Adomian's polynomials [23]. Also variational iteration method needs to find Lagrange multiplier making correction functional stationary in the fractional case [24].

The plan of our paper is as follows: In section 2 we introduce HPM as easy algorithm for solving system of fractional differential equation. Also, some basic definitions and properties of the fractional Riemann-Liouville and Caputo derivatives are briefly mentioned. Section 3 deals with solving and studying new algorithm of fractional coupled nonlinear Schrödinger-Korteweg-de Vries equations and fractional coupled system of diffusion-reaction equation using HPM.

2. PRELIMINARIES

2.1. HOMOTOPY PERTURBATION METHOD

The idea of the HPM and its application in various differential equations are given in [16–20, 22]. We explain the HPM for solving system of differential equations. Consider system of nonlinear differential equations

$$L_i(u,v)+N_i(u,v)=f_i(r),\ r\in\Omega,\ i=1,2, \tag{1}$$

with boundary condition

$$B_1(u,\partial u/\partial n)=0,\quad B_2(v,\partial v/\partial n)=0,$$

where L_i are linear operators, while N_i are nonlinear operators. The He's homotopy perturbation technique defines the homotopys $U(r,p):\Omega\times[0,1]\to\mathbb{R}$ and $V(r,p):\Omega\times[0,1]\to\mathbb{R}$ which satisfy

$$H_i(U,V,p)=(1-p)\ [L_i(U,V)-L_i(v_0,u_0)] \\ +p[L_i(U,V)+N_i(U,V)-f_i(r)]=0, \tag{2}$$

where $p\in[0,1]$ is an impeding parameter, u_0, v_0 are initial approximations which satisfied the boundary conditions. The changing process of p from zero to unity is just that of $U(r,p)$, $V(r,p)$ from u_0, v_0 to $u(r)$, $v(r)$. The basic assumption is that the solutions of Eq. (2) can be expressed as a power series in p

$$U=U_0+p\,U_1+p\,U_2+\ldots,\qquad V=V_0+p\,V_1+p^2\,V_2+\ldots.$$

The approximate solution of Eq. (1), therefore can be readily obtained:

$$u=\lim_{p\to1}U=U_0+U_1+U_2+\ldots,\qquad v=\lim_{p\to1}V=V_0+V_1+V_2+\ldots. \tag{3}$$

The convergence of the series Eq. (3) has been proved in [18,19].

2.2. FRACTIONAL CALCULUS

If $\alpha>0$ then left sided Riemann-Liouville fractional integral of order α is defined [1–10]

$$I_x^\alpha f(x):=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}\,\mathrm{d}t,\qquad 0<x<a. \tag{4}$$

The Caputo derivative is defined as follows

$$D_x^\alpha f(x)=I_x^{n-\alpha}D^nf(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-t)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t)\,\mathrm{d}t,\qquad 0<x<a \tag{5}$$

Let $u(x,t)$ and $n-1\le\alpha<n$, then partial Caputo fractional derivatives is defined as follows:

$$D_t^\alpha u(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha+1}\frac{\partial^{\,n}}{\partial\tau^n}u(x,\tau)\,\mathrm{d}\tau. \tag{6}$$

By inspection we observe that

$$D_x^\alpha(f(x)+g(x))=D_x^\alpha f(x)+D_x^\alpha g(x)$$

and

$$D_x^{\alpha} c = 0, \text{ where } c \text{ is a constant.} \tag{7}$$

In addition of this we have

$$I_t^{\alpha} D_t^{\alpha} u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!}. \tag{8}$$

3. FRACTIONAL COUPLED NONLINEAR SCHRÖDINGER-KORTEWEG-DE VRIES

It is well known the coupled nonlinear Schrödinger-Korteweg-de Vries equations and coupled system of diffusion-reaction equation have many application in physics. In this section we solve fractional coupled nonlinear Schrödinger-Korteweg-de Vries equations and fractional coupled system of diffusion-reaction equation as generalized using HPM.

3.1. EXAMPLE

We consider a coupled nonlinear Schrödinger-Korteweg-de Vries equation

$$\begin{aligned} iD_t^{\alpha} u(x,t) &= u_{xx} + uv, \\ iD_t^{\alpha} v(x,t) &= -6uv_x - v_{xxx} + (|u|^2)_x, \end{aligned} \tag{9}$$

with the initial condition

$$u(x,0) = u_0, \quad v(x,0) = v_0.$$

The system presented above represents an interesting mathematical model having multiple applications in nonlinear optics, chemical and plasma physics as well as in the quantum mechanics. To solve Eq. (9) using HPM, we construct the following homotopys:

$$H_1(U,V,p) = (1-p)\,(iD_t^{\alpha}U) + p\,(iD_t^{\alpha}U - U_{xx} - UV) = 0, \tag{10}$$

$$H_2(U,V,p) = (1-p)\,(iD_t^{\alpha}V) + p\,(iD_t^{\alpha}V + 6UV_{xx} + V_{xxx} - (|U|^2)_x) = 0. \tag{11}$$

Let the solution of Eqs. (10) and (11) be such that

$$U = U_0 + p\,U_1 + p^2 U_2 + \dots, \qquad V = V_0 + pV_1 + p^2 V_2 + \dots. \tag{12}$$

Substituting Eqs. (12) into Eq. (10), and equating the coefficients of the terms with identical powers of p,

$$p^0 : \; iD_t^{\alpha} U_0 = 0,$$

$$p^1 : \; iD_t^{\alpha} U_1 - \frac{\partial^2 U_0}{\partial x^2} - V_0 U_0 = 0,$$

$$p^2: \ iD_t^{\alpha} U_2 - \frac{\partial^2 U_1}{\partial x^2} - U_1 V_0 - U_0 V_1 = 0,$$

$$\vdots$$

$$p^j: \ iD_t^{\alpha} U_j - \frac{\partial^2 U_{j-1}}{\partial x^2} - \sum_{i=0}^{j-1} U_i V_{j-i-1} = 0. \tag{13}$$

We suppose $U_0 = u_0 = u(x,0)$ and $U_i(x,0) = 0, \ (i = 1,2,\ldots)$. Eqs. (13) yield the following relations

$$U_j = \frac{-i}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\frac{\partial^2 U_{j-1}}{\partial x^2} + \sum_{i=0}^{j-1} U_i V_{j-i-1}\right) \mathrm{d}\tau. \tag{14}$$

Also substituting Eq. (12) into Eq. (11) equating the coefficients of the terms with identical powers of p

$$p^0: \ iD_t^{\alpha} V_0 = 0,$$

$$p^1: \ iD_t^{\alpha} V_1 + 6U_0 \frac{\partial V_0}{\partial x} + \frac{\partial^3 V_0}{\partial^3 x} - \frac{\partial |U_0|^2}{\partial x} = 0,$$

$$p^2: \ iD_t^{\alpha} V_2 + 6U_0 \frac{\partial V_1}{\partial x} + 6U_1 \frac{\partial V_0}{\partial x} + \frac{\partial^3 V_1}{\partial^3 x} - \frac{\partial (U_0 \bar{V}_1 + V_1 \bar{U}_0)}{\partial x} = 0,$$

$$\vdots$$

$$p^j: \ iD_t^{\alpha} V_j + 6 \sum_{i=0}^{j-1} U_i \frac{\partial V_{j-i-1}}{\partial x} + \frac{\partial^3 V_{j-1}}{\partial x^3} - \frac{\partial}{\partial x} \sum_{i=0}^{j-1} U_i \bar{V}_{j-i-1} = 0. \tag{15}$$

We suppose $V_0 = v_0 = v(x,0)$ and $V_i(x,0) = 0, \ (i = 1,2,\ldots)$. Eqs. (15) yield the following relations

$$V_j = \frac{-i}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(-6 \sum_{i=0}^{j-1} U_i \frac{\partial V_{j-i-1}}{\partial x} - \frac{\partial^3 V_{j-1}}{\partial x^3} + \frac{\partial}{\partial x} \sum_{i=0}^{j-1} U_i \bar{V}_{j-i-1}\right) \mathrm{d}\tau. \tag{16}$$

Note: If we choose u_0 and v_0 as follows

$$u(x,0) = 6\sqrt{2} e^{i\beta x} k^2 \operatorname{sech}^2(kx), \quad v(x,0) = \frac{\beta + 16k^2}{3} - 6k^2 \tanh^2(kx),$$

where β and k are real constants. We arrive at

$$\begin{aligned} U_0 &= u(x,0) = 6\sqrt{2} e^{i\beta xz} k^2 \operatorname{sech}^2(kx), \\ V_0 &= v(x,0) = \frac{\beta + 16k^2}{3} - 6k^2 \tanh^2(kx), \end{aligned} \tag{17}$$

$$
\begin{aligned}
U_1 = \frac{-it^{\alpha}}{\Gamma(\alpha+1)}\{&6\sqrt{2}e^{ix\beta}i^2k^2\beta^2\operatorname{sech}(kx)^2 - 12\sqrt{2}e^{ix\beta}k^4\operatorname{sech}(kx)^4 \\
&- 24\sqrt{2}e^{ix\beta}ik^3\beta\operatorname{sech}(kx)^2\tanh(kx) + 24\sqrt{2}e^{ix\beta}k^4\operatorname{sech}(kx)^2\tanh(kx)^2 \\
&+ 6\sqrt{2}e^{ix\beta}k^2\operatorname{sech}(kx)^2(\frac{1}{3}16k^2 + \beta - 6k^2\tanh(kx)^2)\},
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
V_1 = \frac{-it^{\alpha}}{\Gamma(\alpha+1)}\{&144e^{2ix\beta}ik^4\beta\operatorname{sech}(kx)^4 - 96k^5\operatorname{sech}(kx)^4\tanh(kx) \\
&+ 432\sqrt{2}e^{ix\beta}k^5\operatorname{sech}(kx)^4\tanh(kx) - 288e^{2ix\beta}k^5\operatorname{sech}(kx)^4\tanh(kx) \\
&+ 48k^5\operatorname{sech}(kx)^2\tanh(kx)^3\},
\end{aligned} \tag{19}
$$

$$\vdots$$

and so on. We were able to obtain the exact solution Eq. (9) as found in [25], namely

$$u(x,t) = 6\sqrt{2}e^{i\theta}k^2\operatorname{sech}^2(k\zeta), \qquad v(x,t) = \frac{\beta+16k^2}{3} - 6k^2\tanh^2(k\zeta) \tag{20}$$

where

$$\theta = (\frac{\beta t}{3} + \beta^2 t + \frac{-10k^2 t}{3} + \beta x), \qquad \zeta = x + 2\beta t \tag{21}$$

3.2. EXAMPLE

Consider Fractional modified nonlinear Korteweg-de Vries equations

$$
\begin{aligned}
D_t^{\alpha}u(x,t) &= \frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3(uv)_x - 3\lambda u_x, \\
D_t^{\alpha}v(x,t) &= -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\lambda v_x,
\end{aligned} \tag{22}
$$

with initial condition

$$u(x,0) = u_0, \quad v(x,0) = v_0.$$

This equation describes interactions of two long waves with different dispersion relations. Constructing the homotopys, we get:

$$
\begin{aligned}
H_1(U,V,p) =&(1-p)(D_t^{\alpha}U) \\
&+ p\big(D_t^{\alpha}U - \frac{1}{2}U_{xxx} + 3U^2U_x - \frac{3}{2}V_{xx} - 3(UV)_x + 3\lambda U_x\big) = 0,
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
H_2(U,V,p) =&(1-p)(D_t^{\alpha}V) \\
&+ p\big(D_t^{\alpha}V + V_{xxx} + 3UV_x + 3U_xV_x - 3U^2V_x - 3\lambda V_x\big) = 0.
\end{aligned} \tag{24}
$$

We can assume the solution of Eq. (23) is as follows:

$$U = U_0 + p\,U_1 + p^2U_2 + \ldots, \qquad V = V_0 + pV_1 + p^2V_2 + \ldots. \tag{25}$$

Substituting Eq. (25) into Eq. (23), and equating the coefficients of the terms with identical powers of p,

$$\begin{aligned}
p^0 &: D_t^\alpha U_0 = 0,\\
p^1 &: D_t^\alpha U_1 - \frac{1}{2}\frac{\partial^3 U_0}{\partial x^3} + 3U_0^2\frac{\partial U_0}{\partial x} - \frac{3}{2}\frac{\partial^2 V_0}{\partial x^2} - 3\frac{\partial}{\partial x}(U_0V_0) + 3\lambda\frac{\partial U_0}{\partial x} = 0,\\
p^2 &: D_t^\alpha U_2 - \frac{1}{2}\frac{\partial^3 U_1}{\partial x^3} + 3(2U_1U_0\frac{\partial U_0}{\partial x} + U_0^2\frac{\partial U_1}{\partial x}) - \frac{3}{2}\frac{\partial^2 V_0}{\partial x^2}\\
&\quad - 3\frac{\partial}{\partial x}(U_1V_0 + U_0V_1) + 3\lambda\frac{\partial U_1}{\partial x} = 0,
\end{aligned}$$

$$\vdots$$

$$p^j:\ iD_t^\alpha U_j - \frac{1}{2}\frac{\partial^3 U_{j-1}}{\partial x^3} + 3\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} U_iU_k\frac{\partial U_{j-k-i-1}}{\partial x} - \frac{3}{2}\frac{\partial^2 V_{j-1}}{\partial x^2} - 3\frac{\partial}{\partial x}\sum_{i=0}^{j-1} U_jV_{j-i-1} + 3\lambda\frac{\partial}{\partial x}U_{j-1} = 0, \quad (26)$$

by assumption $U_0 = u_0 = u(x,0)$ and $U_i(x,0) = 0,\ (i = 1,2,\ldots)$. Eqn. (26) follows that

$$U_j = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\frac{1}{2}\frac{\partial^3 U_{j-1}}{\partial x^3} - 3\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} U_iU_k\frac{\partial U_{j-k-i-1}}{\partial x} + \frac{3}{2}\frac{\partial^2 V_{j-1}}{\partial x^2} + 3\frac{\partial}{\partial x}\sum_{i=0}^{j-1} U_jV_{j-i-1} - 3\lambda\frac{\partial}{\partial x}U_{j-1}\right) d\tau. \quad (27)$$

Also substituting eq. (25) into eq. (24) and equating the coefficients of the terms with identical powers of p

$$\begin{aligned}
p^0 &: D_t^\alpha V_0 = 0,\\
p^1 &: D_t^\alpha V_1 + \frac{\partial^3 U_0}{\partial x^3} + 3U_0\frac{\partial V_0}{\partial x} + 3\frac{\partial V_0}{\partial x}\frac{\partial U_0}{\partial x} - 3U_0^2\frac{\partial V_0}{\partial x} - 3\lambda\frac{\partial V_0}{\partial x} = 0,\\
p^2 &: D_t^\alpha V_2 + \frac{\partial^3 U_1}{\partial x^3} + 3\left(U_1\frac{\partial V_0}{\partial x} + U_0\frac{\partial V_1}{\partial x}\right) + 3\left(\frac{\partial U_0}{\partial x}\frac{\partial V_1}{\partial x} + \frac{\partial U_1}{\partial x}\frac{\partial V_0}{\partial x}\right)\\
&\quad - 3\left(U_0^2\frac{\partial V_1}{\partial x} + 2U_0U_1\frac{\partial V_0}{\partial x}\right) - 3\lambda\frac{\partial V_1}{\partial x} = 0,
\end{aligned}$$

$$\vdots$$

$$p^j:\ D_t^{\alpha}V_j+\frac{\partial^3 U_{j-1}}{\partial x^3}+3\sum_{i=0}^{j-1}U_j\frac{\partial V_{j-i-1}}{\partial x}+3\sum_{i=0}^{j-1}\frac{\partial U_j}{\partial x}\frac{\partial V_{j-i-1}}{\partial x} \tag{28}$$
$$-3\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}U_iU_k\frac{\partial V_{j-k-i-1}}{\partial x}-3\lambda\frac{\partial V_{j-1}}{\partial x}=0,$$

letting $V_0=v_0=v(x,0)$ and $V_i(x,0)=0,\ (i=1,2,\dots)$. According to Eqs. (28) we have

$$V_j=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\Big(-\frac{\partial^3 U_{j-1}}{\partial x^3}-3\sum_{i=0}^{j-1}U_j\frac{\partial V_{j-i-1}}{\partial x}-3\sum_{i=0}^{j-1}\frac{\partial U_j}{\partial x}\frac{\partial V_{j-i-1}}{\partial x}$$
$$+3\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}U_iU_k\frac{\partial V_{j-k-i-1}}{\partial x}+3\lambda\frac{\partial V_{j-1}}{\partial x}\Big)\,\mathrm{d}\tau. \tag{29}$$

Note: If we choose u_0 and v_0 as follows

$$u(x,0)=\frac{c}{2k}+k\tanh(kx),\quad v(x,0)=\frac{\lambda}{2}(1+\frac{k}{c})+c\tanh(kx),$$

where $k,c\neq 0$, and λ are arbitrary constants. We get

$$U_0=u(x,0)=\frac{c}{2k}+k\tanh(kx)\quad\text{and}\quad V_0=\frac{\lambda}{2}(1+\frac{k}{c})+c\tanh(kx), \tag{30}$$

$$U_1=\frac{t^{\alpha}}{\Gamma(\alpha+1)}\{-3k^2\lambda\,\mathrm{sech}(kx)^2-3k^3\,\mathrm{sech}(kx)^2\tanh(kx)$$
$$-3k^2\,\mathrm{sech}(kx)^2(\frac{c}{2k}+k\tanh(kx))^2+\frac{1}{2}(-2k^4\,\mathrm{sech}(kx)^4+4k^4\,\mathrm{sech}(kx)^2\tanh(kx)^2)$$
$$+3(k^2\,\mathrm{sech}(kx)^2(\frac{1}{2}(1+\frac{k}{c})\lambda+c\tanh(kx))+ck\,\mathrm{sech}(kx)^2(\frac{c}{2k}+k\tanh(kx)))\},$$
$$V_1=\frac{t^{\alpha}}{\Gamma(\alpha+1)}\{3ck\lambda\,\mathrm{sech}(kx)^2-3ck^3\,\mathrm{sech}(kx)^4+2k^4\,\mathrm{sech}(kx)^4$$
$$-4k^4\,\mathrm{sech}(kx)^2\tanh(kx)^2-3ck\,\mathrm{sech}(kx)^2(\frac{c}{2k}$$
$$+k\tanh(kx))+3ck\,\mathrm{sech}(kx)^2(\frac{c}{2k}+k\tanh(kx))^2\},$$
$$\vdots$$

and *etc.* We have plotted $|u(x,t)|$ and $|v(x,t)|$ in Fig. 1.

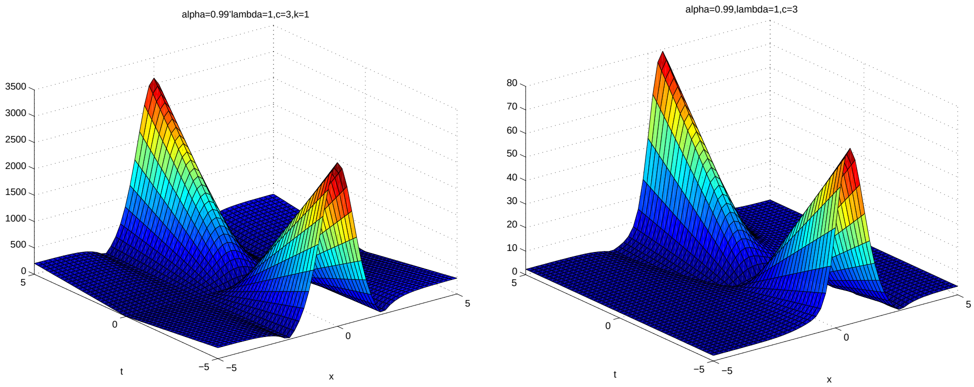

Fig. 1 – Graph of the $|u(x,t)|$ and $|v(x,t)|$ corresponding to the values $\alpha = 0.99$, $c = 3$ and $k = 1$ from left to right.

3.3. EXAMPLE

Consider the fractional coupled system of diffusion-reaction equation

$$\begin{aligned} D_t^{\alpha} u(x,t) &= u(1-u-v) + u_{xx}, \\ D_t^{\alpha} v(x,t) &= v_{xx} - uv, \end{aligned} \tag{31}$$

with initial condition

$$u(x,0) = u_0, \qquad v(x,0) = v_0.$$

This type of equations appear in the field of the chemical reactions, ecology as well as physics. In this case corresponding homotopys are:

$$H_1(U,V,p) = (1-p)(D_t^{\alpha} U) + p(D_t^{\alpha} U - U + U^2 + UV - U_{xx}) = 0 \tag{32}$$

$$H_2(U,V,p) = (1-p)(D_t^{\alpha} V) + p(D_t^{\alpha} V - V_{xx} + UV) = 0. \tag{33}$$

In the same manner we can suppose the solution of Eqns. (32) and (33) be

$$U = U_0 + p\,U_1 + p^2 U_2 + \ldots, \qquad V = V_0 + pV_1 + p^2 V_2 + \ldots. \tag{34}$$

As before, substituting eq. (34) into eq. (32) and equating the coefficients of the terms with identical powers of p,

$$\begin{aligned} p^0 &: D_t^{\alpha} U_0 = 0, \\ p^1 &: D_t^{\alpha} U_1 - U_0 + U_0^2 + U_0 V_0 - \frac{\partial^2 U_0}{\partial x^2} = 0, \\ p^2 &: D_t^{\alpha} U_2 - U_1 + (2U_0 U_1) + (U_1 V_0 + U_0 V_1) - \frac{\partial^2 U_1}{\partial x^2} = 0, \end{aligned} \tag{35}$$

$$\vdots$$

$$p^j : D_t^\alpha U_j - U_{j-1} + \sum_{i=0}^{j-1} U_i U_{j-i-1} + \sum_{i=0}^{j-1} U_i V_{j-i-1} - \frac{\partial^2 U_{j-1}}{\partial x^2} = 0. \tag{36}$$

We choose $U_0 = u_0 = u(x,0)$ and $V_i(x,0) = 0,\ (i = 1,2,\ldots)$. Eq. (36) follows immediately that

$$U_j = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(U_{j-1} - \sum_{i=0}^{j-1} U_i U_{j-i-1} - \sum_{i=0}^{j-1} U_i V_{j-i-1} + \frac{\partial^2 U_{j-1}}{\partial x^2} \right) \mathrm{d}\tau. \tag{37}$$

Similarly, substituting eq. (34) into eq. (33) and equating the coefficients of the terms with identical powers of p

$$\begin{aligned}
p^0 &: D_t^\alpha V_0 = 0,\\
p^1 &: D_t^\alpha V_1 - \frac{\partial^2 U_0}{\partial x^2} + U_0 V_0 = 0,\\
p^2 &: D_t^\alpha V_2 - \frac{\partial^2 U_1}{\partial x^2} + (U_0 V_1 + U_1 V_0) = 0,\\
&\vdots\\
p^j &: D_t^\alpha V_j - \frac{\partial^2 U_{j-1}}{\partial x^2} + \sum_{i=0}^{j-1} U_i V_{j-i-1} = 0,
\end{aligned} \tag{38}$$

where $V_0 = v_0 = v(x,0)$ and $V_i(x,0) = 0,\ (i = 1,2,\ldots)$. We can write solution of equation in virtue of Eqn. (38) as following

$$V_j = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\frac{\partial^2 U_{j-1}}{\partial x^2} - \sum_{i=0}^{j-1} U_i V_{j-i-1} \right) \mathrm{d}\tau. \tag{39}$$

Note: If we choose u_0 and v_0 as follows

$$u(x,0) = \frac{e^{kx}}{[1+e^{0.5kx}]^2}, \qquad v(x,0) = \frac{1}{[1+e^{0.5kx}]}.$$

where k is constant. We have

$$U_0 = u(x,0) = \frac{e^{kx}}{[1+e^{0.5kx}]^2} \quad \text{and} \quad V_0 = v(x,0) = \frac{1}{[1+e^{0.5kx}]}, \tag{40}$$

$$\begin{aligned}
U_1 = \frac{t^\alpha}{\Gamma(\alpha+1)} \{ &- \frac{e^{2kx}}{(1+e^{0.5kx})^4} - \frac{e^{kx}}{(1+e^{0.5kx})^3} + \frac{e^{kx}}{(1+e^{0.5kx})^2} + \frac{1.5e^{2.kx}k^2}{(1+e^{0.5kx})^4}\\
&- \frac{2.5e^{1.5kx}k^2}{(1+e^{0.5kx})^3} + \frac{e^{kx}k^2}{(1+e^{0.5kx})^2} \},
\end{aligned} \tag{41}$$

$$V_1 = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\{-\frac{e^{kx}}{(1+e^{0.5kx})^3} + \frac{1.5e^{2.kx}k^2}{(1+e^{0.5kx})^4} - \frac{2.5e^{1.5kx}k^2}{(1+e^{0.5kx})^3} + \frac{e^{kx}k^2}{(1+e^{0.5kx})^2}\}, \tag{42}$$

$$\vdots$$

and so on. In Fig. 2 we have sketched the $|u(x,t)|$ and $|v(x,t)|$.

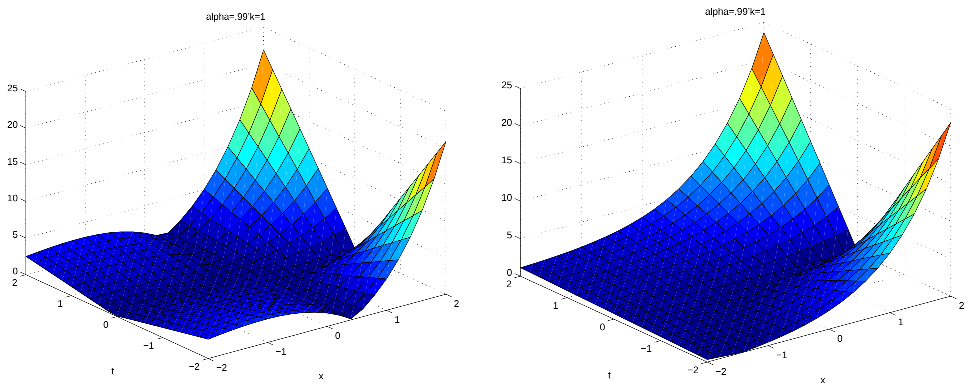

Fig. 2 – Graph of the $|u(x,t)|$ and $|v(x,t)|$ corresponding to the values $\alpha = 0.99$ from left to right.

3.4. EXAMPLE

Consider the fractional coupled system of diffusion-reaction equation

$$\begin{aligned} D_t^{\alpha}u(x,t) &= u(1-u^2-v)+u_{xx}, \\ D_t^{\alpha}v(x,t) &= v(1-u-v)+v_{xx}, \end{aligned} \tag{43}$$

with initial condition

$$u(x,0) = u_0, \qquad v(x,0) = v_0.$$

We construct homotopys as follows:

$$H_1(U,V,p) = (1-p)(D_t^{\alpha}U) + p(D_t^{\alpha}U - U + U^3 + UV - U_{xxx}) = 0 \tag{44}$$

$$H_2(U,V,p) = (1-p)(D_t^{\alpha}V) + p(D_t^{\alpha}V - V + VU + V^2 - V_{xx}) = 0. \tag{45}$$

Let the solution of Eq. (44) be such that

$$U = U_0 + p\,U_1 + p^2U_2 + \ldots, \qquad V = V_0 + pV_1 + p^2V_2 + \ldots. \tag{46}$$

Substituting eq. (46) into eq. (44), and equating the coefficients of the terms with identical powers of p,

$$\begin{aligned}
&p^0 : D_t^\alpha U_0 = 0,\\
&p^1 : D_t^\alpha U_1 - U_0 + U_0^3 + U_0 V_0 - \frac{\partial^3 U_0}{\partial x^3} = 0,\\
&p^2 : D_t^\alpha U_2 - U_1 + (3U_0^2 U_1) + (U_0 V_1 + U_1 V_0) - \frac{\partial^3 U_1}{\partial x^3} = 0,\\
&\quad\vdots\\
&p^j : D_t^\alpha U_j - U_{j-1} + \sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} U_i U_k U_{j-k-i-1} + \sum_{i=0}^{j-1} U_i V_{j-i-1} - \frac{\partial^3 U_{j-1}}{\partial x^3} = 0,
\end{aligned} \tag{47}$$

applying $U_0 = u_0 = u(x,0)$ and $U_i(x,0) = 0,\ (i = 1,2,\ldots)$. We continue in this fashion obtaining

$$U_j = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(U_{j-1} - \sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} U_i U_k U_{j-k-i-1} - \sum_{i=0}^{j-1} U_i V_{j-i-1} + \frac{\partial^3 U_{j-1}}{\partial x^3}\right) d\tau. \tag{48}$$

Substituting eq. (46) into eq. (45) and equating the coefficients of the terms with identical powers of p, we get

$$\begin{aligned}
&p^0 : D_t^\alpha V_0 = 0,\\
&p^1 : D_t^\alpha V_1 - V_0 + V_0 U_0 + V_0^2 - \frac{\partial^2 V_0}{\partial x^2} = 0,\\
&p^2 : D_t^\alpha V_2 - V_1 + (V_1 U_0 + V_0 U_1) + 2V_0 V_1 - \frac{\partial^2 V_1}{\partial x^2} = 0,\\
&\quad\vdots\\
&p^j : D_t^\alpha V_j - V_{j-1} + \sum_{i=0}^{j-1} V_i U_{j-i-1} + \sum_{i=0}^{j-1} V_i V_{j-i-1} - \frac{\partial^2 V_{j-1}}{\partial x^2} = 0.
\end{aligned} \tag{49}$$

Assuming $V_0 = v_0 = v(x,0)$ and $V_i(x,0) = 0,\ (i = 1,2,\ldots)$. In view of the eqs. (49) we obtain

$$V_j = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(V_{j-1} - \sum_{i=0}^{j-1} V_i U_{j-i-1} - \sum_{i=0}^{j-1} V_i V_{j-i-1} + \frac{\partial^2 V_{j-1}}{\partial x^2}\right) d\tau. \tag{50}$$

Note: If we choose u_0 and v_0 as follows

$$u(x,0)=\frac{e^{kx}}{[1+e^{kx}]},\qquad v(x,0)=\frac{1+(3/4)e^{kx}}{[1+e^{kx}]^2},$$

where k is constant, we obtain

$$U_0=u(x,0)=\frac{c}{2k}+k\tanh(kx)\quad\text{and}\quad V_0=v(x,0)=\frac{\lambda}{2}(1+\frac{k}{c})+c\tanh(kx),\tag{51}$$

$$\begin{aligned}U_1=&\frac{t^{\alpha}}{\Gamma(\alpha+1)}\{-\frac{e^{kx}(1+\frac{3e^{kx}}{4})}{1+e^{kx}}+e^{kx}(1+e^{kx})-e^{3kx}(1+e^{kx})^3\\&+7e^{2kx}k^3+e^{kx}(1+e^{kx})k^3\},\end{aligned}\tag{52}$$

$$\begin{aligned}V_1=&\frac{t^{\alpha}}{\Gamma(\alpha+1)}\{-\frac{(1+\frac{3e^{kx}}{4})^2}{(1+e^{kx})^4}+\frac{1+\frac{3e^{kx}}{4}}{(1+e^{kx})^2}-\frac{e^{kx}(1+\frac{3e^{kx}}{4})}{1+e^{kx}}\\&+\frac{6e^{2kx}(1+\frac{3e^{kx}}{4})k^2}{(1+e^{kx})^4}-\frac{3e^{2kx}k^2}{(1+e^{kx})^3}-\frac{2e^{kx}(1+\frac{3e^{kx}}{4})k^2}{(1+e^{kx})^3}+\frac{3e^{kx}k^2}{4(1+e^{kx})^2}\},\end{aligned}\tag{53}$$

$$\vdots$$

and so on. We have shown the graph of the $|u(x,t)|$ and $|v(x,t)|$ *versus* (x,t).

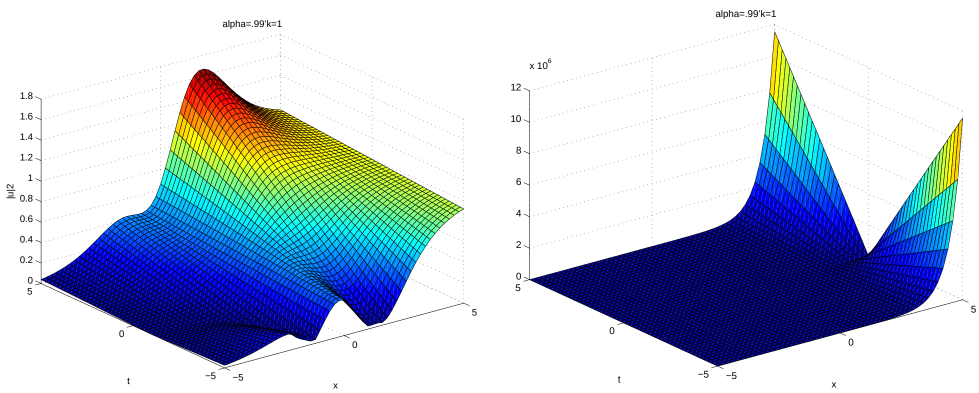

Fig. 3 – Graph of the $|u(x,t)|$ and $|v(x,t)|$ corresponding to the values $\alpha=0.99$ and $k=1$ from left to right.

4. CONCLUSION

In this study, the Homotopy Perturbation Method with new strategies has applied for obtaining approximate analytical solutions of the fractional coupled Schrödinger-Korteweg-de Vries and coupled system of diffusion-reaction equations. In order to increase the accuracy of the approach, higher components of series solution should be taken into account. This algorithm provides accurate numerical solutions without discretization for nonlinear differential equations. In this method, in spite of Adomian decomposition method we don't need to calculate Adomian polynomials or variational iteration method needs to find Lagrange multiplier by making correction functional stationary in the fractional case.

Acknowledgements. The authors would like to thank to J.J. Trujillo and K. Diethelm for their important comments and remarks regarding the revised manuscript.

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