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ON THE ASYMPTOTIC BEHAVIOUR OF SOME ELLIPTIC PROBLEMS IN PERFORATED DOMAINS*

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Abstract. The asymptotic behavior of the solution of a class of elliptic problems modeling diffusion in some periodic perforated media is analyzed. We consider, at the microscale, an elliptic equation with various nonlinear conditions prescribed on the boundary of the perforations and we prove that the effective behavior of the solution of such a problem is governed by another elliptic equation, which, depending on the type of the conditions imposed on the surface of the cavities, can contain extra zero-order terms.

Key words: homogenization, perforated domains, nonlinear boundary conditions.

1. INTRODUCTION

The goal of this paper is to analyze the asymptotic behavior of some nonlinear diffusion problems in a periodically perforated medium.

Let Ω be an open bounded set in $\mathbb{R}^n$. If we perforate it by holes, we obtain an open set Ω^ε, the *perforated domain,* ε representing a small parameter related to the characteristic size of the perforations. We shall be concerned with the case in which the perforations are identical and periodically distributed and their size is of the order of ε.

In such a domain, we shall analyze the effective behavior of the solution of the following problem:

$$\begin{cases} -K\Delta u^\varepsilon + \beta(x, u^\varepsilon) = f \quad \text{in} \quad \Omega^\varepsilon, \\ -K\dfrac{\partial u^\varepsilon}{\partial \nu} = \varepsilon^{1+a} \quad g(x, u^\varepsilon) + \varepsilon^{1+b} h(\dfrac{x}{\varepsilon}) \quad \text{on} \quad S^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

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where ν is the exterior unit normal to Ω^ε, $a,\ b \geq 0,\ f \in L^2(\Omega)$, S^ε is the boundary of the holes, $\partial\Omega$ is the external boundary of Ω and K is a constant diffusion coefficient. We can consider that Ω is a periodic structure, consisting of two parts: a fluid phase Ω^ε and a solid skeleton (grains), $\Omega \setminus \overline{\Omega^\varepsilon}$, ε representing a small parameter related to the characteristic size of the grains. The nonlinear problem (1)–(3) describes the stationary flow of a fluid confined in Ω^ε, of concentration u^ε, reacting inside Ω^ε and, also, on the boundary of the grains.

We shall consider that the functions β, g and h in (1) are given (see, for details, Section 2).

The existence and uniqueness of a weak solution of (1) is given by the classical theory of semilinear monotone problems (see [1], [6] and [9]).

If $a = b = 0$, we shall prove that the solution u^ε, properly extended to the whole of Ω, converges weakly in $H_0^1(\Omega)$ to the unique solution of the following homogenized problem:

$$\begin{cases} -\sum_{\mathrm{i,j=1}}^{n} q_{ij} \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \dfrac{|\partial T|}{|Y^*|} g(x,u) + \dfrac{|\partial T|}{|Y^*|} M_{\partial T}(h) + \beta(x,u) = f \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{3}$$

Here, $Q = ((q_{ij}))$ is the standard homogenized matrix, whose entries are defined as follows:

$$q_{ij} = \delta_{ij} + \frac{1}{|Y^*|} \int_{Y^*} \frac{\partial \chi_j}{\partial y_i} \mathrm{d}y, \tag{4}$$

in terms of the functions χ_j, solutions of the so-called cell problems

$$\begin{cases} -\Delta\chi_i = 0 \ \text{ in } \ Y^*, \\ \dfrac{\partial(\chi_i + y_i)}{\partial\nu} = 0 \ \text{ on } \ \partial T, \\ \chi_i \ \ Y-\text{periodic}. \end{cases} \tag{5}$$

In (3), T is the elementary hole, $Y^* = Y \setminus \overline{T}$, Y is the representative cell and $M_{\partial T}(h)$ is the mean value of h over ∂T (see Section 2).

If $a,\ b > 0$, then the solution u^ε, extended to the whole of Ω, converges weakly in $H_0^1(\Omega)$ to the unique solution of the problem:

$$\begin{cases} -\sum_{\mathrm{i,j=1}}^{n} q_{ij} \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \beta(x,u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6}$$

where the effective matrix is given again by (4)-(5).

Let us notice that in (1) we can replace $-K\Delta u^{\varepsilon}$ by a strong elliptic operator $-\mathrm{div}(A^{\varepsilon}\nabla u^{\varepsilon})$. So, we can consider the case of a general medium, having discontinuous properties, represented by a coercive periodic matrix with rapidly oscillating coefficients. The matrix A^{ε} is defined in terms of a symmetric matrix $A \in L^{\infty}_{\#}(\Omega)^{n\times n}$, whose entries are Y-periodic, bounded and measurable real functions. Here, we have used the symbol $\#$ to denote periodicity properties. We assume that there exist $0 < \delta < \gamma$ such that

$$\delta\left|\zeta\right|^2 \le A(y)\zeta \cdot \zeta \le \gamma\left|\zeta\right|^2 ,$$

for any $\zeta, y \in \mathbb{R}^n$. We denote by $A^{\varepsilon}(x)$ the value of $A(y)$ at the point $y = x/\varepsilon$.

In this case, the solution u^{ε} converges weakly in $H_0^1(\Omega)$ to the unique solution of the following homogenized problem:

$$\begin{cases} -\sum_{\mathrm{i,j=1}}^{n} q_{ij}^0 \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \dfrac{\left|\partial T\right|}{\left|Y^*\right|} g(x,u) + \dfrac{\left|\partial T\right|}{\left|Y^*\right|} M_{\partial T}(h) + \beta(x,u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where, this time, the effective matrix $Q^0 = ((q_{ij}^0))$ is given by:

$$q_{ij}^0 = \frac{1}{\left|Y^*\right|}\int_{Y^*} (a_{ij}(y) + a_{ik}(y)\frac{\partial \chi_j}{\partial y_k})\mathrm{d}y,$$

in terms of the functions χ_j, solutions of the cell problems

$$\begin{cases} -\mathrm{div}_y A(y)(D_y\chi_j + e_j) = 0 \text{ in } Y^*, \\ A(y)(D_y\chi_j + e_j)\cdot \nu = 0 \text{ on } \partial T, \\ \chi_j \in H^1_{\#Y}(Y^*), \ \int_{Y^*} \chi_j = 0, \end{cases}$$

where e_j are the elements of the canonical basis in $\mathbb{R}^n$.

Also, it is to be noticed that we can also treat the case in which the nonlinear term on the boundary of the reactive obstacles is described by various maximal monotone graphs, the limit behavior of such problems being governed by variational inequalities. We can deal, for instance, with the case of a single-valued maximal monotone graph with $g(x,0)=0$, *i.e.* with the case in which g is the subdifferential of a convex lower semicontinuous function G. We can treat in a similar manner the case of a multi-valued maximal monotone graph g, which includes interesting semilinear boundary-value problems, such as Signorini's problem ([7–9]).

Similar problems have been considered by many authors. We mention only [3, 4, 5].

The results of this paper constitute a generalization of some of the results obtained in [9], by considering space-dependent nonlinear terms, both in the interior of the domain and on its walls, and also, by assuming various scalings in the conditions imposed on the boundaries of the obstacles. The appearance of nonlinear terms on the surface of the obstacles creates more difficulties in passing to the limit, since we are dealing with highly oscillating boundaries. Let us mention that numerical simulations will be performed in a forthcoming paper.

The structure of our paper is the following one: first, let us mention that we shall just focus on the case $n \geq 3$, the case $n=2$ being similar. In Section 2, we introduce some useful notation and assumptions and we give the main result. In Section 3, we give the proof of the main convergence result of this paper.

Finally, notice that throughout the paper, by C we shall denote a generic fixed strictly positive constant, whose value can change from line to line.

2. PRELIMINARIES AND THE MAIN RESULT

Let Ω be a bounded connected open set in $\mathbb{R}^n$,with boundary $\partial\Omega$ of class C^2. Let $Y=[0,l_1[\times\{0,l_2[\times\ldots\times[0,l_n[$ be the representative cell in $\mathbb{R}^n$ and T, the elementary hole, be an open subset of Y with boundary ∂T of class C^2, such that $\overline{T}\subset Y$. We shall denote by $T^{\varepsilon,k}$ the translated image of εT by εkl, $k\in\mathbb{Z}^n$. Also, we shall denote by T^ε the set of all the holes contained in Ω and by $\Omega^\varepsilon=\Omega\setminus\overline{T^\varepsilon}$. Hence, Ω^ε is a periodically perforated domain with holes of the same size as the period. We shall also use the following notation:

$$Y^*=Y\setminus\overline{T},\ \ S^\varepsilon=\partial T^\varepsilon,\ \ \theta=\frac{|Y^*|}{|Y|}.$$

Also, we shall denote by χ^ε the characteristic function of the domain Ω^ε.

As already mentioned, we are interested in studying the behavior of the solution, in such a perforated domain, of problem (1).

We shall consider that the functions β, g and h in (1) are given. More precisely, we assume that β is continuous, monotonously non-decreasing and such that $\beta(x,0)=0,$ for any $x\in\Omega$ and the function g is continuously differentiable, monotonously non-decreasing, with $g(x,0)=0,$ for any $x\in\Omega$. The function h is supposed to be Y - periodic in $L^2(\partial T)$. Moreover, its mean value

$$M_{\partial T}(h)=\frac{1}{|\partial T|}\int_{\partial T} h(y)\mathrm{d}\sigma.$$

is assumed to be different from zero.

We shall also suppose that there exist a positive constant C and two exponents q and r such that

$$|\beta(x,\ v|\le C(1+|v|^q), \tag{7}$$

$$\left|\frac{\partial g}{\partial v}\right|\le C(1+|v|^q), \tag{8a}$$

$$\left|\frac{\partial g}{\partial x_i}\right|\le C(1+|v|^r),\ 1\le i\le n, \tag{8b}$$

with $0<q<n/(n-2)$ if $n\ge 3$ and $0<q<\infty$ if $n=2$ and with $0<r<n/(n-2)+q$ if $n\ge 3$ and $0<r<\infty$ if $n=2$.

As important examples of such functions, we can take

$$g(x,v)=\frac{\delta(x)v}{1+\gamma(x)v}\ \text{(Langmuir kinetics)},$$

with δ and γ positive smooth functions with bounded first derivatives, and

$$\beta(x,v)=\psi(x)|v|^{p-1}v\quad 0<p<1\ \text{(Freundlich kinetics)},$$

with ψ a positive smooth bounded given function.

Let us introduce the functional space

$$V^{\varepsilon}=\{\ v\in H^1(\Omega^{\varepsilon})\ \mid\ v=0\ \text{ on }\ \partial\Omega\},$$

with

$$\|v\|_{V^{\varepsilon}}=\|\nabla v\|_{L^2(\Omega^{\varepsilon})}.$$

The weak formulation of problem (1–3) is

$$\text{Find } u^\varepsilon \in V^\varepsilon \text{ such that}$$

$$K\int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla\varphi \mathrm{d}x + \varepsilon^{1+a}\int_{S^\varepsilon} g(x,u^\varepsilon)\varphi \mathrm{d}\sigma + \varepsilon^{1+b}\int_{S^\varepsilon} h(\frac{x}{\varepsilon})\varphi \mathrm{d}\sigma \tag{9}$$
$$+\int_{\Omega^\varepsilon} \beta(x,u^\varepsilon)\varphi \mathrm{d}x = \int_{\Omega^\varepsilon} f\varphi \mathrm{d}x,\ \forall\varphi \in V^\varepsilon.$$

By classical existence results (see [1]), there exists a unique weak solution of problem (9).

In order to formulate our main convergence theorem, let us recall the following well-known results ([2–3]):

LEMMA 2.1. *There exists a linear continuous extension operator*

$$P^\varepsilon \in L(L^2(\Omega^\varepsilon); L^2(\Omega)) \cap L(V^\varepsilon; H_0^1(\Omega))$$

$$\|P^\varepsilon v\|_{L^2(\Omega)} \le C\|v\|_{L^2(\Omega^\varepsilon)}$$

and

$$\|\nabla P^\varepsilon v\|_{L^2(\Omega)} \le C\|\nabla v\|_{L^2(\Omega^\varepsilon)},$$

for any $v \in V^\varepsilon$.

LEMMA 2.2. *There exists a positive constant* C , *independent of* ε , *such that*

$$\|v\|_{L^2(\Omega^\varepsilon)} \le C\|\nabla v\|_{L^2(\Omega^\varepsilon)},\ \text{for any } v \in V^\varepsilon.$$

The main result of this paper is the following one:

THEOREM 2.3. *Let* u^ε *be the unique solution of the problem* (9). *If* $a = b = 0$, *then there exists an extension* $P^\varepsilon u^\varepsilon$ of u^ε *into all* Ω *such that* $P^\varepsilon u^\varepsilon \to u$ *weakly in* $H_0^1(\Omega)$ *and* u *is the unique solution of*:

$$\begin{cases} -\sum_{\mathrm{i,j=1}}^{n} q_{ij} \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \dfrac{|\partial T|}{|Y^*|} g(x,u) + \dfrac{|\partial T|}{|Y^*|} M_{\partial T}(h) + \beta(x,u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{10}$$

Here, $Q = ((q_{ij}))$ is the standard homogenized matrix, defined by (4–5).

3. PROOF OF THE MAIN RESULT

Proof of THEOREM 2.3. Let u^ε be the solution of the variational problem (9) and let $P^\varepsilon u^\varepsilon$ be the extension given by Lemma 2.1. Taking $\varphi = u^\varepsilon$ as a test function in (9), it is not difficult to see that $P^\varepsilon u^\varepsilon$ is bounded in $H_0^1(\Omega)$. So, by extracting a subsequence, one can assume that there exists $u \in H_0^1(\Omega)$ such that

$$P^\varepsilon u^\varepsilon \xrightarrow{w} u \quad \text{weakly in} \quad H_0^1(\Omega). \tag{11}$$

In order to to identify the limit equation satisfied by u, we have to pass to the limit in (9). We shall start with the case $a = b = 0$.

Let us introduce (see [4]), for any $k \in L^p(\partial T)$, $1 \le p \le \infty$, the linear form μ_k^ε on $W_0^{1,q}(\Omega)$ defined by

$$\langle \mu_k^\varepsilon, \varphi \rangle = \varepsilon \int_{S^\varepsilon} k(\frac{x}{\varepsilon})\varphi \mathrm{d}\sigma, \;\; \forall \varphi \in W_0^{1,q}(\Omega),$$

with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. From [4], we know that

$$\mu_k^\varepsilon \to \mu_k \quad \text{strongly in} \quad (W_0^{1,q}(\Omega))', \tag{12}$$

where

$$\langle \mu_k, \varphi \rangle = \mu_k \int_\Omega \varphi \mathrm{d}x,$$

with

$$\mu_k = \frac{1}{|Y|} \int_{\partial T} k(y) \mathrm{d}\sigma.$$

Moreover, if k is constant, we have

$$\mu_k^\varepsilon \to \mu_k \tag{13}$$

strongly in $W_0^{-1,\infty}(\Omega)$ and we shall denote by μ^ε the above introduced measure in the particular case in which $k = 1$.

On the other hand, let us notice that, exactly like in [5] and [9], one can prove that for any $\varphi \in C_0^\infty(\Omega)$ and for any $z^\varepsilon \to z$ weakly in $H_0^1(\Omega)$, we get

$$\begin{aligned} &\varphi g(x,z^{\varepsilon}) \to \varphi g(x,z) \ \text{ strongly in } \ L^{\bar{q}}(\Omega), \\ &\varphi\beta(x,z^{\varepsilon}) \to \varphi\beta(x,z) \ \text{ strongly in } \ L^{\bar{q}}(\Omega), \end{aligned} \tag{14}$$

where

$$\bar{q} = \frac{2n}{q(n-2)+n}.$$

From (13) (with $k=1$) and (14) written for $z^{\varepsilon} = P^{\varepsilon}u^{\varepsilon}$, we obtain

$$\left\langle \mu_k^{\varepsilon}, \varphi g(x, P^{\varepsilon}u^{\varepsilon}) \right\rangle \to \frac{|\partial T|}{|Y|} \int_{\Omega} \varphi g(x,u)\mathrm{d}x, \ \forall \varphi \in C_0^{\infty}(\Omega). \tag{15}$$

Let $\varphi \in C_0^{\infty}(\Omega)$. From (9), we get

$$\begin{aligned} &K\int_{\Omega^{\varepsilon}} \nabla u^{\varepsilon} \cdot \nabla\varphi \mathrm{d}x + \varepsilon\int_{S^{\varepsilon}} g(x,u^{\varepsilon})\varphi\mathrm{d}x + \varepsilon\int_{S^{\varepsilon}} h\left(\frac{x}{\varepsilon}\right)\varphi\mathrm{d}x + \\ &+\int_{\Omega^{\varepsilon}} \beta(x, P^{\varepsilon}u^{\varepsilon})\varphi\mathrm{d}x = \int_{\Omega^{\varepsilon}} f\varphi\mathrm{d}x. \end{aligned} \tag{16}$$

Now, we can pass to the limit, with $\varepsilon \to 0$, in all the terms of (16). For the first one, we have

$$\lim_{\varepsilon\to 0} K\int_{\Omega^{\varepsilon}} \nabla u^{\varepsilon} \cdot \nabla\varphi\mathrm{d}x = \frac{|Y^*|}{|Y|}\int_{\Omega} Q\nabla u \cdot \nabla\varphi\mathrm{d}x. \tag{17}$$

For the second term, using (15), we get

$$\lim_{\varepsilon\to 0} \varepsilon\int_{S^{\varepsilon}} g(x,u^{\varepsilon})\varphi\mathrm{d}x = \frac{|\partial T|}{|Y|}\int_{\Omega} g(x,u)\varphi\mathrm{d}x. \tag{18}$$

For the third one, using (12), we obtain

$$\lim_{\varepsilon\to 0} \varepsilon\int_{S^{\varepsilon}} h(\frac{x}{\varepsilon})\varphi\mathrm{d}x = \frac{|\partial T|}{|Y|} M_{\partial T}(h)\int_{\Omega} \varphi\mathrm{d}x. \tag{19}$$

On the other hand, we know that $\chi_{\Omega^{\varepsilon}} \to \frac{|Y^*|}{|Y|}$ weakly in any $L^{\sigma}(\Omega)$ with $\sigma \geq 1$. In particular, defining q^* such that

$$\frac{1}{\bar{q}} + \frac{1}{q^*} = 1,$$

we see that $q^* \geq 1$ and, consequently,

$$\chi_{\Omega^\varepsilon} \to \frac{|Y^*|}{|Y|} \text{ weakly in } L^{q^*}(\Omega).$$

Hence, we obtain:

$$\lim_{\varepsilon\to 0}\int_\Omega \chi_{\Omega^\varepsilon}\beta(x,u^\varepsilon)\varphi \mathrm{d}x = \frac{|Y^*|}{|Y|}\int_\Omega \beta(x,u)\varphi \mathrm{d}x. \tag{20}$$

It is not difficult to pass to the limit in the right-hand side of (16). Indeed, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \chi_{\Omega^\varepsilon} f\varphi \mathrm{d}x = \frac{|Y^*|}{|Y|}\int_\Omega f\varphi \mathrm{d}x. \tag{21}$$

Putting together (17–21), we have

$$\frac{|Y^*|}{|Y|}\int_\Omega Q\nabla u\cdot\nabla\varphi \mathrm{d}x + \frac{|\partial T|}{|Y|}\int_\Omega g(x,u)\varphi \mathrm{d}x + \frac{|\partial T|}{|Y|}M_{\partial T}(h)\int_\Omega \varphi \mathrm{d}x +$$
$$+\frac{|Y^*|}{|Y|}\int_\Omega \beta(x,u)\varphi \mathrm{d}x = \frac{|Y^*|}{|Y|}\int_\Omega f\varphi \mathrm{d}x, \quad \forall\varphi\in C_0^\infty(\Omega).$$

Hence,

$$-\frac{|Y^*|}{|Y|}\mathrm{div}(Q\nabla u) + \frac{|\partial T|}{|Y|}g(x,u) + \frac{|\partial T|}{|Y|}M_{\partial T}(h) + \frac{|Y^*|}{|Y|}\beta(x,u) = \frac{|Y^*|}{|Y|}f \text{ in } \Omega.$$

Therefore, we get exactly the limit equation (3). Since $u\in H_0^1(\Omega)$ (*i.e.* $u=0$ on $\partial\Omega$) and u is uniquely determined, the whole sequence $P^\varepsilon u^\varepsilon$ converges and Theorem 2.3. is proved.

If $a,\ b>0$, then it is easy to see that the terms containing g and h disappear (they converge to zero when $\varepsilon\to 0$), and, therefore, the limit behavior of the solution of problem (1) is governed in this case by the problem (6).

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