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ON THE DISCRETE SUMUDU TRANSFORM

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Abstract. In this paper, we define the Sumudu transform on an arbitrary time scale. Starting from this definition we define the discrete Sumudu transform. We prove the initial and final value problems and study the basic properties of this transform. We also present the discrete Sumudu transform of some basic functions.

Key words: Time scales, Sumudu transform, Convolution.

1. INTRODUCTION

There are several integral transforms such as the Laplace, Millen, Hankel and Fourier transforms that are used to solve differential equations which appear in many fields of sciences and engineering. In the early 1990's, Watugala [1,2] introduced the Sumudu transform and applied it to solve ordinary differential equations. Watugala's work was followed by Weerakoon who introduced the complex inversion formula for the Sumudu transform [3, 4]. The fundamental properties of this transform, which is thought to be an alternative to the Laplace transform were then established in many articles [5–8].

The Sumudu transform is defined over the set of functions

$$A = \{f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty)\}, \tag{1}$$

by

$$F(u) = \mathbb{S}\{f(t)\}(u) = \frac{1}{u}\int_0^\infty f(t)e^{-\frac{t}{u}}dt, \quad u \in (-\tau_1, \tau_2). \tag{2}$$

Although the Sumudu transform of a function has a deep connection to its Laplace transform, the Sumudu transform may be used to solve intricate problems in engineering and applied sciences, that can hardly be solved when the Laplace transform is used. Moreover, some properties of Sumudu transform makes it more advantageous than the Laplace transforms. Some of these properties are:
- the Sumudu transform of a Heaviside step function is a also Heaviside step function

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in the transformed domain;
- $\mathbb{S}\{t^n\} = n!u^n$;
- $\lim_{t\to\infty} f(t) = \lim_{u\to\infty} F(u)$;
- $\lim_{t\to 0} f(t) = \lim_{u\to 0} F(u)$;
- if $c > 0$, $\mathbb{S}\{f(ct)\} = F(cu)$.

Recently, it was proved that by using the Sumudu transform, one can transform the two dimensional transport equation into a Fredholm integral equation [15]. In [14], the authors applied the Sumudu transform to fractional differential equations which have many applications in many fields of sciences (see [16] and the references therein).

Starting with a general definition of the Laplace transform on an arbitrary time scales, the concept of the h-Laplace and consequently the discrete Laplace transformed were specified in [9]. The theory of time scales was initiated by Stefan Hilger [10]. This theory is a tool that unifies the theories of continuous and discrete time systems. It is a subject of recent studies on many different fields in which dynamic process can be described with discrete or continuous models.

In this paper, starting from the definition of the Sumudu transform on a general time scales, we define the discrete Sumudu transform and present its basic properties.

Our paper is organized as follows: In section 2 we introduce basic concepts concerning the calculus of time scales. In section 3 we define the the discrete Sumudu transform and present its basic properties. Section 4 is devoted to the conclusion.

2. PRELIMINARIES ON TIME SCALES

A time scale set $\mathbb{T}$ is a non empty closed subset of $\mathbb{R}$ [12].

The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \,\forall\, t \in \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t)(t) = sup\{s \in \mathbb{T} : s < t\} \;\forall\, t \in \mathbb{T}$ where $inf\ \emptyset = sup\ \mathbb{T}$ and $sup\ \emptyset = inf\ \mathbb{T}$.

A point $t \in \mathbb{T}$ is called right-dense, left-dense, right-scattered and left-scattered if $\sigma(t) = t,\;\; \rho(t) = t,\;\; \sigma(t) > t$ and $\rho(t) < t$ respectively.

The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$.

The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise it is $\mathbb{T}$ without this left-scattered maximum.

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^\kappa$ if there is a number $f^\Delta(t)$ with the property that $\forall \varepsilon > 0$ there is a neighborhood $U = (t-\delta, t+\delta)\bigcap \mathbb{T}$ of t for some $\delta > 0$ such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \;\forall\, s \in U.$$

The number $f^\Delta(t)$ is called the delta derivative of f on $\mathbb{T}^\kappa$. We have the following theorem

Theorem 2.1. *Assume that* $f:\mathbb{T}\to\mathbb{R}$ *is a function and let* $t\in\mathbb{T}^{\kappa}$*. Then we have:*

(i) If f *is differentiable at* t*, then* f *is continuous at* t*.*

(ii) If f *is continuous at* t *and* t *is right-scattered, then* f *is differentiable at* t *with* $f^{\Delta}(t)=\dfrac{f(\sigma(t))-f(t)}{\mu(t)}$.

(iii) If t *is right-dense, then* f *is differentiable at* t *iff the limit* $\lim\limits_{s\to t}\dfrac{f(t)-f(s)}{t-s}$ *exists. In this case* $f^{\Delta}(t)=\lim\limits_{s\to t}\dfrac{f(t)-f(s)}{t-s}$.

(iv) If f *is differentiable at* t*, then* $f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t)$*.*

The following formulas hold for delta differentiable functions:

$$f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t),$$
$$(fg)^{\Delta}(t)=f^{\Delta}(t)g^{\sigma}(t)+f(t)g^{\Delta}(t),$$

where $f^{\sigma}(t)=f(\sigma(t))$.

A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous if it is continuous at right-dense and its left-sided limit exists at left-dense points $t\in\mathbb{T}$. The set of all rd-continuous functions is denoted by C_{rd} and the set of all differentiable functions with rd-continuous derivative is denoted by C^1_{rd}.

Theorem 2.2. *If* $f\in C_{rd}$*, then* f *possesses an antiderivative.*

That is there exists a function F with $F^{\Delta}=f$, and in this case an integral is defined by $\int_s^t f(\tau)\Delta\tau=F(t)-F(s)$.

2.1. THE EXPONENTIAL FUNCTION

Let $\mathbb{T}$ be a time scale with σ as the forward jump operator and Δ as the delta differentiation operator.

Definition 2.3. A function $p:\mathbb{T}\to\mathbb{C}$ is called regressive if $1+\mu(t)p(t)\neq 0$, $\forall t\in\mathbb{T}$.

The set $\mathcal{R}$ of all regressive and rd-continuous function forms an abelian group under the addition $\oplus$ defined by

$$(p+q)(t)=p(t)+q(t)+\mu(t)p(t)q(t),\quad \forall t\in\mathbb{T}.$$

The additive inverse of $p \in \mathcal{R}$, denoted by $\ominus p$ is

$$(\ominus p)(t) = -\frac{p(t)}{1+\mu(t)p(t)}$$

Theorem 2.4. *Let $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. Then the I.V.P*

$$y^{\Delta} = p(t)y, \qquad y(t_0) = 1, \tag{3}$$

has a unique solution on $\mathbb{T}$.

The solution of the I.V.P. (3) is called the exponential function and is denoted by $e_p(.,t_0)$. For more details on time scales and exponential functions we refer to [9,11].

3. THE DISCRETE SUMUDU TRANSFORM

Let $\mathbb{T}$ be a time scale such that $\sup\mathbb{T} = \infty$ and let $t_0 \in \mathbb{T}$ be a fixed point. Let $u \neq 0$ be a complex constant such that $u + \mu(t) \neq 0$, $\forall t \in \mathbb{T}$ or $\frac{1}{u} \in \mathcal{R}$.

Definition 3.1. Suppose $f : [t_0, \infty) \to \mathbb{C}$ is a locally integrable function. Then the Sumudu transform of f is defined by

$$\mathbb{S}\{f(t)\}(u) = \frac{1}{u}\int_{t_0}^{\infty} f(t)e_{\ominus\frac{1}{u}}(\sigma(t),t_0)\Delta t, \text{ for } u \in \mathcal{D}(f), \tag{4}$$

where $\mathcal{D}(f)$ is the set of complex numbers u such that the improper integral in (4) exists.

When $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$, $\ominus\frac{1}{u} = -\frac{1}{u}$ and thus we get the definition of the usual Sumudu transform (2). If we consider $\mathbb{T} = h\mathbb{Z} = \{hk | k \in \mathbb{Z}\}$ then $\sigma(t) = t+h$ and $\mu(t) = h$, where h is a fixed real number. The Δ-derivative of a function in this case is

$$f^{\Delta}(t) = \frac{f(t+h)-f(t)}{h}, \quad \forall\, t \in h\mathbb{Z}.$$

The unique solution of the I.V.P.

$$y^{\Delta} = \frac{1}{u}y, \qquad y(t_0) = 1$$

is

$$y(t) = e_{\frac{1}{u}}(t,t_0) = \left(1+\frac{h}{u}\right)^{\frac{t-t_0}{h}}.$$

Since $\ominus\frac{1}{u} = -\frac{1}{u+h}$, the solution of the I.V.P.

$$y^{\Delta} = \ominus\frac{1}{u}y, \qquad y(t_0) = 1$$

is

$$y(t) = e_{\ominus\frac{1}{u}}(t,t_0) = \left(\frac{u}{u+h}\right)^{\frac{t-t_0}{h}}.$$

Thus, according to (4), we have

$$\mathbb{S}\{f(t)\}(u) = \frac{h}{u}\sum_{t=t_0}^{\infty} f(t)\left(\frac{u}{u+h}\right)^{\frac{t+h-t_0}{h}} = \frac{h}{u}\sum_{t=0}^{\infty} f(t+t_0)\left(\frac{u}{u+h}\right)^{\frac{t+h}{h}}. \quad (5)$$

Setting $t = kh, t_0 = k_0h$, we get

$$\mathbb{S}\{f(t)\}(u) = \frac{h}{u}\sum_{k=0}^{\infty} f(kh+k_0h)\left(\frac{u}{u+h}\right)^{k+1}, \quad (6)$$

where $k, k_0 \in \mathbb{Z}$. Finally, replacing h by 1 we reach the following definition of the discrete Sumudu transform.

Definition 3.2. If $f : \mathbb{N}_0 \to \mathbb{C}$ is a function, then its discrete Sumudu transform is defined by

$$\mathbb{S}_d\{f(k)\}(u) = \frac{1}{u}\sum_{k=0}^{\infty} f(k)\left(\frac{u}{u+1}\right)^{k+1}, \quad (7)$$

for all values of $u \neq -1$ such that the series converges.

If $R = \limsup |f(k)|^{\frac{1}{k}}$, then we have one of the following three cases:

(i) If $0 < R < \infty$, then the series (7) converges for $\left|\frac{u+1}{u}\right| > R$ and diverges elsewhere;

(ii) If $R = 0$, then the series (7) converges for all u except possibly when $u = -1$;

(iii) If $R = \infty$, the series diverges everywhere.

Below we present some properties of the discrete Sumudu transform.

Theorem 3.3. *If* $\mathbb{S}_d\{f(k)\}(u) = F(u)$ *for* $\left|\frac{u+1}{u}\right| > A \geq R$, *then for the same values of* u

$$\mathbb{S}_d\{f(k+1)\}(u) = \frac{u+1}{u}F(u) - \frac{f(0)}{u} \quad (8)$$

and in general

$$\mathbb{S}_d\{f(k+m)\}(u) = \left(\frac{u+1}{u}\right)^m F(u) - \sum_{n=0}^{m-1} \frac{(u+1)^{m-n-1}}{u^{m-n}} f(n). \quad (9)$$

Proof.

$$\mathbb{S}_d\{f(k+1)\}(u) = \frac{1}{u}\sum_{k=0}^{\infty} f(k+1)\left(\frac{u}{u+1}\right)^{k+1} = \frac{1}{u}\sum_{k=0}^{\infty} f(k+1)\left(\frac{u}{u+1}\right)^{k+1}$$
$$= \frac{1}{u}\sum_{k=1}^{\infty} f(k)\left(\frac{u}{u+1}\right)^{k} = \frac{1}{u}\left[\sum_{k=0}^{\infty} f(k)\left(\frac{u}{u+1}\right)^{k} - f(0)\right]$$
$$= \frac{1}{u}\left[\frac{u+1}{u}\sum_{k=0}^{\infty} f(k)\left(\frac{u}{u+1}\right)^{k+1} - f(0)\right] = \frac{u+1}{u}F(u) - \frac{f(0)}{u}.$$

The identity (9) may be proved easily by induction. □

Theorem 3.4. *If* $\mathbb{S}_d\{f(k)\}(u) = F(u)$ *exists, then*

$$f(0) = \lim_{u\to 0} F(u) \tag{10}$$

and

$$\lim_{k\to\infty} f(k) = \lim_{u\to\infty} F(u). \tag{11}$$

Proof.

$$\lim_{u\to 0} F(u) = \lim_{u\to 0}\frac{1}{u}\left[\frac{u}{u+1}f(0) + \left(\frac{u}{u+1}\right)^2 f(1) + \left(\frac{u}{u+1}\right)^3 f(2)\cdots\right]$$
$$= \lim_{u\to 0}\left[\frac{1}{u+1}f(0) + \frac{u}{(u+1)^2}f(1) + \frac{u^2}{(u+1)^3}f(2)\cdots\right] = f(0).$$

To prove (11), consider $\mathbb{S}_d\{f(k+1)-f(k)\}(u)$. On using Theorem 3.3 we have

$$u\mathbb{S}_d\{f(k+1)-f(k)\}(u) = F(u) - f(0).$$

On the other hand

$$u\mathbb{S}_d\{f(k+1)-f(k)\}(u) = \sum_{k=0}^{\infty}[f(k+1)-f(k)]\left(\frac{u}{u+1}\right)^{k+1}$$
$$= \lim_{n\to\infty}\sum_{k=0}^{n}[f(k+1)-f(k)]\left(\frac{u}{u+1}\right)^{k+1}$$
$$= \lim_{n\to\infty}\frac{u}{u+1}\left\{-f(0) + \left[1 - \frac{u}{u+1}\right]f(1) + \left[\frac{u}{u+1} - \left(\frac{u}{u+1}\right)^2\right]f(2)\right.$$
$$\left.+\cdots\left[\left(\frac{u}{u+1}\right)^{n-1} - \left(\frac{u}{u+1}\right)^{n}\right]f(n) + \left(\frac{u}{u+1}\right)^{n} f(n+1)\right\}.$$

Therefore,

$$\lim_{u\to\infty} F(u)-f(0)=\lim_{u\to\infty}\lim_{n\to\infty}\frac{u}{u+1}\Bigg\{-f(0)+\left[1-\frac{u}{u+1}\right]f(1)+\left[\frac{u}{u+1}-\left(\frac{u}{u+1}\right)^2\right]f(2)$$

$$+\cdots+\left[\left(\frac{u}{u+1}\right)^{n-1}-\left(\frac{u}{u+1}\right)^{n}\right]f(n)+\left(\frac{u}{u+1}\right)^{n}f(n+1)\Bigg\}=\lim_{n\to\infty}f(n+1)-f(0).$$

Thus, (11) holds. □

Lemma 3.5. *If $A>R$, the series defined in (7) is uniformly convergent for values of u such that $\left|\frac{u+1}{u}\right|\geq A$.*

Proof. Since $A>R$, $\exists\,\epsilon>0$ such that $A=R+\epsilon$. On the other hand, since $R=\limsup|f(k)|^{\frac{1}{k}}$, for the same ϵ, $\exists\, n\in\mathbb{N}$ such that

$$|f(k)|\leq(R+\epsilon)^k,\quad \forall k\geq n.$$

Thus

$$\left|\sum_{k=n}^{\infty}f(k)\left(\frac{u}{u+1}\right)^{k+1}\right|\leq\sum_{k=n}^{\infty}|f(k)|\left|\frac{u}{u+1}\right|^{k+1}\leq$$

$$\sum_{k=n}^{\infty}\left(\frac{R+\epsilon}{A}\right)^{k+1}=\frac{A}{A-R-\epsilon}\left(\frac{R+\epsilon}{A}\right)^{n+1}\to 0 \text{ as } n\to\infty.$$

□

Below we present the uniqueness theorem:

Theorem 3.6. *Let $F(u)$ be the discrete Sumudu transform of $f(k)$. If $F(u)\equiv 0$ for $\left|\frac{u+1}{u}\right|>R$, then $f(k)\equiv 0$, $\forall\, k\in\mathbb{N}_0$.*

Proof. Since $F(u)\equiv 0$ we have

$$f(0)+\left(\frac{u}{u+1}\right)f(1)+\left(\frac{u}{u+1}\right)^2f(2)+\left(\frac{u}{u+1}\right)^3f(3)+\cdots=0. \tag{12}$$

By Lemma 3.5, since the series in (12) is uniformly convergent we can take term by term limits. Taking the limit as $u\to 0$ in (12), we get $f(0)=0$. Now multiplying (12) by $\frac{1}{u}$ and then taking the limit as $u\to 0$, we get $f(1)=0$. Similarly, multiplying the remaining part of (12) by $\frac{1}{u}$ and then taking the limit as $u\to 0$, we get $f(2)=0$. In a similar way we get $f(3)=f(4)=\cdots=0$. Thus $f(k)=0$, $\forall\, k\in\mathbb{N}_0$. □

Therefore if $\mathbb{S}_d\{f(k)\}(u)=F(u)$ and $\mathbb{S}_d\{g(k)\}(u)=G(u)$ exist such that $F(u)\equiv G(u)$, then $G(u)-F(u)\equiv 0$ and thus $g(k)-f(k)\equiv 0$. Hence $f(k)=g(k)$.

The convolution of two functions $f,g:\mathbb{N}_0\to\mathbb{C}$ is defined [9] by

$$(f*g)(k)=\sum_{m=0}^{k-1}f(k-m-1)g(m),\quad \text{for } k\in\mathbb{N}_0, \tag{13}$$

where when $k=0$, $(f*g)(k)=0$. Below we present the discrete Sumudu transform of the convolutions of two functions.

Theorem 3.7. *Let $f,g:\mathbb{N}_0\to\mathbb{C}$ be given functions. If $\mathbb{S}_d\{f(k)\}(u)=F(u)$ exists for $\left|\frac{u+1}{u}\right|>A_1$ and $\mathbb{S}_d\{g(k)\}(u)=G(u)$ exists for $\left|\frac{u+1}{u}\right|>A_2$, then*

$$\mathbb{S}_d\{(f*g)(t)\}(u)=uF(u)G(u),\ \text{for}\ \left|\frac{u+1}{u}\right|>\max\{A_1,A_2\}. \tag{14}$$

Proof.

$$\begin{aligned}
uF(u)G(u)&=\frac{1}{u}\sum_{i=0}^{\infty}f(i)\left(\frac{u}{u+1}\right)^{i+1}\sum_{i=0}^{\infty}g(j)\left(\frac{u}{u+1}\right)^{j+1}\\
&=\frac{u}{(u+1)^2}\sum_{i=0}^{\infty}f(i)\left(\frac{u}{u+1}\right)^{i}\sum_{i=0}^{\infty}g(j)\left(\frac{u}{u+1}\right)^{j}\\
&=\frac{u}{(u+1)^2}\sum_{k=0}^{\infty}\sum_{m=0}^{k}f(m)g(k-m)\left(\frac{u}{u+1}\right)^{k}\\
&=\frac{u}{(u+1)^2}\sum_{k=1}^{\infty}\sum_{m=0}^{k-1}f(m)g(k-m-1)\left(\frac{u}{u+1}\right)^{k-1}\\
&=\frac{1}{u+1}\sum_{k=1}^{\infty}\sum_{m=0}^{k-1}f(m)g(k-m-1)\left(\frac{u}{u+1}\right)^{k}\\
&=\frac{1}{u+1}\sum_{k=0}^{\infty}\sum_{m=0}^{k-1}f(m)g(k-m-1)\left(\frac{u}{u+1}\right)^{k}\\
&=\frac{1}{u}\sum_{k=0}^{\infty}(f*g)(k)\left(\frac{u}{u+1}\right)^{k+1}=\mathbb{S}_d\{(f*g)(k)\}(u).
\end{aligned}$$

□

In the following theorem, we present the Sumudu transform of Δf:

Theorem 3.8. *If $\mathbb{S}_d\{f(k)\}(u)=F(u)$ for $\left|\frac{u+1}{u}\right|>A$, then for the same values* *of* u

$$\mathbb{S}_d\{\Delta f(k)\}(u) = \frac{u+1}{u}F(u) - \frac{f(0)}{u} \tag{15}$$

and in general

$$\mathbb{S}_d\{\Delta^n f(k)\}(u) = u^{-n}\left[F(u) - \sum_{i=0}^{n-1} u^k \left|\Delta^k f(k)\right|_{k=0}\right], \tag{16}$$

where $\Delta f(k) = f(k+1) - f(k)$ *and* $\Delta^0 f(k) = f(k)$.

Proof. (15) can be proved easily by using the definition of $\Delta f(k)$ and the shifting theorem 3.3, while (16) can be proved by induction. □

Finally below we present the discrete Sumudu transform of some elementary functions:

$$\mathbb{S}_d\{(1+\lambda)^k\} = \frac{1}{1-\lambda u} \text{ for } \left|\frac{(1+\lambda)u}{u+1}\right| < 1,$$

$$\mathbb{S}_d\{\cos(ak)\} = \frac{(1-\cos a)u+1}{2(1-\cos a)u^2+2(1-\cos a)u+1},$$

$$\mathbb{S}_d\{\sin(ak)\} = \frac{u\sin a}{2(1-\cos a)u^2+2(1-\cos a)u+1},$$

$$\mathbb{S}_d\{k^{(n)}\} = n!u^n, \text{ where } k^{(n)} = k(k-1)\cdots(k-n+1),$$

$$\mathbb{S}_d\left\{\sum_{m=0}^{k-1} f(m)\right\} = u\mathbb{S}_d\{f(k)\}.$$

4. CONCLUSION

In this paper, the discrete Sumudu transform was defined and its basic properties were presented. We believe that this transform is an alternative in solving difference equations, not only to the discrete Laplace transform, but to the Z-transform as well.

More properties of this transform will be discussed in another work.

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