*Romanian Reports in Physics, Vol. 63, Supplement, P. 1302–1315, 2011*

*Dedicated to Professor Marin Ivaşcu's 80$^{th}$ Anniversary*

# DUST ACOUSTIC SOLITONS IN DUSTY PLASMAS WITH GRAIN SIZE DISTRIBUTION

D. GRECU, A. T. GRECU

Department of Theoretical Physics,
Horia Hulubei National Institute for Physics and Nuclear Engineering,
30 Reactorului str., RO-077125, POB-MG6, Măgurele-Bucharest, Romania, EU
*E-mail*: dgrecu@theory.nipne.ro

*Received July 4, 2011*

*Abstract.* A Korteweg-de Vries (KdV) equation describing the evolution of the nonlinear dust-acoustic waves (DAWs) is deduced for a dusty plasma with an arbitrary distribution of grain sizes. The coefficients of the KdV equation are expressed in terms of different mean values of the dust grain radius $r_d$. These are calculated explicitly for three simple distribution functions: (i) a rectangular distribution, (ii) a $\delta$-distribution for grains with two different radii and (iii) two non-overlapping rectangular distributions. The choice of these simplified distributions allows us to obtain clear analytical expressions and the influence of the distribution function on the KdV soliton is easily seen. As a general rule the coefficients of the KdV equation are always greater than in the case of a mono-sized dusty plasma and consequently the one-soliton solution has a lower amplitude and an increased width.

*Key words*: dusty plasma, grain size distribution, dust acoustic soliton.

## 1. INTRODUCTION

A dusty plasma is a complex system of electrons, different species of ions, neutral atoms and massive particulates of dust. The difference from a usual plasma is due to the presence of the dust components. When the mean distance between the dust particles is smaller than the Debye screening length the charged dust grains contribute to the collective motion of the plasma. Even in the simplest case of a single type of single ionized positive ions and in the absence of neutral atoms, the presence of dust grains introduces new behaviors which have to be taken into account in plasma studies. The dust particles can have various shapes, dimensions, composition and texture, their behavior being strongly influenced by all these properties. We shall consider the simplest model of dust particles made from the same non-magnetic material, having smooth, spherical forms of radii $r_d$ which are distributed according to a specific distribution function $f(r_d)$. Also we shall neglect any external, ionizing agent so that the dust grains will be negatively charged due to the fluxes of electrons and ions falling onto their surface.

The dusty plasma is a common component of many astrophysical systems like interstellar and circumstellar clouds, solar nebula, the comet tails, planetary rings and even in Earth's atmosphere. The study of these systems represented one of the major impulses responsible for the tremendous development of the field in the last three decades. Another reason is the presence of dusty plasma in many laboratory devices and experiments like `dc` and `rf` discharges, plasma processing reactors and fusion plasma devices. We shall not enter into more details as further informations can be obtained from scores of monographies and review papers (see [1–5] and references therein).

One of the main effect of the charged dust grains on the plasma properties is the existence of a class of very low frequency waves, the so-called *Dust Acoustic Waves* – DAWs. They were predicted by Rao *et al.* [6], and appear due to collective motion of the dust fluid. In a hydrodynamic description and using a multiple scales analysis the nonlinear DAWs are described by the well-known Korteweg-de Vries (KdV) equation. They were observed experimentally [1, 7, 8] in the frequency range of tens of Hertz and can be recorded on video camera. The linear and nonlinear (solitons) DAWs were since studied by many authors in different physical conditions, but most of them discussed only mono-sized dusty plasmas (see [3, 9–17] and references therein; this selected list does not contain works on DAWs in the presence of magnetic fields). In a less ideal model the dust grains have many different sizes both in space plasmas and laboratory experiments. In space plasmas it is widely accepted that dust particle radii have a power law distribution [1, 18]

$$f(r_d) = \begin{cases} K r_d^{-\beta}, & r_{min} < r_d < r_{max} \\ 0, & \text{otherwise} \end{cases}$$
$$K = \frac{1-\beta}{r_{max}^{\beta-1} - r_{min}^{\beta-1}},$$

where $\beta = 4.5$ for the F-ring of Saturn and $\beta = 6 \div 7$ for the G-ring while $\beta = 3.4$ in the cometary environment. In laboratory experiments it was found [19] that a Gaussian (normal) distribution is appropriate

$$f(r_d) = K \exp\left(-\frac{(r-r_0)^2}{2\sigma^2}\right)$$
$$K = \frac{1}{\sqrt{\pi/2\sigma}} \frac{1}{1+\Phi(r_0/2\sigma)},$$

where $r_0$ is the most probable value, $\sigma$ the width of the distribution and

$$\Phi(y) = \int_0^y \exp(-x^2)\,\mathrm{d}x$$

the error function [19].

Several authors have investigated the influence of the dust grain size distribution on the properties of the dust acoustic waves [19–30]. In the present paper, a dusty plasma composed of electrons, one species of single ionized positive ions and dust particles of same nature is considered. The dust grains are assume of spherical shape with a (normalized) distribution function of dust particle radii $f(r_d)$. We shall consider that the charging time of the dust grains is several orders of magnitude smaller than the time characterizing the DAWs evolution, so in a first approximation we shall neglect the dust charge fluctuations.

In the next section, using a hydrodynamic description of the system, a KdV equation describing the nonlinear excitations of the system is derived applying the asymptotic procedure known as the multiple scales method or reductive perturbative method. Although the method is well-known and used in many papers a brief review is necessary here in order to introduce several mean values of the dust grain radius powers and express the final result (the coefficients of the KdV equation) in terms of these values. In section 3 some very simple expressions for the distribution function $f(r_d)$ are considered, namely

- rectangular distribution

$$f(r_d) = \begin{cases} h, & |r_d - r_0| \le \dfrac{d}{2} \\ 0, & \text{otherwise} \end{cases}, \tag{1}$$

  where $hd = 1$.

- $\delta$-distribution of two dust grain species with radii $r_1$ and $r_2$

$$f(r_d) = c_1\delta(r_d - r_1) + c_2\delta(r_d - r_2), \quad c_1 + c_2 = 1 \tag{2}$$

- two non-overlapping rectangular distributions

$$f(r_d) = \begin{cases} h_1, & |r_d - r_1| \le \dfrac{d_1}{2} \\ h_2, & |r_d - r_2| \le \dfrac{d_2}{2} \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

$$h_1 d_1 = c_1, \quad h_2 d_2 = c_2, \quad c_1 + c_2 = 1, \quad \frac{d_1 + d_2}{2} < |r_2 - r_1|.$$

These are highly simplified distributions which were used because they lead to easily interpretable results and the dependence of the one-soliton solution of the KdV evolution equation on the parameters of the distribution function is straightforward.

Through-out the rest of the paper, the mean values of quantities depending on the dust grain radius will be denoted by $\langle \ldots \rangle$. Thus, if $\eta(r_d)$ is such a physical quantity, we have

$$\langle \eta \rangle = \int \eta(r_d) f(r_d)\, \mathrm{d}r_d.$$

## 2. BASIC EQUATIONS & MULTIPLE SCALE ANALYSIS

Let us denote by $n_e(r,t)$, $n_i(r,t)$, $n(r,t;r_d)$ the number densities of electrons, positive ions and dust particles (of a given radius $r_d$) respectively, and by capital letters $N_e$, $N_i$; $N_d(r_d)$ their corresponding equilibrium values. If $N_0$ is the total number density of dust grains we have $N_0 = \int N(r_d)\, \mathrm{d}r_d$. Therefore, the equilibrium dust number density can be expressed in terms of the dust grain size distribution as $N(r_d) = N_0 f(r_d)$. The dust charge will also depend on the dust particle size and, for a given radius $r_d$, it writes $q(r_d) = -eZ(r_d)$, where $Z(r_d)$ is the number of negative elementary charges accumulated on the dust grain surface. Thus the neutrality condition, at equilibrium, writes

$$N_i = N_e + \int N(r_d) Z(r_d)\, \mathrm{d}r_d = N_e + N_0 \langle Z \rangle, \tag{4}$$

where $\langle Z \rangle$, the mean number of charges on the dust grains, is

$$\langle Z \rangle = \int Z(r_d) f(r_d)\, \mathrm{d}r_d. \tag{5}$$

The charging process of a dust grain is a complex phenomenon. Even neglecting any external ionizing factors, the process retains its complexity and further simplifying approximations are needed to obtain a simpler view. Ignoring several processes like secondary electron emission, thermionic and field emission and so on, the dust grains are charging through the electron and ionic currents falling on them and their charge is negative at equilibrium due to the higher mobility of the electrons. In the widely used Orbit Limited Motion (OLM) approximation, the ionic and electronic currents collected by a spherical dust grain of radius $r_d$ have simple expressions [1, 31]

$$I_i = 4\pi r_d^2 N_i e \sqrt{\frac{k_B T_i}{2\pi m_i}} \left(1 - \frac{e\Phi_d}{k_B T_i}\right),$$
$$I_e = -4\pi r_d^2 N_e e \sqrt{\frac{k_B T_e}{2\pi m_e}} \exp\left(\frac{e\Phi_d}{k_B T_e}\right).$$

Here $\Phi_d$ is the floating potential of the dust grain, defined as the difference between the grain potential and the plasma one, $\Phi_d = \Phi_g - \varphi$ and related to the dust grain charge by $q_d = C\Phi_d$, $C$ being the capacitance of the spherical dust grain

$$C = 4\pi\varepsilon_0 r_d \exp\left(-r_d/\lambda_D\right) \simeq 4\pi\varepsilon_0 r_d,$$

as in the case of a dusty plasma, $r_d \ll \lambda_D$. Here $\lambda_D$ is the dusty plasma Debye screening length. Then, the currents $I_i$, $I_e$ write

$$\begin{aligned} I_i &= 4\pi r_d^2 N_i e \sqrt{\frac{k_B T_i}{2\pi m_i}} \left(1 + \frac{1}{4\pi\varepsilon_0} \frac{e^2 Z(r_d)}{r_d k_B T_i}\right), \\ I_e &= -4\pi r_d^2 N_e e \sqrt{\frac{k_B T_e}{2\pi m_e}} \exp\left(-\frac{1}{4\pi\varepsilon_0} \frac{e^2 Z(r_d)}{r_d k_B T_e}\right). \end{aligned} \tag{6}$$

At equilibrium, one has $I_i + I_e = 0$ and all dust grains, regardless of their radius, will be charged at the same floating potential. Consequently, we have the following very important relation

$$\frac{Z(r_d)}{r_d} = \frac{\langle Z \rangle}{\langle r_d \rangle}, \tag{7}$$

in which the mean value $\langle Z \rangle$ is found as the solution of the equation

$$\sqrt{\frac{m_e}{m_i}}\left(1 + z\right) = \sqrt{\theta} \exp\left(-z/\theta\right) \left(1 - Pz\right), \tag{8}$$

where

$$z = \frac{1}{4\pi\varepsilon_0} \frac{\langle Z \rangle e^2}{\langle r_d \rangle k_B T_i}, \qquad P = 4\pi N_0 \langle r_d \rangle \lambda_{Di}^2. \tag{9}$$

Here we denoted $\theta = T_e/T_i$, the ratio of electron fluid to ion fluid temperature and $\lambda_{Di}$ is the ion Debye screening length, $\lambda_{Di}^2 = \varepsilon_0 k_B T_i/(N_i e^2)$. Using these notations the neutrality condition (at equilibrium) becomes

$$\frac{N_e}{N_i} = 1 - zP, \tag{10}$$

so the quantity $(1 - zP)$ is a measure of the depletion of the electron gas due to the charging process of the dust grains.

The DAWs are associated only to the motion of the dust fluid, so in a hydrodynamic description we will assume that the electron and ion fluids are in equilibrium with the local, plasma potential $\varphi(x,t)$, thus their number densities are given by Boltzmann distributions. In order to reduce the problem to a numerical one we must introduce appropriate units. Thus, all the densities will be expressed in units of their equilibrium values, the plasma potential in units of $k_B T_i/e$, the time in units of $\omega_{pd}^{-1}$, where $\omega_{pd} = \left(\dfrac{N_0 e^2 \langle Z \rangle^2}{\varepsilon_0 \langle m_d \rangle}\right)^{1/2}$ is the plasma frequency of the dust fluid composed of grains with a mean mass $\langle m_d \rangle$ and a negative mean charge $(-e\langle Z \rangle)$, the space coordinate $x$ in units of $\lambda_d = \left(\dfrac{\varepsilon_0 k_B T_d}{N_0 e^2 \langle Z \rangle}\right)^{1/2}$, the Debye screening length of the dust fluid, and the dust fluid velocity $v(x,t;r_d)$ in units of thermal velocity $v_{Td} = \left(\dfrac{k_B T_d \langle Z \rangle}{m_d}\right)^{1/2}$, $\lambda_d \omega_{pd} = v_{Td}$. Then, the equations describing the dust fluid dynamics are

$$\begin{gathered} \frac{\partial}{\partial t} n(r_d) + \frac{\partial}{\partial x}\left(n(r_d) v(r_d)\right) = 0, \\ \left(\frac{\partial}{\partial t} + v(r_d)\frac{\partial}{\partial x}\right) v(r_d) = Q(r_d)\frac{\partial \varphi}{\partial x}, \end{gathered} \tag{11}$$

where

$$Q(r_d) = \frac{\langle m_d \rangle}{m(r_d)} \frac{Z(r_d)}{\langle Z \rangle} = \frac{\langle r_d^3 \rangle}{\langle r_d \rangle r_d^2}. \tag{12}$$

The densities of the electron and ion fluids are, respectively

$$n_e(x,t) = \exp\left(\frac{\varphi}{\theta}\right), \qquad n_i(x,t) = \exp(-\varphi), \tag{13}$$

where $\theta = T_e/T_i$ and for simplicity we assumed $T_d = T_i$. To these we have to add the Poisson equation, which writes

$$\frac{\partial^2 \varphi}{\partial x^2} = \mu_e n_e - \mu_i n_i + \frac{1}{\langle r_d \rangle} \int r_d n(r_d)\, \mathrm{d}r_d, \tag{14}$$

where

$$\mu_j = \frac{N_j}{N_0 \langle Z \rangle}, \quad j = e, i.$$

In these dimensionless variables the neutrality condition becomes

$$\mu_i = 1 + \mu_e. \tag{15}$$

The standard multiple scale method [32] is used to analyze this system of equations. One introduces the stretched variables

$$\xi = \varepsilon^{1/2}(x - u_0 t), \quad \tau = \varepsilon^{3/2} t,$$

with $u_0$ a speed of the space-time reference frame which will be determined later ($\varepsilon \ll 1$). All the quantities $n_e$, $n_i$, $n(r_d)$, $v(rd)$, $\varphi$ are expanded in power series with respect to the small parameter $\varepsilon$ around their equilibrium values, namely

$$\begin{aligned} n_j(\xi,\tau) &= 1 \qquad + \varepsilon n_j^{(1)}(\xi,\tau) + \varepsilon^2 n_j^{(2)}(\xi,\tau) + \ldots, \quad j = e, i \\ n(\xi,\tau;r_d) &= f(r_d) + \varepsilon n^{(1)}(\xi,\tau;r_d) + \varepsilon^2 n^{(2)}(\xi,\tau;r_d) + \ldots, \\ v(\xi,\tau;r_d) &= \qquad \varepsilon v^{(1)}(\xi,\tau;r_d) + \varepsilon^2 v^{(2)}(\xi,\tau;r_d) + \ldots, \\ \varphi(\xi,\tau) &= \qquad \varepsilon \varphi^{(1)}(\xi,\tau) + \varepsilon^2 \varphi^{(2)}(\xi,\tau) + \ldots. \end{aligned} \tag{16}$$

From the Boltzmann distributions of $n_e$ and $n_i$ we get, in the first two orders of approximation

$$\begin{aligned} n_e^{(1)} &= \frac{1}{\theta}\varphi^{(1)}, & n_i^{(1)} &= -\varphi^{(1)}, \\ n_e^{(2)} &= \frac{1}{\theta}\varphi^{(2)} + \frac{1}{2\theta^2}\left[\varphi^{(1)}\right]^2, & n_i^{(2)} &= -\varphi^{(2)} + \frac{1}{2}\left[\varphi^{(1)}\right]^2. \end{aligned} \tag{17}$$

From the equation of continuity and the equation of motion (11), in the first orders of $\varepsilon$, we obtain

$$\begin{aligned} -u_0 n^{(1)} + f(r_d) v^{(1)} &= 0, \\ -u_0 v^{(1)} &= Q(r_d)\varphi^{(1)}, \end{aligned} \tag{18}$$

which lead to the following expressions of $n^{(1)}$ and $v^{(1)}$ in terms of $\varphi^{(1)}$

$$v^{(1)} = -\frac{1}{u_0} Q(r_d)\varphi^{(1)}, \qquad n^{(1)} = -\frac{1}{u_0^2} f(r_d) Q(r_d)\varphi^{(1)}. \tag{19}$$

Introducing these in the Poisson equation, the terms of first order in $\varepsilon$ allow us to find the following expression for the velocity $u_0$

$$u_0^2 = \frac{D}{\mu_i + \frac{\mu_e}{\theta}}, \quad D = \frac{\langle r_d^3 \rangle}{\langle r_d \rangle^2}\left\langle \frac{1}{r_d} \right\rangle, \tag{20}$$

where the mean values are calculated over the distribution function $f(r_d)$.

In the next order in $\varepsilon$ the equation of continuity yields

$$-u_0 \frac{\partial n^{(2)}}{\partial \xi} + \frac{\partial n^{(1)}}{\partial \tau} + f(r_d)\frac{\partial v^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi}\left(n^{(1)} v^{(1)}\right) = 0, \tag{21}$$

and from the momentum equation we get

$$-u_0 \frac{\partial v^{(2)}}{\partial \xi} + \frac{\partial v^{(1)}}{\partial \tau} + v^{(1)}\frac{\partial v^{(1)}}{\partial \xi} = Q(r_d)\frac{\partial \varphi^{(2)}}{\partial \xi}. \tag{22}$$

In order $\varepsilon^2$ the Poisson equation writes

$$\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} = \left(\mu_i + \frac{\mu_e}{\theta}\right)\varphi^{(2)} + \frac{1}{\langle r_d \rangle}\int r_d n^{(2)}(r_d)\,\mathrm{d}r_d - \frac{1}{2}\left(\mu_i - \frac{\mu_e}{\theta}\right)\left[\varphi^{(1)}\right]^2. \tag{23}$$

Eliminating the second order quantities, $n^{(2)}$, $v^{(2)}$, $\varphi^{(2)}$, from this set of equations, the final result is the following KdV equation for $\varphi^{(1)}(\xi,\tau)$

$$\frac{\partial \varphi^{(1)}}{\partial \tau} + a\frac{\partial^3 \varphi^{(1)}}{\partial \xi^3} - b\varphi^{(1)}\frac{\partial \varphi^{(1)}}{\partial \xi} = 0, \tag{24}$$

where

$$a = \frac{u_0}{2}\frac{1}{\mu_i + \frac{\mu_e}{\theta}} = \frac{u_0^3}{2D},$$

$$b = \frac{u_0}{2}\left(\frac{3}{u_0^2}\bar{D} - \frac{\mu_i - \frac{\mu_e}{\theta}}{\mu_i + \frac{\mu_e}{\theta}}\right), \quad \bar{D} = \frac{\langle r_d^3\rangle \left\langle \frac{1}{r_d^3}\right\rangle}{\langle r_d\rangle \left\langle \frac{1}{r_d}\right\rangle}.$$

For dust grains of the same dimension $f(r_d) = \delta(r_d - r_0)$, both coefficients $D$ and $\bar{D}$ are equal to unity and

$$u_0^2 = \frac{1}{\mu_i + \frac{\mu_e}{\theta}}, \quad a \to a_0 = \frac{u_0^3}{2}, \quad b \to b_0 = a\left[\frac{3}{u_0^4} - \left(\mu_i - \frac{\mu_e}{\theta}\right)\right],$$

the well known expressions of the coefficients of the KdV equation [1].

As $b$ is always a positive quantity, the one soliton solution of the KdV equation (Dust Acoustic Soliton – DAS) writes

$$\varphi^{(1)} = -\varphi_m \operatorname{sech}^2\left[(\xi - V\tau)/\Delta\right], \tag{25}$$

where $\varphi_m = \dfrac{3V}{b}$ and $\Delta^2 = \dfrac{4a}{V}$. This corresponds to a compressive ($n^{(1)} \sim -\varphi^{(1)}$) bell-shaped wave of amplitude $\varphi_m$ and width $\Delta$ propagating with the velocity $V$. It is obvious that as the wave velocity increases so does the amplitude but its width is decreasing. Also the dependence of the soliton characteristics ($\varphi_m$, $\Delta$) on the KdV equation coefficients ($a$, $b$) is evident, namely an increase of $a(b)$ coefficient leads to an increase(decrease) of the soliton width(amplitude). Therefore, a detailed study of the influence of the grain radius distribution function on these coefficients is necessary in order to assess the behavior of nonlinear DAWs in real dusty plasmas.

### 3. DUST GRAIN RADIUS DISTRIBUTION EFFECT ON DAS PARAMETERS

We start the analysis by considering the influence of the dust grain radius distribution function on the mean value $\langle Z\rangle$. For the rectangular distribution function (1) the mean radius is $\langle r_d\rangle = r_0$ and according to the general results of the previous section $\langle Z\rangle = Z_0$, $Z_0$ corresponding to the dust grains with the same radius $r_0$. Therefore, in this case $\langle Z\rangle$ is not influenced by the width of the distribution function. For the other two distribution functions, (2), (3), if we denote by $c$ the concentration of the dust grains with larger radius ($r_2 > r_1$, $c_2 = c$, $c_1 = 1 - c$), $\langle r_d\rangle$ is a linearly increasing function of $c$ from $r_1$ ($c = 0$) to $r_2$ ($c = 1$), $\langle r_d\rangle = r_1 + (r_2 - r_1)c$. We expect that also $\langle Z\rangle$ will be a monotonously increasing function of $c$ from $Z_1$ to $Z_2$, where these two values correspond to the equilibrium charge of dust grains with radius $r_1$ and $r_2$, respectively. In figure 1 we give the dependence of the parameter $z$ on $\log P$ to illustrate the behavior of $\langle Z\rangle$ for two values of the ratio $\theta$. One can see

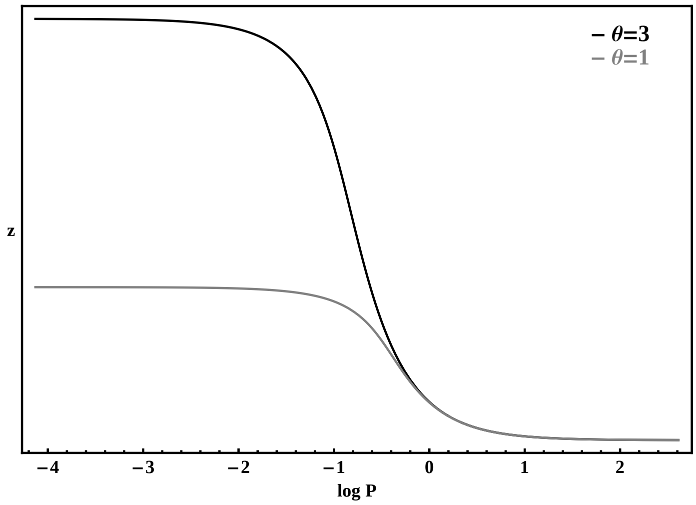


Fig. 1 – Dependence of $z$ on $\log P$ in an $\mathrm{Ar}^{+}$ plasma for two values of the parameter $\theta = T_e/T_i$: $\theta = 1$ (*lower curve*); $\theta = 3$ (*upper curve*).

that $z(\log P)$ is a monotonously decreasing function of $c$. In the intermediate region where $z$ has a rapid variation, differentiating equation (8) with respect to $c$, we get

$$\frac{\mathrm{d}z}{\mathrm{d}c} = -\frac{Pz\frac{r_2-r_1}{\langle r_d\rangle}}{(1-Pz)\left(\frac{1}{\theta}+\frac{1}{1+z}\right)+P} < 0, \tag{26}$$

where we used $\dfrac{\mathrm{d}P}{\mathrm{d}c} = P\dfrac{r_2-r_1}{\langle r_d\rangle}$, $\dfrac{\mathrm{d}\langle r_d\rangle}{\mathrm{d}c} = (r_2-r_1)$. Then using the definition (9) of $z$, we have

$$\frac{\mathrm{d}}{\mathrm{d}c}\ln\langle Z\rangle = \frac{\mathrm{d}}{\mathrm{d}c}\ln\langle r_d\rangle + \frac{\mathrm{d}}{\mathrm{d}c}\ln z = \frac{r_2-r_1}{\langle r_d\rangle}\left[1-\frac{1}{\frac{1-Pz}{P}\left(\frac{1}{\theta}+\frac{1}{1+z}\right)+1}\right] > 0, \tag{27}$$

which shows that $\langle Z\rangle$ is a monotonously increasing function of $c$.

Small values of $P$ correspond to small number densities $N_0$ of the dust grains, which are well separated (screened) and no longer interacting (dust in plasma). In this limit the charge of each grain is that of an isolated one (determined by the size of the particle). From figure 1 one sees that its value is strongly dependent on the ratio $\theta = T_e/T_i$. As the density of dust grains increase, the particles start behaving as a fluid (dusty plasma) and the charge on each grain decreases rapidly to a value determined by the plasma potential and particle size distribution. Then, for high density $N_0$, the charge attains another plateau corresponding to dusty plasmas with completely depleted electron gas.

The main effect of the dust grain radius distribution is seen on the quantities $D$ and $\bar{D}$ which appear in the definition of $u_0$ and the coefficients $a$ and $b$ of the KdV equation. In the following we shall analyze these constants for the distribution functions (1)-(3).

### 3.1. RECTANGULAR DISTRIBUTION

As mentioned above, in this case $\langle r_d \rangle = r_0$. The other mean values of interest are given by

$$\langle r_d^3 \rangle = r_0^3(1+\lambda^2),$$
$$\left\langle \frac{1}{r_d} \right\rangle = \frac{1}{r_0}\frac{1}{2\lambda}\ln\frac{1+\lambda}{1-\lambda}, \quad \left\langle \frac{1}{r_d^3} \right\rangle = \frac{1}{r_0^3}\frac{1}{(1-\lambda^2)^2}, \tag{28}$$

where we denoted $\lambda = d/2r_0$, $0 \leq \lambda < 1$. Then

$$D = \frac{1}{2\lambda}(1+\lambda^2)\ln\frac{1+\lambda}{1-\lambda} \tag{29}$$

and for small $\lambda$ a series expansion gives

$$D \simeq (1+\lambda^2)\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{2k+1}.$$

For $\bar{D}$ we obtain

$$\bar{D} = \left(\frac{1+\lambda^2}{1-\lambda^2}\right)^2. \tag{30}$$

In the limit $\lambda \to 0$, both $D$ and $\bar{D}$ become equal to unity while they are monotonously growing with $\lambda$ otherwise. Consequently, the quantities $u_0$, $a$, $b$, characterizing the KdV equation, are monotonous increasing functions of $\lambda$.

### 3.2. $\delta$-DISTRIBUTION OF TWO GRAIN SPECIES

In this case the result will depend both on the ratio $\gamma = r_2/r_1 > 1$ of higher to lower radius and on the concentration $c_2 = c$ ($c_1 = 1-c$) of the more massive dust grains. We obtain

$$\langle r_d \rangle = r_1\left[1+(\gamma-1)c\right], \quad \langle r_d^3 \rangle = r_1^3\left[1+(\gamma^3-1)c\right],$$
$$\left\langle \frac{1}{r_d} \right\rangle = \frac{1}{r_1}\left[1-\frac{\gamma-1}{\gamma}c\right], \quad \left\langle \frac{1}{r_d^3} \right\rangle = \frac{1}{r_1^3}\left[1-\frac{\gamma^3-1}{\gamma^3}c\right]. \tag{31}$$

All these mean values have a linear dependence on the concentration $c$, but a stronger influence from the ratio $\gamma$. For $D$ and $\bar{D}$ the following expressions are obtained

$$D = \frac{1+(\gamma^3-1)c}{[1+(\gamma-1)c]^2}\left(1-\frac{\gamma-1}{\gamma}c\right), \quad \bar{D} = \frac{1+(\gamma^3-1)c}{1+(\delta-1)c}\frac{1-\frac{\gamma^3-1}{\gamma^3}c}{1-\frac{\gamma-1}{\gamma}c}. \tag{32}$$

As expected, both expression in (32) are equal to unity for the extreme values of $c$. In the figure 2 we give the dependence of $D$ and $\bar{D}$ on the concentration for two values of $\gamma$. As both $\mu_i$ and $\mu_e$ contain $\langle Z\rangle$ at the denominator, the quantity $\left(\mu_i + \dfrac{\mu_e}{\theta}\right)$ is

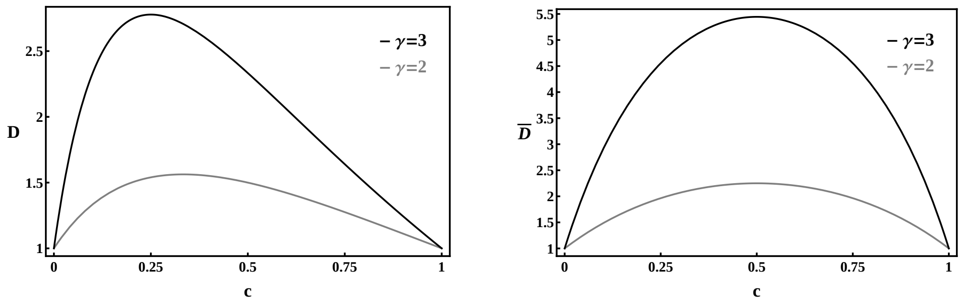


Fig. 2 – Dependence of $D$ (*left*) and $\bar{D}$ (*right*) on the concentration $c$ for two values of $\gamma$: $\gamma = 2$ (*lower curve*); $\gamma = 3$ (*upper curve*).

a slowly decreasing function of $c$ and $\left(\mu_i - \dfrac{\mu_e}{\theta}\right) / \left(\mu_i + \dfrac{\mu_e}{\theta}\right)$ does not depend on the concentration $c$. Then, it is easily seen that both coefficients $a$ and $b$ of the KdV equation will depend on $c$ in a similar way as $D$ and $\bar{D}$ (see fig. 2).

### 3.3. NON-OVERLAPPING RECTANGULAR DISTRIBUTION FOR TWO RADII

In this case (3) the result will depend not only on the ratio $\gamma = r_2/r_1 > 0$ and the concentration $c_2 = c$ but also on the width parameters $\lambda_1 = \dfrac{d_1}{2r_1}$ and $\lambda_1 = \dfrac{d_2}{2r_2}$. In these variables the non-overlapping condition writes $\lambda_1 + \gamma\lambda_2 < \gamma - 1$.For the mean values involved in our discussion we get

$$\begin{gathered}
\langle r_d\rangle = r_1\left[1+(\gamma-1)c\right], \quad \langle r_d^3\rangle = r_1^3\left[(1+\lambda_1^2)(1-c)+\gamma^3(1+\lambda_2^2)c\right], \\
\left\langle \frac{1}{r_d}\right\rangle = \frac{1}{r_1}\left[\frac{1}{2\lambda_1}\ln\frac{1+\lambda_1}{1-\lambda_1}(1-c)+\frac{1}{\gamma}\frac{1}{2\lambda_2}\ln\frac{1+\lambda_2}{1-\lambda_2}c\right] \\
\left\langle \frac{1}{r_d^3}\right\rangle = \frac{1}{r_1^3}\left[\frac{1-c}{(1-\lambda_1^2)^2}+\frac{1}{\gamma^3}\frac{c}{(1-\lambda_2^2)^2}\right].
\end{gathered} \tag{33}$$

For small $\lambda_1$ and $\lambda_2$, the mean value $\left\langle \dfrac{1}{r_d}\right\rangle$ writes

$$\left\langle \frac{1}{r_d}\right\rangle = \frac{1}{r_1}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left[\lambda_1^{2k}(1-c)+\frac{1}{\gamma}\lambda_2^{2k}c\right].$$

Then all these mean values depend linearly on the concentration $c$.

For the quantities $D$ and $\bar{D}$ we get

$$
\begin{aligned}
D &= \frac{(1+\lambda_1^2)(1-c)+\gamma^3(1+\lambda_2^2)c}{[1+(\gamma-1)c]^2}\left[\frac{1}{2\lambda_1}\ln\frac{1+\lambda_1}{1-\lambda_1}(1-c)+\frac{1}{2\lambda_2}\ln\frac{1+\lambda_2}{1-\lambda_2}\frac{c}{\gamma}\right] \\
\bar{D} &= \frac{(1+\lambda_1^2)(1-c)+\gamma^3(1+\lambda_2^2)c}{1+(\gamma-1)c}\,\frac{\frac{1-c}{(1-\lambda_1^2)^2}+\frac{1}{\gamma^3}\frac{c}{(1-\lambda_2^2)^2}}{\frac{1}{2\lambda_1}\ln\frac{1+\lambda_1}{1-\lambda_1}(1-c)+\frac{1}{\gamma}\frac{1}{2\lambda_2}\ln\frac{1+\lambda_2}{1-\lambda_2}c},
\end{aligned} \tag{34}
$$

in which the logarithmic functions transform, as above, into power series for small values of $\lambda_1$ and $\lambda_2$. In the limit $\lambda_1,\ \lambda_2 \to 0$ the expressions of $D$ and $\bar{D}$ for the $\delta$-distribution function (discussed in the previous paragraph) are obtained. In figure 3 the dependence of $D$ and $\bar{D}$ on the concentration $c$ is represented for a fixed ratio $\gamma$ and different sets of values ($\lambda_1$,$\lambda_2$). All have the same qualitative behavior with the

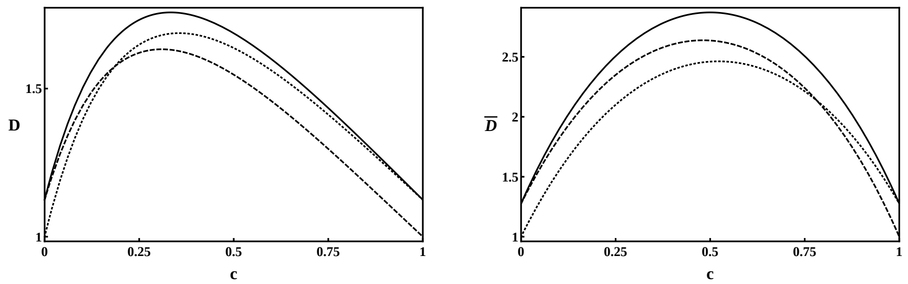


Fig. 3 – Dependence of $D$ (*left*) and $\bar{D}$ (*right*) on $c$ for different values of ($\lambda_1$,$\lambda_2$) for a given $\gamma = 2$: $\lambda_1 = 0.3$, $\lambda_2 = 0$ (*dashed curve*); $\lambda_1 = 0$, $\lambda_2 = 0.3$ (*dotted curve*); $\lambda_1 = \lambda_2 = 0.3$ (*full curve*).

concentration $c$ as presented in figure 3, but at the extremities ($c = 0$, $c = 1$) they are determined by the given set of ($\lambda_1, \lambda_2$) values.

## 4. CONCLUSIONS

The influence of the dust size distribution function on the coefficients of the KdV equation is determined analytically for simple expressions of the equilibrium distribution function. This effect is given by simple expressions depending only on mean values of various powers of the dust grain radius $r_d$ which are contained in the coefficients $D$ and $\bar{D}$. These coefficients are equal to unity for a mono-sized dusty plasma, but when the dust fluid is a mixture they are always greater than one. The consequences for the soliton properties are straightforward, namely, compared to the mono-sized case the velocity $u_0$ increases, the amplitude $\varphi_m$ is smaller and the pulse width $\Delta$ is greater. We can say that for a dusty plasma with multiple size dust grains the soliton properties are "*degraded*", in accordance with the results of previous works [19–30].

The previous discussion, based on a multiple scale analysis and leading to the KdV equation, is appropriate for the description of nonlinear DAWs of finite but small amplitudes. For large or arbitrary amplitude DAWs a different approach is needed. Looking for solitary wave solutions (all the quantities depend only on a simple variable $\zeta = x - Mt$, where $M$ is known as the Mach number) the equations (11), (13), (14) are easily integrated. One obtains [1], [33]

$$\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta}\right)^2 + V(\varphi) = 0, \tag{35}$$

where $V(\varphi)$ is the Sagdeev's potential, which in our case writes

$$V(\varphi) = \mu_i\left[1-\exp(-\varphi)\right] + \mu_e\theta\left[1-\exp(\varphi/\theta)\right] + M^2\left[1 - \frac{1}{\langle r_d^3\rangle}\int r_d^2\sqrt{r_d^2 + \frac{\langle r_d^3\rangle}{\langle r_d\rangle}\frac{2\varphi}{M^2}}\, f(r_d)\,\mathrm{d}r_d\right]. \tag{36}$$

For $f(r_d) = \delta(r_d - r_0)$ this transforms into the well-known result for mono-sized dusty plasma [1]

$$V(\varphi) = \mu_i\left[1-\exp(-\varphi)\right] + \mu_e\theta\left[1-\exp(\varphi/\theta)\right] + M^2\left[1-\sqrt{1+\frac{2\varphi}{M^2}}\right]. \tag{37}$$

In general, for various distribution functions $f(r_d)$, the integration in (36) is more difficult and details as well as a complete discussion of the results will be presented elsewhere.

Another extension of the results obtained in the present paper is to take into account the variation of the dust grain charge. As mentioned in *Introduction* there are many situations, especially in laboratory experiments, when the charging time of the dust grains is several orders of magnitude smaller than the hydrodynamic time characterizing the DAWs. Based on this observation, in all our discussion the charge on the dust grain was considered constant. A better approximation is to assume that during the slow movement of the DAWs the charge on the dust grain has enough time to readjust itself to the slowly varying potential $\varphi(x,t)$. Thus the "*local potential approximation*" (LPA) was used by us [34, 35] to study the influence of the charge variation on the coefficients of the KdV equation. In this approach the equation (8), valid for the equilibrium state, is extended and generalized by considering the local values of $n_i(x,t)$, $n_e(x,t)$, $Z_d(x,t;r_d)$. This extension, containing the supplemental influence of the dust grain charge variation for a dusty plasma with dust size distribution is under work and the results will be published elsewhere.

***Acknowledgements.*** Support through CNCSIS program IDEI-571/2008 is kindly acknowledged by one of the authors, A.T. Grecu.

REFERENCES


1. P. K. Shukla, A. A. Mamun, *Introduction to Dusty Plasma Physics* (IOP Publishing Ltd., Bristol, Philadelphia, 2002).
2. F. Verheest, *Waves in Dusty Space Plasmas* (Kluwe Acad. Press, Dordrecht, 2000).
3. P. K. Shukla, B. Eliasson, Reviews of Modern Physics **81**, 25 (2009).
4. V. E. Fortov, A. V. Ivlev, S. A. Khrapak, G. E. Morfill, Phys. Rep. **421**, 1–103 (2005).
5. P. K. Shukla, Phys. Plasmas **8**(5), 1791 (2001).
6. N. N. Rao, P. K. Shukla, M. Y. Yu, Planet. Space Sci. **38**(4), 543–546 (1990).
7. A. Barkan, R. L. Merlino, N. D'Angelo, Phys. Plasmas **2**(10), 3563 (1995).
8. E. J. Thomas, R. Fisher, R. L. Merlino, Phys. Plasmas **14**, 123701 (2007).
9. A. A. Mamun, R. A. Cairns, P. K. Shukla, Phys. Lett. A **373**, 2355–2359 (2009).
10. Y.-N. Nejoh, Phys. Plasmas **4**(8), 2813 (1997).
11. J. X. Ma, J. Liu, Phys. Plasmas **4**(2), 253–255 (1997).
12. B. S. Xie, K. F. He, Z. Q. Huang, Phys. Lett. A **247**, 403 (1997).
13. W. S. Duan, Chaos, Solitons and Fractals **14**, 503 (2003).
14. W. S. Duan, Chaos, Solitons and Fractals **14**, 1315 (2002).
15. W. S. Duan, Phys. Plasmas **8**, 3583 (2001).
16. S. I. Popel, M. Y. Yu, V. N. Tsytovich, Phys. Plasmas **3**, 4313 (1996).
17. A. V. Ivlev, G. Morfill, Phys. Rev. E **63**, 026412 (2001).
18. P. Meuris, Planet. Space Sci. **41**, 1171 (1997).
19. M. A. Raadu, IEEE Trans. Plasma Sci. **29**, 182 (2001).
20. A. Brattli, O. Havnes, F. Melandsø, J. Plasma Phys. **58**(4), 691–704 (1997).
21. W. S. Duan, Y. R. Shi, Chaos, Solitons and Fractals **18**, 321 (2003).
22. M. Shafiq, M. A. Raadu, Phys. Lett. A **305**, 79–86 (2002).
23. G. J. He, W. S. Duan, D. X. Tian, Phys. Plasmas **15**, 043702 (2008).
24. M. Ishak-Boushaki, S. Bahamida, R. Annou, Phys. Plasmas **10**(8), 3418 (2003).
25. M. M. Lin, S. W. Duan, Physica Scripta **79**, 320 (2007).
26. W. S. Duan, J. Parkes, Phys. Rev. E **68**, 067402 (2003).
27. M. M. Lin, W. S. Duan, Phys. Plasmas **11**, 5710 (2004).
28. S. K. El-Labany, F. M. Safi, W. M. Moslem, Planet. Space Sci. **55**, 2192 (2007).
29. V. V. Yaroshenko, G. Jacobs, F. Verheest, Phys. Rev. E **64**, 036401 (2001).
30. F. Verheest, V. V. Yaroshenko, G. Jacobs, P. Meuris, Physica Scripta **64**, 494 (2001).
31. J. E. Allen, Physica Scripta **45**, 497–503 (1992).
32. H. Washimi, T. Taniuti, Phys. Rev. Lett. **17**(19), 996–998 (1966).
33. A. A. Mamun, Astrophys. Space Sci. **268**, 443 (1999).
34. A. T. Grecu, D. Grecu, Rom. J. Phys. **53**, 1131 (2008).
35. A. T. Grecu, *Nonlinear Processes in Field Theory and Astrophysics* (Ph.D. Thesis, Bucharest, 2010).