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# ON THE PROBLEM OF THE TWO-BODIES OF VARIABLE MASS

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*Abstract*. In this paper we intend to present an analytical solution of the problem which refers to the increase of the major semi-axis of the planetary orbits in the case of the two-bodies in the second-order of approximation. The obtained results are compared with those reported by Vescan [9] for the first-order approximation and with those obtained numerically from the established basic equations.

*Key words*: gravitation-stars: variable mass-mass loss-celestial mechanics-two body problem; methods: analytical and numerical results.

## 1. INTRODUCTION

Many studies were done in the 20th century on celestial mechanics which lead to systems of differential equations that model the two-body problem in the case of variable mass. An example is the problem studied by the Romanian physicist Vescan [9] who studied in his PhD thesis the influence of variation of solar mass (as a result of solar radiation) about the major semi-axis of the planetary orbits. This interesting problem is an astronomical application of the two-body problem in the case of variable mass, a problem which was studied for the first time by the Swedish astronomer Gylden [1].

Problems connected by the motion of celestial bodies with variable mass, which refer to the decrease of the mass as a result of corpuscular star radiation have had a great importance in the last several decades. In these studies it is usually considered that the law of mass variation of a star (particularly of the Sun) is of the following form

$$\frac{dM}{dt} = -\alpha M^n, \tag{1}$$

where $\alpha$ is a constant and $n$ is a positive real number. We notice that the case $n = 1$ corresponds to the law of mass variation used by Vescan [9] and the case $n = 2$ was studied by Meshcherskii [5]. Many papers which have studied the laws of mass variation of the central body and the problem of the two-bodies with decreasing mass can be found in the literature, such as, for example, by Jeans [4], Hadjidemetriou [2, 3], Schröder and Smith [8]. Some refined analytical solutions are given in the paper Pal *et al.* [7].

In this article we will use the linear expression of the mass variation to establish the differential equation of the trajectory, the form of planetary orbits and the increase of the major semi-axis in the approximation of the second-order. The obtained analytical solution is compared with the results reported by Vescan [9] and with those resulted from the integration of equations using numerical methods. The obtained results show a very good agreement between them.

## 2. THE BASIC EQUATIONS

By analogy with the radioactivity, Vescan [9] has considered the law of variation of mass solar of the form

$$\mu = \mu_0 e^{-qt}, \tag{2}$$

where $q$ is a very small constant, $t$ is the time and $\mu_0$ is the mass of the Sun at the initial time (initial moment) $t_0$. By development in series and limiting to the first two terms, it results in the following linear expression

$$e^{-qt} = 1 - qt. \tag{3}$$

Then, the Newton's law of attraction between the Sun and the planet of mass $m$ can be written as

$$F = -k\frac{m\mu_0}{r^2}(1 - qt). \tag{4}$$

In order to obtain the equation of the planetary orbit, Vescan [9] has used Picard's successive approximation method and obtained the following equation of the trajectory

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}\left(1 + \frac{qa^2\theta}{\sqrt{ka\mu_0}}\right). \tag{5}$$

This equation shows that the trajectory of one planet has the form of an elliptical spiral or an increasing rosette. The increase of the major semi-axis during the revolution $\Delta a_V$ obtained by Vescan [9] is

$$\Delta a_v = \frac{2\pi q a^3}{\sqrt{ka\mu_0}}. \tag{6}$$

## 3. DIFFERENTIAL EQUATION OF THE TRAJECTORY

Because the attraction force is a central one, we have

$$r^2\dot{\theta} = C, \tag{7}$$

where $C$ is the constant area which has the expression

$$C = \sqrt{ka\mu_0(1-e^2)}, \tag{8}$$

Then, the Newton's law (4) becomes

$$F = -\frac{km\mu_0}{r^2}\left[1 - \frac{q}{c}\int_0^\theta r^2(x)\,dx\right], \tag{9}$$

where

$$r(\theta) = \frac{b^2}{a(1+e\cos\theta)} \tag{10}$$

is an even function. The expression of $r^2(\theta)$ developed in a Fourier trigonometrically series is given by

$$r^2 = \frac{b^4}{a^2}(1+e\cos\theta)^{-2} = \frac{b^4}{a^2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+e\cos x)^{-2}dx + \sum_{n=1}^{\infty}\frac{\cos n\theta}{\pi}\int_{-\pi}^{\pi}(1+ +e\cos x)^{-2}\cos nx\,dx\right]. \tag{11}$$

For the approximation of the second order ( $n = 2$ ), we have

$$\int_{-\pi}^{\pi}(1+e\cos x)^{-2}\,dx = \frac{2\pi}{(1-e^2)^{\frac{3}{2}}},$$

$$\int_{-\pi}^{\pi}\frac{\cos x}{(1+e\cos x)^2}\,dx = -\frac{2e\pi}{(1-e^2)^{\frac{3}{2}}},$$

$$\int_{-\pi}^{\pi}\frac{\cos 2x}{(1+e\cos x)^2}\,dx = \frac{2\pi\left[2(1-e^2)^{\frac{3}{2}}+3e^2-2\right]}{(1-e^2)^{\frac{3}{2}}e^2}. \tag{12}$$

On the other hand, from (9) we get

$$\int_0^\theta r^2\,d\theta = \frac{b^4}{a^2(1-e^2)^{\frac{3}{2}}}\left[\theta - 2e\sin\theta + \frac{2(1-e^2)^{\frac{3}{2}}+3e^2-2}{e^2}\sin 2\theta\right]. \tag{13}$$

Using Binet's formula

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{F\,r^2}{m\,C^2} \tag{14}$$

together with the equations (9) and (13) it obtains

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{k\mu_0}{C^2}\left[1-\frac{qb^4}{C\,a^2(1-e^2)^{\frac{3}{2}}}\left(\theta-2e\sin\theta+\frac{2(1-e^2)^{\frac{3}{2}}+3e^2-2}{e^2}\sin 2\theta\right)\right], \quad (15)$$

and it represents the differential equation of trajectory.

## 4. THE FORM OF THE PLANETARY ORBITS AND THE EXTENSION OF THE MAJOR SEMI-AXIS

In order to solve equation (15) we look for a solution of the form

$$\frac{1}{r}=\frac{k\mu_0}{C^2}(1+e\cos\theta)\,\psi(\theta) \quad (16)$$

where $\psi(\theta)$ is a correction factor. Using (15) and (16) it is found that the function $\psi(\theta)$ must satisfy the ordinary differential equation

$$(1+e\cos\theta)\psi''-2e\sin\theta\psi'+\psi=1-\frac{qb^4}{C\,a^2(1-e^2)^{\frac{3}{2}}}\left(\theta-2e\sin\theta+\frac{2(1-e^2)^{\frac{3}{2}}+3e^2-2}{e^2}\sin 2\theta\right), \quad (17)$$

where the prime denotes differentiation with respect to $\theta$. Further, using the form of equation (17), we assume that its solution is given by

$$\psi=M+\alpha\theta+\frac{\beta}{2}\theta^2+\frac{\gamma}{3}\theta^3, \quad (18)$$

where M, $\alpha, \beta$ and $\gamma$ are unknown functions of $\theta$ having very small numerical values for $\theta\in(0,2\pi)$. Substituting (18) into (17), we get, by the identification of $\theta^3$, $\theta$, $\sin\theta$ and $\cos\theta$ the following algebraic system

$$M+\beta=1,$$

$$\alpha+2\gamma=-\frac{qb^4}{C\,a^2(1-e^2)^{\frac{3}{2}}}=-\frac{qa^2}{\sqrt{ka\mu_0}},\ \beta+2\gamma\theta=0,$$

$$\alpha+\beta\theta=-\frac{qa^2}{\sqrt{ka\mu_0}}\left(1-\frac{2(1-e^2)^{\frac{3}{2}}+3e^2-2}{e^2}\cos\theta\right). \quad (19)$$

The solution of this system is

$$M=1-\beta.$$

$$\alpha = -\frac{qb^4}{C\,a^2(1-e^2)^{\frac{3}{2}}}\left[1 - \frac{2(1-e^2)^{\frac{3}{2}} + 3e^2 - 2}{e^3}\cdot\frac{\cos\theta}{1+\theta^2}\right];$$

$$\beta = \frac{qb^4[2(1-e^2)^{\frac{3}{2}} + 3e^2 - 2]}{C\,a^2(1-e^2)^{\frac{3}{2}}e^3}\cdot\frac{\theta\cos\theta}{1+\theta^2};$$

$$\gamma = -\frac{qb^4[2(1-e^2)^{\frac{3}{2}}+3e^2-2]}{C\,a^2(1-e^2)^{\frac{3}{2}}e^3}\cdot\frac{\cos\theta}{2(1+\theta^2)}. \tag{20}$$

Thus (18) becomes

$$\Psi = 1 - \frac{qa^2}{\sqrt{ka\mu_0}}\theta + \frac{qa^2}{\sqrt{ka\mu_0}}\cdot\frac{2(1-e^2)^{\frac{3}{2}}+3e^2-2}{e^3}\cdot\frac{\theta^3\cos\theta}{3(1+\theta^2)}. \tag{21}$$

From (16) and (21) we get the form of the planetary orbits:

$$r(\theta) \approx \frac{a(1-e^2)}{1+e\cos\theta}\left[1 + \frac{qa^2\theta}{\sqrt{ka\mu_0}} - \frac{qa^2}{\sqrt{ka\mu_0}}\cdot\frac{2(1-e^2)^{\frac{3}{2}}+3e^2-2}{e^3}\cdot\frac{\theta^3\cos\theta}{3(1+\theta^2)}\right]. \tag{22}$$

From the expression (22) it can be seen that one more term appears besides the result obtained by Vescan [9] in his equation (5). Thus, the extension of the major semi-axis during a revolution is given by

$$\Delta a_1 = \Delta a_v\left[1 + \frac{2(1-e^2)^{\frac{3}{2}}+3e^2-2}{e^3}\cdot\frac{\pi^2(13+9\pi^2)}{3(1+\pi^2)(1+9\pi^2)}\right]. \tag{23}$$

## 5. NUMERICAL RESULTS

On the other side, we determine numerically the quantity

$$\Delta a = a(2\pi) - a(0), \tag{24}$$

using the following system (see Muchametkalieva and Omarov [6]; Hadjidemetriou, J. [2, 3])

$$\frac{da}{dt} = -\frac{a(1+2e\cos\theta+e^2)}{1-e^2}\cdot\frac{1}{\mu}\cdot\frac{d\mu}{dt};$$

$$\frac{de}{dt} = -(e+\cos\theta)\cdot\frac{1}{\mu}\cdot\frac{d\mu}{dt};$$

$$\frac{d\omega}{dt} = -\frac{\sin\theta}{e}\cdot\frac{1}{\mu}\cdot\frac{d\mu}{dt};$$

$$\frac{d\theta}{dt}=\frac{C(1+e\cos\theta)^2}{a^2(1-e^2)^2}+\frac{\sin\theta}{e}\cdot\frac{1}{\mu}\cdot\frac{d\mu}{dt};$$

$$\frac{d\mu}{dt}=-q\mu, \tag{25}$$

where $a$, $e$, $\omega$, $\theta$ and $\mu$ represent the major semi-axis, the eccentricity, the argument of the perihelion, the true anomaly and the mass of the Sun.

The system (25) was integrated numerically for a revolution using a Fehlberg fourth-fifth order Runge-Kutta method (MAPLE, rkf45) for the values of the quantities $q$, $K$ and $\mu_0$ considered by Vescan [9]

$$q=5\cdot10^{-21}s^{-1},k=6{,}65\cdot10^{-8}\frac{cm^3}{g\cdot s^2},\mu_0=2\cdot10^{33}g. \tag{26}$$

The initial values (at the time $t=0$) of the parameters $a$, $e$, $\omega$ and $\theta$ are given in Table 1.

## 6. CONCLUSIONS

The aim of this paper is to determine analytically the expressions of $\Delta a_1$ given by (23) and numerically the expression of $\Delta a$. The results are presented in Table 2 where there are included also the analytical results obtained by Vescan [9] given by Eqs. (6). Comparing the results given in this table, obtained using the analytical and numerical solutions it can be concluded that there is a good agreement between them. Therefore, we are confident that the system (25) can be used also for the present problem. On the other hand, from the numerical integration of the system (25) it has been found that the variation of the eccentricity is oscillating on time. But, the amplitude of the oscillation is very small (taking as on example the Earth the difference between the maximum and minimum for ten years is of order $10^{-14}$ and for one thousand years it is $10^{-11}$), that means that the variation of the eccentricity is very small. It means that the eccentricity is constant. This results is in very good agreement with the that obtained by Jeans [4].

*Table 1*

Values of $a_0$, $e_0$, $\omega_0$ and $\theta_0$ at the initial time, $t=0$

| Planet | $a_0=a(0)$ (cm) | $e_0=e(0)$ | $\omega_0=\omega(0)$ (rad/s) | $\theta_0=\theta(0)$ (rad) |
|---|---|---|---|---|
| Mercury | $5.8\cdot10^{12}$ | 0.205631752 | $\frac{2\pi}{365.25\cdot24\cdot3600\cdot0.241}$ | 0 |
| Venus | $1.08\cdot10^{13}$ | 0.006771882 | $\frac{2\pi}{365.25\cdot24\cdot3600\cdot0.615}$ | 0 |

*Table 1 (continued)*

| | | | | |
|---|---|---|---|---|
| Earth | $1.5\cdot10^{13}$ | 0.016708617 | $\frac{2\pi}{365.25\cdot24\cdot3600}$ | 0 |
| Mars | $2.3\cdot10^{13}$ | 0.09340062 | $\frac{2\pi}{365.25\cdot24\cdot3600\cdot1.881}$ | 0 |
| Jupiter | $7.8\cdot10^{13}$ | 0.048494851 | $\frac{2\pi}{365.25\cdot24\cdot3600\cdot11.862}$ | 0 |
| Saturn | $1.4\cdot10^{14}$ | 0.055508622 | $\frac{2\pi}{365.25\cdot24\cdot3600\cdot29.458}$ | 0 |
| Uranus | $2.9\cdot10^{14}$ | 0.046295899 | $\frac{2\pi}{365.25\cdot24\cdot3600\cdot84.013}$ | 0 |
| Neptune | $4.5\cdot10^{14}$ | 0.008988095 | $\frac{2\pi}{365.25\cdot24\cdot3600\cdot164.794}$ | 0 |
| Pluto | $6.0\cdot10^{14}$ | 0.248 | $\frac{2\pi}{365.25\cdot24\cdot3600\cdot247.686}$ | 0 |

*Table 2*

| Planet | $\Delta a_v$ Eq. (6) (cm) | $\Delta a_1$ Eq. (23) (cm) | $\Delta a$ Present numerical results (cm) |
|---|---|---|---|
| Mercury | 0.2206957839 | 0.2324573618 | 0.22168 |
| Venus | 1.044199404 | 1.045353061 | 1.0488 |
| Earth | 2.373844135 | 2.384145714 | 2.3690 |
| Mars | 6.911043753 | 7.077387973 | 6.8046 |
| Jupiter | 146.3728602 | 148.2003297 | 145.9688 |
| Saturn | 631.7483878 | 640.7767262 | 652.952 |
| Uranus | 3901.387136 | 3947.887234 | 3838.938 |
| Neptune | 11701.87184 | 11729.51855 | 11701.188 |
| Pluto | 24021.61369 | 25570.68070 | 23075.000 |

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