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FARADAY WAVES IN BOSE-EINSTEIN CONDENSATES SUBJECT TO ANISOTROPIC TRANSVERSE CONFINEMENT

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Abstract. In this study we introduce a variational model for the dynamics of Faraday waves in longitudinally-inhomogeneous Bose-Einstein condensates subject to anisotropic transverse confinement and periodic modulations of the scattering length. We derive a set of variational equations under the assumption that the period of the Faraday wave is smaller than the longitudinal width of the condensate and determine analytically the dispersion relation for large condensates through a Mathieu-type analysis of the variational equations. Finally, we show the emergence of Faraday waves in a realistic anisotropic ^{87}Rb Bose-Einstein condensate and discuss possible extensions of the variational treatment to binary condensates.

Key words: Bose-Einstein condensates, variational treatment, Faraday waves.

1. INTRODUCTION

Following a series of inceptive theoretical investigations into extended parametric resonances in quantum gases [1–6], pattern formation has become a fashionable research topic after the experimental observation of Faraday waves in cigar-shaped Bose-Einstein condensates (BECs) [7] and vertically-vibrated ^{4}He cells [8], and the achievement of periodic modulations of the scattering length of a ^{7}Li BEC [9]. Surveying the recent literature for bosonic systems one notices the study of resonances and the nonlinear correction to the frequencies of the collective modes of a BEC [10], the path-integral formalism for the dynamics of bosonic systems subject to time-dependent potentials [11, 12], the properties of Faraday patterns in low-density [13], high-density [14] and dipolar BECs [15], the suppression of Faraday waves in BECs loaded onto shallow optical lattices [16], the parametric excitation of "scars" in BECs [17], the emergence of non-autonomous bright and dark solitons [18], and, quite interestingly, the removal of excitations in BECs subject to time-dependent periodic potentials [19]. Similar interest has been shown to two-component bosonic systems where the emergence of Faraday patterns has been mapped perturbatively to a set of coupled Mathieu equations [20] and sinusoidal excitations were shown to exist [21]. On the fermionic side, Faraday waves have been analyzed in detail in superfluid fermionic gases [22, 23] and it has been shown that the collective modes of 1D fermionic systems can be amplified by means of parametric resonances to the extent of observing a clear spin-charge separation [24].

Much of the appeal in investigating the nonlinear dynamics of BECs comes from the well-established accuracy of the Gross-Pitaevskii (GP) equation that governs the $T = 0$ K dynamics of the condensate [25]. For precise numerical computations the three-dimensional version of the GP equation is used [26, 27], while for analytic results it is usually more convenient to work with a simplified one- or two-dimensional version of the equation. For highly-elongated condensates such as the ones in Refs. [7, 9] one can use a wide range of effectively one-dimensional non-polynomial Schrödinger equations derived from the full GP equation after integrating out the radial dynamics (see Refs. [28–30] for the low-density regime and Refs. [14, 31, 32] for high-density condensates). Similar equations have been derived for pancake-shaped condensates by integrating out the longitudinal dynamics (see Refs. [28, 33] for the main results) and we expect in the near future related results for binary condensates. Despite the numerous equations available, pattern formation in non-polynomial Schrödinger equations is studied either numerically or perturbatively and there is very little insight on the interplay between the collective modes of the condensate and the dynamics of the excited pattern.

Another way of gaining analytical insight into the dynamics of BECs is through variational approaches. These techniques usually rely on a set of envelopes (Gaussian in the low-density regime [34–36], q-Gaussian in the high-density regime [37–40]) that describe the bulk of the condensate, to which various nonlinear waves can be grafted, depending on the specific problem under scrutiny. Variational methods have had notable success in describing the ground-state properties of BECs and the frequencies of the collective modes (see Ref. [25, ch. 6] for an overview), have been fruitfully used to control the dynamics of solitons [41, 42] and have been recently introduced to describe the emergence of surface waves in BECs subject to time-dependent modulations of the scattering length [43].

In this article we study by variational means the emergence of Faraday waves in longitudinally-inhomogeneous BECs subject to anisotropic transverse confinement and periodic modulations of the scattering length. To this end we derive a set of variational (ordinary differential) equations for the widths of the condensate and the components of the Faraday wave and determine analytically (for large condensates) the period of the excited wave through a Mathieu-type analysis. Solving numerically the variational equations we show the emergence of Faraday waves in a realistic highly-elongated ^{87}Rb BEC.

2. VARIATIONAL TREATMENT

We build the variational equations starting from the habitual three-dimensional GP Lagrangian density [36] (with $m = \hbar = 1$)

$$\mathcal{L}(\mathbf{r},t) = \frac{i}{2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) + \frac{1}{2}|\nabla\psi|^2 + V(\mathbf{r},t)|\psi|^2 + \frac{g(t)N}{2}|\psi|^4 \qquad (1)$$

considering a trapping potential of the form

$$V(\mathbf{r},t) = \frac{1}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2 + \Omega_z^2 z^2\right) \qquad (2)$$

and a hybrid trial wave function which includes a surface wave grafted onto a set of Gaussian envelopes that describe the bulk of the condensate, namely

$$\begin{aligned}\psi(\mathbf{r},t) \;=\;& f(\ldots)\exp\left(-\frac{x^2}{2w_x^2(t)} + ix^2\alpha_x(t)\right)\exp\left(-\frac{y^2}{2w_y^2(t)} + iy^2\alpha_y(t)\right)\\ &\times\exp\left(-\frac{z^2}{2w_z^2(t)} + iz^2\alpha_z(t)\right)(1 + (u(t) + iv(t))\cos kz)\,. \qquad (3)\end{aligned}$$

The function $f(\ldots)$ depends on k, the condensate's widths and their associated phases and is determined from the normalization condition $\int d\mathbf{r}\,|\psi|^2 = 1$. Our ansatz stems from the variational treatment of low-density BECs [35, 36] and is particularly well suited to describe the emergence of surface waves due to the exponential cut-off of the wave function away from the center of the cloud. The q-Gaussian envelopes provide better results as far as the collective modes are concerned [40], but they are of finite spatial extent and (leaving aside the problematic analytic tractability of the Lagrangian) one has spurious resonances between the longitudinal extent of the cloud and the period of the excited wave. Please notice that we are considering the general case of a condensate without cylindrical symmetry, as Ω_x is usually different from Ω_y.

Computing the Lagrangian $L(t) = \int d\mathbf{r}\mathcal{L}(\mathbf{r},t)$ is relatively straightforward but at the end of the calculation one finds that $L(t)$ contains terms proportional to $\exp\left(-k^2 w_z^2(t)/8\right)$ and higher powers of it. Assuming that the longitudinal width of the condensate is larger than the period of the excited wave we make a series expansion of the Lagrangian considering $\exp\left(-k^2 w_z^2(t)/8\right)$ as the small parameter and truncate the expansion to leading order. One can now easily compute the Lagrangian, which is, however, fairly complex and we do not show it. To first order in $u(t)$ and $v(t)$ the Euler-Lagrange equations can be cast as

$$\dot{w}_i(t) = 2w_i(t)\alpha_i(t), \tag{4}$$

$$\dot{\alpha}_i(t) = -\frac{\Omega_i^2}{2} + \frac{1}{2w_i^4(t)} + \frac{Ng(t)}{4\sqrt{2}\pi^{3/2}w_i^3(t)w_j(t)w_k(t)} - 2\alpha_i^2(t), \tag{5}$$

where $i \in \{x,y,z\}$, $j \in \{y,z,x\}$, $k \in \{z,x,y\}$, and

$$\dot{v}(t) = -u(t)\left(\frac{k^2}{2} + \frac{Ng(t)}{\sqrt{2}\pi^{3/2}w_x(t)w_y(t)w_z(t)}\right), \tag{6}$$

$$\dot{u}(t) = \frac{k^2}{2}v(t). \tag{7}$$

The main message conveyed by these equations is that for small-amplitude surface waves the collective dynamics of the condensate is left unperturbed by the surface wave, the correction being of the second order. The equations for the widths of the condensate can be written as

$$\ddot{w}_i(t) + \Omega_i^2 w_i(t) = \frac{1}{w_i^3(t)} + \frac{Ng(t)}{2\sqrt{2}\pi^{3/2}w_i^2(t)w_j(t)w_k(t)}, \tag{8}$$

where $i \in \{x,y,z\}$, $j \in \{y,z,x\}$ and $k \in \{z,x,y\}$. While these equations do not generally have simple analytic solutions, for large condensates the second term on the right hand side of equation (8) is considerably higher than the first, therefore outside of resonances the dynamics of the widths is well approximated by

$$w_x(t) \approx \left(\frac{Ng(t)\Omega_y}{2^3\pi^3\Omega_x^3 w_z(t)}\right)^{1/4}, \tag{9}$$

$$w_y(t) \approx \left(\frac{Ng(t)\Omega_x}{2^3\pi^3\Omega_y^3 w_z(t)}\right)^{1/4}. \tag{10}$$

In writing the previous two relations we have implicitly considered a highly elongated condensate whose longitudinal extent is roughly constant. The justification of this assumption will be seen in the next section, where we show that the emergence of the Faraday waves has a negligible impact on the longitudinal extent of the condensate. It is straightforward to check that the previous approximations yield

$$\dot{w}_z(t) = 2w_z(t)\alpha_z(t) \tag{11}$$

and

$$\dot{\alpha}_z(t) = -\frac{\Omega_z^2}{2} + \frac{1}{w_z^4(t)} + \frac{\sqrt{Ng(t)\Omega_x\Omega_y}}{2(2\pi)^{3/4}w_z^{5/2}(t)}. \tag{12}$$

Modulating $g(t)$ as $g(t) = g \cdot (1 + \epsilon \sin(\omega t))$ the dynamics of the surface wave is given by

$$\ddot{u}(t)+u(t) \quad \left[\frac{k^2}{\omega^2}+\frac{2^{5/4}k^2\sqrt{Ng(1+\epsilon\sin(2t)\Omega_x\Omega_y)}}{\omega^2\pi^{3/4}\sqrt{w_z(t)}}\right] = \quad 0. \tag{13}$$

Considering the longitudinal width of the condensate to be constant, the previous equation reduces to a Mathieu equation for small values of ϵ, namely

$$\ddot{u}(t)+u(t)(a(k,\omega)+\epsilon b(k,\omega)\sin(2t)) = 0, \tag{14}$$

where $a(k,\omega)$ and $b(k,\omega)$ yield from the series expansion of the function under the square root. The waves observed experimentally correspond to *the most unstable solutions* of equation (14), and for positive values of $b(k,\omega)$ the dispersion relation of these solutions is given by $a(k,\omega) = 1$ [44], namely

$$k = \frac{2^{1/8}}{\pi^{3/8}w_z^*}\left[\sqrt{Ng\Omega_x\Omega_y+\pi^{3/2}\omega^2 w_z^*}-\sqrt{Ng\Omega_x\Omega_y}\right]^{1/2}, \tag{15}$$

where w_z^* is the longitudinal width of the condensate at $t = 0$. For small values of ϵ one can easily check that equation (14) has solutions of the form $\sin(\sqrt{a}t)$ and $\cos(\sqrt{a}t)$, therefore the most unstable solutions have a frequency half that of the parametric drive and are usually referred to as Faraday waves in honor of Faraday's classic study on the behavior of "fluids in contact with vibrating surfaces" [45]. The dynamics of the system becomes highly nontrivial for ω in a vicinity of either Ω_x or Ω_y, as these are the main resonant frequencies of the condensate and approximations (9) and (10) break down. A similar behavior is expected for ω in a vicinity of $2\Omega_x/n^2$ and $2\Omega_y/n^2$, where n is an integer, though these resonances become weaker for larger values of n and our variational equations should predict results close to those obtained from the full numerical simulations.

3. RESULTS AND DISCUSSIONS

We consider a realistic ^{87}Rb condensate of $N = 5\cdot 10^6$ atoms loaded into a magnetic trap with $\{\Omega_x,\Omega_y,\Omega_z\} = \{200(2\pi)\text{ Hz}, 100(2\pi)\text{ Hz}, 7(2\pi)\text{ Hz}\}$ and a parametric drive of $\omega = 250(2\pi)$ Hz and $\epsilon = 0.3$. Setting $\dot{\alpha}_z(t) = \alpha_z(t) = 0$ in equation (12) one finds that the equilibrium value of the longitudinal width is around 47 μm, while equation (15) indicates that k is close to $38\cdot 10^4$ m^{-1} (which is consistent with the experimental results in Ref. [7] for cylindrically symmetric condensates).

As $\exp\left(-k^2w_z^2/8\right)$ is very close to 0, our series expansion of the Lagrangian is valid and one can safely use the variational equations. We determine the equilibrium widths of the condensate by solving equations (5) with $\dot{\alpha}_i(t) = \alpha_i(t) = 0$, where $i \in \{x,y,z\}$, using a Newton-Raphson method [46]. To see the emergence of the Faraday waves we solve equations (4)-(7) using a 4-5 embedded Runge-Kutta method

[47] with k computed from (15). As the full three-dimensional density profile can not be easily represented due to the exponential cut-off of the wave function far from the center of the condensate, we look at

$$\phi(y,z,t) = \int_{-\infty}^{\infty} dx \, |\psi(x,y,z,t)|^2 \,. \tag{16}$$

In Figs. 1, 2 and 3 we plot the $\phi(y,z,t)$ density profile at $t_1 = 100$ ms (Fig. 1),

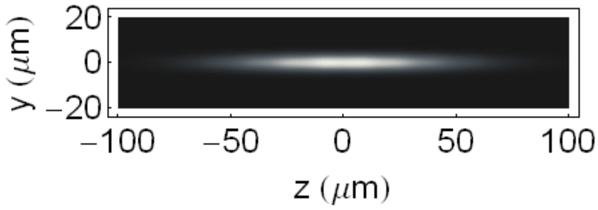

Fig. 1 – Integrated density profile of a ^{87}Rb condensate with $N = 5 \cdot 10^6$ atoms confined by a trap with $\{\Omega_x, \Omega_y, \Omega_z\} = \{200(2\pi)\ \text{Hz}, 100(2\pi)\ \text{Hz}, 7(2\pi)\ \text{Hz}\}$ and subject to a parametric drive $g(t) = g \cdot (1 + \epsilon \sin(wt))$, where $\omega = 250(2\pi)$ Hz. The snapshot is taken at $t = 100$ ms. The bright regions of the density profile are the ones with the most number of atoms.

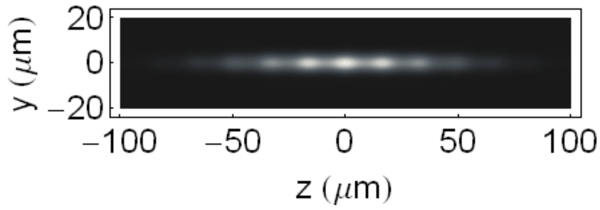

Fig. 2 – Integrated density profile of a ^{87}Rb condensate at $t = 200$ ms. Same setting as in Fig. 1. The bright regions of the density profile are the ones with the most number of atoms.

$t_2 = 200$ ms (Fig. 2) and $t_3 = 250$ ms (Fig. 3), while in Fig. 4 we show the transverse density profile of the condensate at $z = 0$. The longitudinal extent of the condensate remains constant throughout the simulation at about 200 µm, which justifies our assumption in the previous section. The transverse extent of the condensate is about 10 µm on the y axis and 5 µm on the x axis, which is consistent with the trapping frequencies. The Faraday waves emerge relatively slowly (when compared to existing experimental data for cylindrically symmetric condensates) mainly because of the Gaussian transverse ansatz which only allows for a dynamical rescaling of the initial transverse density profile. This delayed emergence of the Faraday waves has also been observed in the non-polynomial Schrödinger equations that rely on a Gaussian radial ansatz [13].

Summarizing, we have investigated by variational means the emergence of

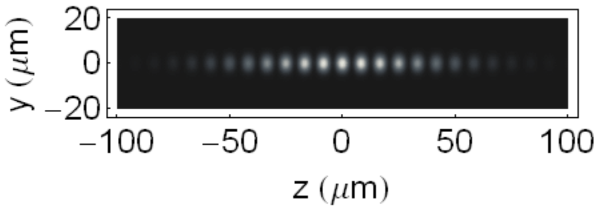

Fig. 3 – Integrated density profile of a ^{87}Rb condensate at $t = 250$ ms. Same setting as in Fig. 1. The bright regions of the density profile are the ones with the most number of atoms.

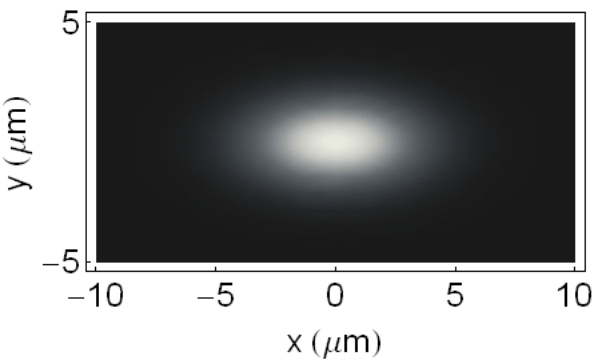

Fig. 4 – Transverse density profile of a ^{87}Rb condensate at $t = 200$ ms and $z = 0$. Same setting as in Fig. 1. The bright regions of the density profile are the ones with the most number of atoms.

Faraday waves in BECs subject to anisotropic transverse confinement and periodic modulations of the scattering length. We have derived analytically the dispersion relation for dense, highly-elongated condensates and have shown the time evolution of the density profile of a realistic ^{87}Rb condensate similar to that in Ref. [7]. Our results could be easily extended to binary non-miscible condensates subject to anisotropic transverse confinement, as in this setting the individual wave functions are well-localized and have minimal overlap. Extending these results to miscible binary condensates is, however, not transparent at the moment, as miscible condensates exhibit symbiotic pairs of bright-dark solitonic ground states. In any case, the interplay between the Faraday waves in the two components of the condensate is an open research topic and non-miscible binary systems seem like the ideal playground to investigate the subject.

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