Romanian Reports in Physics, Vol. 64, No. 2, P. 357–366, 2012

CHIRAL SOLITONS WITH BOHM POTENTIAL USING G'/G METHOD AND EXP-FUNCTION METHOD

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Received November 28, 2011

Abstract. This paper studies the chiral nonlinear Schrödinger's equation with Bohm potential. There are two approaches that are used to carry out the integration of the governing equation. They are the G'/G method and the exp-function method. Finally the traveling wave hypothesis is used to obtain solution in terms of doubly periodic function where in the limiting case topological soliton solutions are retrieved.

PACS Codes: 02.30.Ik; 02.30.Jr; 02.30.Mv.

Key words: chiral solitons, integrability.

1. INTRODUCTION

The study of chiral soliton has been going on for the past few decades [1-10]. These chiral solitons appear in Quantum Mechanics, particularly in the area of Quantum Hall Effect. The governing equation is the chiral nonlinear Schrödinger's equation (CNLSE). This equation was studied earlier. The traveling wave solution was obtained for this equation in 1998 and both bright and dark soliton solutions are obtained [9].

Later, the Bohm potential was introduced in 2004 [8]. Subsequently, several studies were done with the so called perturbed CNLSE where the Bohm potential was treated as the perturbation term. The soliton perturbation theory was applied to obtain the adiabatic dynamics of soliton parameters. Later, the several integrability techniques were applied to integrate the perturbed CNLSE. These include the semi-inverse variational method, Lie symmetry analysis as well as as the traveling wave hypothesis. Recently, the generalized form of the CNLSE was also studied in 2011 [3] where the ansatz method was used to carry out the integration of the generalized CNLSE. In this paper, the conserved quantities were also studied using the multiplier approach [3].

In this paper, there will be two more methods of integrability will be applied to integrate the perturbed CNLSE. They are the G'/G method as well as the exp-function method. Finally, the snoidal wave solution will also be obtained along with its limiting case that leads to the shock wave solution.

2. GOVERNING EQUATIONS

The dimensionless form of the perturbed CNLSE that is going to be studied in this paper is given by [1-3, 7, 8]

$$\mathrm{i}q_t + aq_{xx} + \mathrm{i}b\left(qq_x^* - q^*q_x\right)q = \mathrm{i}\alpha q\frac{|q|_{xx}}{|q|}. \tag{1}$$

In (1), on the left-hand side, the first term represents the evolution term which governs time evolution of the wave profile, while a is the coefficient of dispersion and the coefficient of b is the derivative coupling and b is known as nonlinear coupling constant. This term changes sign under the parity, $x \to -x$. It is due to this property, (1) is known as the chiral NLSE. This kind of nonlinearity is also known as the *current density*.

On right-hand side of (1), the coefficient of α is called the Bohm potential, also known as the internal self-potential that was introduced by de Broglie and explored by Bohm to introduce the hidden variable theory [8]. Therefore, it produces quantum behavior so that all quantum features are related to its special properties.

In the following three sections, the integration of the perturbed CNLSE will be studied. The techniques of integrability that will be utilized are the G'/G method, exp-method and the traveling wave hypothesis. These are respectively going to be studied in the following sections.

3. G'/G METHOD

In this section, the G'/G method will be used to carry out the integration of the CNLSE with Bohm potential. The method integrability will first be described and then subsequently it will be applied to integrate the CNLSE with Bohm potential.

3.1. DESCRIPTION OF THE METHOD

The objective of this section is to outline the use of the G'/G-expansion method for solving certain nonlinear partial differential equations (PDEs). A nonlinear PDE

$$P(q, q_x, q_t, q_{xx} q_{xt}, q_{tt}, \cdots) = 0 \tag{2}$$

using wave transformation

$$q(x,t) = Q(\xi), \xi = B(x - vt), \tag{3}$$

can be converted to

$$P(Q, BQ', -vBQ', B^2Q'', -vB^2Q'', v^2B^2Q'', \cdots) = 0. \tag{4}$$

Let the solution of eq. (4) can be expressed by a polynomial in $\frac{G'}{G}$ as follows:

$$Q(\xi) = \sum_{l=0}^{m} a_l (\frac{G'}{G})^l, \;\; a_m \neq 0, \tag{5}$$

where a_l, $l = 0,1,\cdots,m$ are constants to be determined later and $G(\xi)$ satisfies the second order linear ordinary differential equation (LODE):

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda \frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0. \tag{6}$$

where λ and μ are arbitrary constants. The positive integer m in (5) can be determined by considering the homogeneous balance the highest order derivatives and highest order nonlinear appearing in ODE (4). Substituting eq. (5) along with eq. (6) into eq. (4), collecting all terms with the same power of G'/G together, the left-hand side of eq. (4) is converted into another polynomial in G'/G. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_0, ..., a_m, \lambda, v$ and μ by using Maple. Assuming that the unknown constants can be obtained by solving the algebraic equations. Since the general solutions of the second order LODE (6) have been well known for us, then substituting $a_0, ..., a_m, v$ and general solutions of eq. (4) into (5) we have more traveling wave solutions of the nonlinear evolution equation (2).

3.2. APPLICATION TO CNLSE

The CNLSE with Bohm given by (1) is going to be studied with the initial condition

$$q(x,0) = g(x)e^{i(-\kappa x + \theta)}$$

and boundary condition

$$|q(x,t)| \to 0 \;\; as \;\; |x| \to \pm\infty.$$

Substituting

$$q(x,t) = g(x - vt)e^{i(-\kappa x+\omega t+\theta)} \tag{7}$$

into (1) and decomposing into real and imaginary parts respectively yields

$$\omega g - a\left(g'' - \kappa^2 g\right) + 2\lambda\kappa g^3 = 0 \tag{8}$$

and

$$g'(v + 2ak) + \alpha g'' = 0, \tag{9}$$

where the function g represents the soliton shape and v is the velocity of the soliton. From the phase component, κ is the soliton frequency, ω is the soliton wave number, while θ is the phase constant and $g' = \mathrm{d}g/\mathrm{d}s$, $g'' = \mathrm{d}^2 g/\mathrm{d}s^2$, with $s = x - vt$.

It needs to be noted that equations (8) and (9) are to be solved together in order to integrate (1). From (9), it is possible to obtain

$$g(x,t) = -\left(\frac{\alpha}{v + 2ak}\right)\exp\left\{-\left(\frac{v + 2ak}{\alpha}\right)\right\}. \tag{10}$$

From (10), it is possible to determine the velocity of the soliton by solving for v in terms of the remaining parameters, where the soliton expression for the function g is given later in (12). Now, multiplying (8) by g' and integrating, yields

$$(\omega + ak^2)g^2 + \lambda k g^4 - a(g')^2 = h, \tag{11}$$

where h is the constant of integration.

Balancing g^4 and $(g')^2$ in (11) gives

$$4m = 2m + 2$$

so that

$$m = 1.$$

Suppose that the solutions of (11) can be expressed by a polynomial in G'/G as follows:

$$g(s) = a_0 + a_1\left(\frac{G'}{G}\right),\ a_1 \neq 0. \tag{12}$$

Substituting eqs. (12) into eq. (11), collecting the coefficients of $\left(\frac{G'}{G}\right)^i$ and setting it to zero gives a system of algebraic equations that solving it with Maple gives the following solutions.

$$a_0 = \frac{\lambda a_1}{2},\ k = \frac{a}{ba_1^2},$$

$$h = \frac{a_1^8 \omega^2 b^4 + 2a_1^4 \omega b^2 a^3 + a^6}{-4aa_1^6 b^4}, \tag{13}$$

$$\mu = \frac{2a_1^4 \omega b^2 + a_1^4 ab^2 \lambda^2 + 2a^3}{4b^2 a_1^4 a}.$$

where a, b, ω and λ are arbitrary constants.

Substituting eqs. (13) and general solution of eq. (6) into eq. (12) we have three types of traveling wave solutions of the CNLSE as follows:

– when $\Delta = \lambda^2 - 4\mu = \frac{-2(a_1^4 \omega b^2 + a^3)}{a_1^4 ab^2} > 0,$

$$g(s) = \beta a_1 \frac{c_1 \mathrm{sh}(\beta s) + c_2 \mathrm{ch}(\beta s)}{c_1 \mathrm{ch}(\beta s) + c_2 \mathrm{sh}(\beta s)}, \tag{14}$$

where $\alpha = \frac{1}{2}\sqrt{\frac{-2(a_1^4 \omega b^2 + a^3)}{a_1^4 ab^2}};$

– when $\Delta = \lambda^2 - 4\mu = \frac{-2(a_1^4 wb^2 + a^3)}{a_1^4 ab^2} < 0,$

$$g(s) = \beta a_1 \frac{-c_1 \sin(\beta s) + c_2 \cos(\beta s)}{c_1 \cos(\beta s) + c_2 \sin(\beta s)}, \tag{15}$$

where $\beta = \frac{1}{2}\sqrt{\frac{2(a_1^4 \omega b^2 + a^3)}{a_1^4 ab^2}}.$

– when $\Delta = \lambda^2 - 4\mu = \frac{-2(a_1^4 \omega b^2 + a^3)}{a_1^4 ab^2} = 0,$

$$g(s) = \frac{c_1 a_1}{c_2 + c_1 s}. \tag{16}$$

4. EXP-FUNCTION METHOD

This section will integrate the CNLSE by the exponential function method that is abbreviated as exp-function approach. The details are in the following subsection below.

4.1. DESCRIPTION OF THE METHOD

The exp-function method is based on the assumption that traveling wave solutions of eq. (11) can be expressed in the following form:

$$g(s)=\frac{\sum_{n=-p_1}^{p_2} a_n \mathrm{e}^{ns}}{\sum_{m=-p_3}^{p_4} b_m \mathrm{e}^{ms}}, \tag{17}$$

where p_1, p_2, p_3, and p_4 are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. To determine the values of p_1 and p_3, we balance the linear term of highest order in eq. (11) with the highest order nonlinear term. Similarly to determine the values of p_2 and p_4, we balance the linear term of lowest order in eq. (11) with the lowest order nonlinear term.

4.2. APPLICATION TO CNLSE

In this part we want to solve eq. (17) with exp-function method. Using the ansatz (17) for $g^4(s)$ and $\left(g'\right)^2(s)$, gives

$$\left(g'\right)^2(s)=\frac{c_1\mathrm{e}^{-2(p_1+p_3)s}+\cdots+c_2\mathrm{e}^{2(p_2+p_4)s}}{c_3\mathrm{e}^{-4p_3s}+\cdots+c_4\mathrm{e}^{4p_4s}}, \tag{18}$$

and

$$g^4(s)=\frac{d_1\mathrm{e}^{-4p_1s}+\cdots+d_2\mathrm{e}^{4p_2s}}{d_3\mathrm{e}^{-4p_3s}+\cdots+d_4\mathrm{e}^{4p_4s}}, \tag{19}$$

where $c_i, d_i,\ i=1,\cdots,4$ are obtained easily by simple calculations.

Balancing the highest order of exp-function in the eqs. (18) and (19) gives

$$2p_2+2p_4-4p_4=4p_2-4p_4,$$

which leads

$$p_2=p_4.$$

Similarly, for determining p_1 and p_3 in eq.(17), balancing the lowest order of exp-function in the eqs.(18) and (8) gives

$$-2p_1 - 2p_3 + 4p_3 = -4p_1 + 4p_3,$$

which leads

$$p_1 = p_3.$$

Since the final solution dose not depend on the values of p_1, p_2, p_3 and p_4, so for simplicity we let $p_1 = p_3 = 1$ and $p_2 = p_4 = 1$. Therefore eq. (19) becomes

$$g(s) = \frac{a_{-1}\mathrm{e}^{-s} + a_0 + a_1\mathrm{e}^{s}}{b_{-1}\mathrm{e}^{-s} + b_0 + b_1\mathrm{e}^{s}}. \tag{20}$$

Substituting eq. (20) into eq. (11), and equating the coefficients of all powers of e^{ns} to zero yields a system of algebraic equations for $a_{-1}, a_0, a_1, b_{-1}, b_0, b_1, \omega, k$ and h. Solving the system with the Maple gives the following solutions.

- **First set:**

$$a_{-1} = b_{-1} = 0, b_1 = \frac{-a_1 b_0}{a_0}, h = 0, \omega = \frac{a(8b^2 a_0^4 + a^2 b_0^4)}{-16(b^2 a_0^4)}, \tag{21}$$

$$k = \frac{ab_0^2}{4(a_0^2 b)}, \quad h = \frac{aa_0^2}{-4b_0^2},$$

where a_0, a_1 and b_0 are free parameters.

- **Second set:**

$$b_{-1} = b_1 = a_0 = 0, \quad h = \frac{4aa_1 a_{-1}}{b_0^2}, \quad \omega = a, \quad k = 0, \tag{22}$$

where a_{-1}, a_1 and b_0 are free parameters.

- **Third set:**

$$b_{-1} = b_0 = a_0 = a_1 = 0, \quad \omega = 4a, \quad k = 0, \quad h = 0, \tag{23}$$

where a_{-1} and b_1 are free parameters.

- **Fourth set:**

$$a_{-1} = \frac{a_0 b_0}{b_1}, \quad b_{-1} = a_1 = 0, \quad \omega = a, \quad k = 0, \quad h = 0, \tag{24}$$

where a_0, b_1 and b_0 are free parameters.

- **Fifth set:**

$$a_{-1} = a_1 = b_0 = 0, \quad \omega = \frac{-a(-b^2 a_0^4 + 16a^2 b_1^2 b_{-1}^2)}{(b^2 a_0^4)}, \tag{25}$$

$$k = \frac{-4ab_1 b_{-1}}{(ba_0^2)}, \quad h = 0,$$

where b_{-1}, b_1 and a_0 are free parameters.

- **Sixth set:**

$$a_{-1} = b_1 = a_0 = b_0 = 0, \quad \omega = 4a, \quad k = 0, \quad h = 0, \tag{26}$$

where a_1 and b_{-1} are free parameters.

- **Seventh set:**

$$a_1 = b_1 = 0, \quad b_{-1} = \frac{a_0 b_0}{a_{-1}}, \quad \omega = a, \quad k = 0, \quad h = 0, \tag{27}$$

where a_0, a_{-1} and b_0 are free parameters.

- **Eighth set:**

$$a_0 = b_0 = 0, \quad b_1 = \frac{-a_1 b_{-1}}{a_{-1}}, \quad \omega = \frac{-a(2b^2 a_{-1}^4 + a^2 b_{-1}^4)}{(b^2 a_{-1}^4)}, \tag{28}$$

$$k = \frac{ab_{-1}^2}{(ba_{-1}^2)}, \quad h = \frac{-aa_{-1}^2}{b_{-1}^2},$$

where a_{-1}, a_1 and b_{-1} are free parameters.

- **Ninth set:**

$$a_1 = \frac{(a_0^2 b_{-1}^2 - b_0^2 a_{-1}^2)}{4(a_{-1} b_{-1}^2)}, \quad b_1 = \frac{(a_0^2 b_{-1}^2 - b_0^2 a_{-1}^2)}{-4(a_{-1}^2 b_{-1})}, \tag{29}$$

$$\omega = \frac{a(8b^2 a_{-1}^4 + a^2 b_{-1}^4)}{-16(b^2 a_{-1}^4)}, \quad k = \frac{ab_{-1}^2}{4(ba_{-1}^2)}, \quad h = \frac{aa_{-1}^2}{-4b_{-1}^2},$$

where a_{-1}, b_{-1} and b_0 are free parameters.

5. TRAVELING WAVE HYPOTHESIS

The starting point for this section is equation (11) which is the corresponding ODE corresponding to the traveling wave hypothesis. Now (11) can be re-written as

$$\left(g'\right)^2 = a_1 g^4 + a_2 g^2 + a_3, \tag{30}$$

where

$$a_1 = \frac{\lambda\kappa}{a}, \tag{31}$$

$$a_2 = \frac{\omega + a\kappa^2}{a} \tag{32}$$

and

$$a_3 = -\frac{h}{a}. \tag{33}$$

Suppose g_j, for $1 \le j \le 4$, are the roots of the biquadratic equation

$$a_1 g^4 + a_2 g^2 + a_3 = 0, \tag{34}$$

then by the fundamental theorem of algebra (34) implies

$$a_1(g - g_1)(g - g_2)(g - g_3)(g - g_4) = 0. \tag{35}$$

Introducing the transformation

$$g = \frac{g_2(g_1 - g_4)w^2 - g_1(g_2 - g_4)}{(g_1 - g_4)w^2 - (g_2 - g_4)} \tag{36}$$

the corresponding ODE for w is

$$\left(w'\right)^2 - \left(1 - w^2\right)\left(1 - k^2 w^2\right), \tag{37}$$

where

$$k^2 = \frac{(g_2 - g_3)(g_1 - g_4)}{(g_1 - g_3)(g_2 - g_4)} \tag{38}$$

so that the solution of (37) is given in terms of the snoidal waves as

$$w = \operatorname{sn}\left(\sqrt{a_1}M; k\right), \tag{39}$$

where

$$M = \frac{\sqrt{(g_2 - g_4)(g_1 - g_3)}}{2}. \tag{40}$$

Then, in the limiting case, (39) leads to

$$\lim_{k \to 1} w = \tanh\left(\sqrt{a_1} M\right), \tag{41}$$

which is the shock wave solution.

6. CONCLUSIONS

This paper studies the chiral NLSE in presence of Bohm potential. In presence of this term, several mathematical tools and techniques are applied to extract several kinds of solution to this nonlinear evolution equation. The mathematical methods are exponential function method, G'/G approach as well as the traveling wave hypothesis. These lead to several types of solutions including the periodic functions, rational solutions, snoidal waves and in its limiting case shock waves. These variety of solutions will be extremely useful in carrying out further analysis of the chiral NLSE with Bohm potential. Thus, these solutions open up a wide arena of research possibilities in the are of Nuclear Physics.

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