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ENVELOPE SOLITONS, PERIODIC WAVES AND OTHER SOLUTIONS TO BOUSSINESQ-BURGERS EQUATION

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Abstract. This paper retrieves the topological soliton and cnoidal wave solutions to the Boussinesq-Burgers equations. By using the Jacobi elliptic function method, we find the exact periodic solutions for the considered model. Exact travelling wave solutions which include new envelope solitary and shock wave solutions are obtained. The conditions of the existence of the derived solutions are presented.

Key words: topological soliton, cnoidal wave solutions, Jacobi elliptic function method, Boussinesq-Burgers equations.

1. INTRODUCTION

Recently, searching for solitary wave solutions of nonlinear models by means of different methods has been one of the most excited advances of nonlinear science and theoretical physics. These exact solutions when they exist can help one to well understand the mechanism of the complicated physical phenomena and dynamical processes modelled by these nonlinear evolution equations [1]. The existence of solitary wave solutions implies perfect balance between nonlinearity and dispersion which usually requires rather specific conditions and cannot be established in general [2]. The most important property of the solitary wave solution is that it is extremely stable, which led to their later name “soliton”, namely, their emerging from multi-solitary wave collisions with the same shapes and velocities with which they entered [3]. During the past few decades soliton theory has been applied to numerous practical and fundamental problems in areas as diverse as hydrodynamics, plasma, nonlinear optics, molecular biology, field theory, and astrophysics [4].

It is well known that there are several systematic approaches to obtain explicit solutions of soliton equations, such as the inverse scattering transformation (IST), the subsidiary ordinary differential equation method (sub- ODE method for short) [5-8], solitary wave ansatz method [9, 10], sine-cosine method [11, 12], Hirota bilinear method [13, 14], F-expansion method [15], the Jacobi elliptic function expansion method [1, 16] and so on. Without these modern methods of integrability, many such equations would not have been solved, thus leaving many scientific questions unanswered [9]. Among the various methods, the Jacobi elliptic function expansion method has been proved to be one of the most useful method to get explicit solutions of some single and coupled soliton equations [1, 16].

In this paper, we consider the Boussinesq-Burgers soliton equation [17, 18]

$$u_t = -2uu_x + \frac{1}{2}v_x, \tag{1}$$

$$v_t = \frac{1}{2}u_{xxx} - 2(uv)_x, \tag{2}$$

where x and t respectively represent the normalized space and time, the subscripts denote the derivatives, $u(x,t)$ is the horizontal velocity field (at the leading order it is the depth-averaged horizontal field) and $v(x,t)$ denotes the height of the water surface above a horizontal bottom [18]. It should be noted that this model describes the propagation of shallow water waves [18].

A problem of our interest consists in solving a family of Boussinesq-Burgers soliton equation of the form:

$$u_t + \alpha uu_x + \beta v_x = 0, \tag{3}$$

$$v_t + \gamma(uv)_x + \delta u_{xxx} = 0, \tag{4}$$

where α, β, γ and δ are nonzero real constants. If setting $\alpha = 2, \beta = -1/2$, $\gamma = 2$ and $\delta = -1/2$, the system of Eqs. (3) and (4) reduces to the model equations (1) and (2).

In this paper, we use the Jacobi elliptic function method to construct the exact periodic solutions of a family of Boussinesq-Burgers equations. As a result, some exact travelling wave solutions are obtained which include new envelope solitary and shock wave solutions. Importantly, these exact solutions can be useful to understand the dynamical process modelled by the considered model. We will see that the proposed method is concise and effective to solve coupled NLPDEs. Conditions for the existence of solitary wave solutions have also been reported.

2. EXACT SOLUTIONS

2.1. ANSATZ 1

In this section, we illustrate the applications of the Jacobi elliptic sine function expansion method to the Boussinesq-Burgers soliton equations (3) and (4).

Let us first assume the travelling wave solution of Eqs. (3) and (4) in the form

$$u(x,t)=U(\xi),\ \ v(x,t)=V(\xi),\ \ \xi=k(x-ct), \tag{5}$$

where k and c are the wave number and wave speed, respectively. Substituting (5) into Eqs. (3) and (4), we have

$$-kcU' + \alpha kUU' + \beta kV' = 0, \tag{6}$$

$$-kcV' + \gamma k\left(UV\right)' + \delta k^3U''' = 0, \tag{7}$$

where the prime denotes the derivative with respect to the variable ξ.

Integrating the ODE (6) once yields the relationship between V and U as follows:

$$V=\frac{c}{\beta}U-\frac{\alpha}{2\beta}U^2+K,\ \ \beta\neq 0, \tag{8}$$

where K is an integration constant. The substitution of (8) into the second equation (7) yields the ODE:

$$\left[\gamma K-\frac{c^2}{\beta}\right]U' + \frac{\left(\alpha+2\gamma\right)c}{\beta}UU' - \frac{3\alpha\gamma}{2\beta}U^2U' + \delta k^2U''' = 0. \tag{9}$$

Integrating (9) once yields

$$\left[\gamma K-\frac{c^2}{\beta}\right]U + \frac{\left(\alpha+2\gamma\right)c}{2\beta}U^2 - \frac{\alpha\gamma}{2\beta}U^3 + \delta k^2U'' = 0. \tag{10}$$

By the Jacobi elliptic function expansion method, $U(\xi)$ can be expressed as a finite series of Jacobi elliptic function, $\mathrm{sn}\xi$, i.e., the ansatz [1, 16]

$$U\left(\xi\right)=\sum_{i=0}^{n}a_i\mathrm{sn}^i\left(\xi\right), \tag{11}$$

is made and its highest degree is [1, 16]

$$O\left[U\left(\xi\right)\right]=n. \tag{12}$$

Here, in (11), a_i are constants to be determined later and $\mathrm{sn}\left(\xi\right)=\mathrm{sn}(\xi\,|\,m)$ is the Jacobi elliptic sine function where m is the modulus of the elliptic function and n is a parameter which can be found by balancing the highest-order linear term with the nonlinear terms.

Note that the Jacobi elliptic functions possess properties of triangular functions [1]

$$\begin{gathered}\mathrm{sn}^2\left(\xi\right)+\mathrm{cn}^2\left(\xi\right)=1,\\ \mathrm{dn}^2\left(\xi\right)+m^2\mathrm{sn}^2\left(\xi\right)=1,\\ \left(\mathrm{sn}\left(\xi\right)\right)'=\mathrm{cn}\left(\xi\right)\mathrm{dn}\left(\xi\right),\\ \left(\mathrm{cn}\left(\xi\right)\right)'=-\mathrm{sn}\left(\xi\right)\mathrm{dn}\left(\xi\right),\\ \left(\mathrm{dn}\left(\xi\right)\right)'=-m^2\mathrm{sn}\left(\xi\right)\mathrm{cn}\left(\xi\right).\end{gathered} \tag{13}$$

Here, $\mathrm{cn}\left(\xi\right)$ and $\mathrm{dn}\left(\xi\right)$ are, respectively, the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind.

Balancing the highest-order derivative U'' with the nonlinear U^3 term in Eq. (10), we get $n=1$ in Eq. (11). Accordingly, Eq. (11) modifies to:

$$U\left(\xi\right)=a_0+a_1\mathrm{sn}\left(\xi\right) \tag{14}$$

where a_0 and a_1 are coefficients to be determined later. From (14), we get

$$\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}=-\left(1+m^2\right)a_1\mathrm{sn}\left(\xi\right)+2m^2a_1\mathrm{sn}^3\left(\xi\right), \tag{15}$$

$$U^2=a_0{}^2+2a_0a_1\mathrm{sn}\left(\xi\right)+a_1{}^2\mathrm{sn}^2\left(\xi\right), \tag{16}$$

$$U^3=a_0{}^3+3a_0{}^2a_1\mathrm{sn}\left(\xi\right)+3a_0a_1{}^2\mathrm{sn}^2\left(\xi\right)+a_1{}^3\mathrm{sn}^3\left(\xi\right). \tag{17}$$

Let us take in Eq. (10)

$$\eta_1=\frac{\gamma K\beta-c^2}{\beta\delta k^2},\ \ \eta_2=\frac{\left(\alpha+2\gamma\right)c}{2\beta\delta k^2},\ \ \eta_3=\frac{\alpha\gamma}{2\beta\delta k^2}. \tag{18}$$

Thus, we can write Eq. (10) as an ordinary differential equation in the form

$$U''+\eta_1U+\eta_2U^2-\eta_3U^3=0. \tag{19}$$

Substituting (15)-(17) into (19) and equating the coefficients of all powers of $\text{sn}^i(\xi)$ ($i = 0,1,2,3$) yields a set of algebraic equations for a_0, a_1:

$$\eta_1 a_0 + \eta_2 a_0^2 - \eta_3 a_0^3 = 0, \tag{20}$$

$$-\left(1+m^2\right)a_1 + \eta_1 a_1 + 2\eta_2 a_0 a_1 - 3\eta_3 a_0^2 a_1 = 0, \tag{21}$$

$$\eta_2 a_1^2 - 3\eta_3 a_0 a_1^2 = 0, \tag{22}$$

$$2m^2 a_1 - \eta_3 a_1^3 = 0. \tag{23}$$

Solving the system of algebraic equations, we can distinguish two cases namely:

2.1.1. Case 1

We have: $$a_0 = 0,\ m = \sqrt{\eta_1 - 1},\ \eta_2 = 0,\ a_1 = \sqrt{\frac{2m^2}{\eta_3}}. \tag{24}$$

If we substitute the parameters η_1, η_2 and η_3 in Eq. (18) into Eq. (24), we get

$$a_0 = 0,\ m = \sqrt{\frac{\gamma K\beta - c^2}{\beta\delta k^2} - 1},\ \alpha + 2\gamma = 0,\ a_1 = 2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}, \tag{25}$$

which give the dependent coefficients in terms of the model coefficients α, β, γ and δ.

Substituting (25) into (14) and using Eqs. (5) and (8), we find that the Boussinesq-Burgers equations (3) and (4) admit a new exact periodic solutions in the form:

$$u(x,t) = 2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\text{sn}\left[k(x-ct)\right], \tag{26}$$

$$v(x,t) = \frac{2mkc}{\beta}\sqrt{\frac{\beta\delta}{\alpha\gamma}}\text{sn}\left[k(x-ct)\right] - \frac{2m^2k^2\delta}{\gamma}\text{sn}^2\left[k(x-ct)\right] + K. \tag{27}$$

As $m \to 1$, then $\text{sn}(\xi) = \tanh(\xi)$, thus the solitary wave solutions (26) and (27) are obtained as follows:

$$u(x,t) = 2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\tanh\left[k(x-ct)\right], \tag{28}$$

$$v(x,t) = \frac{2mkc}{\beta}\sqrt{\frac{\beta\delta}{\alpha\gamma}}\tanh\left[k(x-ct)\right] - \frac{2m^2k^2\delta}{\gamma}\tanh^2\left[k(x-ct)\right] + K, \tag{29}$$

which are shock wave solutions of the family of Boussinesq-Burgers equations (3) and (4).

2.1.2. Case 2

We have:
$$a_0 = \frac{\eta_2 + \sqrt{4\eta_3\eta_1 + \eta_2^2}}{2\eta_3}, \tag{30}$$

$$a_0 = \frac{\eta_2 + \sqrt{\eta_2^2 + 3\eta_1\eta_3 - 3\eta_3\left(1+m^2\right)}}{3\eta_3}, \tag{31}$$

$$a_0 = \frac{\eta_2}{3\eta_3}, \tag{32}$$

$$a_1 = \sqrt{\frac{2m^2}{\eta_3}}. \tag{33}$$

Consequently, one obtains from Eqs. (30–32) an important constraint equation between the parameters η_1, η_2 and η_3 as

$$\frac{\eta_2 + \sqrt{4\eta_3\eta_1 + \eta_2^2}}{2\eta_3} = \frac{\eta_2 + \sqrt{\eta_2^2 + 3\eta_1\eta_3 - 3\eta_3\left(1+m^2\right)}}{3\eta_3} = \frac{\eta_2}{3\eta_3}. \tag{34}$$

We can see that the coefficient a_0 will be determined from Eq. (30) or (31) or (32). Let us for example express the coefficient a_0 as function of the model coefficients α, β, γ and δ by making use of Eqs. (18) and (32):

$$a_0 = \frac{\left(\alpha + 2\gamma\right)c}{3\alpha\gamma}. \tag{35}$$

On the other hand, if we substitute the parameter η_3 in Eq. (18) into Eq. (33), we get the coefficient a_1 as function of the model coefficients α, β, γ and δ as

$$a_1 = 2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}, \tag{36}$$

which forces the constraint relation

$$\alpha\beta\gamma\delta > 0. \tag{37}$$

Substituting Eqs. (35) and (36) into (14) and using Eqs. (5) and (8), we get a final solutions in the form

$$u(x,t)=\frac{(\alpha+2\gamma)c}{3\alpha\gamma}+2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{sn}\left[k(x-ct)\right], \tag{38}$$

$$\begin{aligned}v(x,t)=&\frac{(\alpha+2\gamma)c^2}{3\alpha\beta\gamma}+\frac{2mkc}{\beta}\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{sn}\left[k(x-ct)\right]-\\&-\frac{\alpha}{2\beta}\left\{\frac{(\alpha+2\gamma)c}{3\alpha\gamma}+2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{sn}\left[k(x-ct)\right]\right\}^2+K,\end{aligned} \tag{39}$$

which are another pair of exact periodic solutions of the family of Boussinesq-Burgers equations (3) and (4). When $m\to 1$, then $\mathrm{sn}(\xi)=\tanh(\xi)$, and thus Eqs. (38) and (39) are respectively degenerated into the following forms

$$u(x,t)=\frac{(\alpha+2\gamma)c}{3\alpha\gamma}+2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\tanh\left[k(x-ct)\right], \tag{40}$$

$$\begin{aligned}v(x,t)=&\frac{(\alpha+2\gamma)c^2}{3\alpha\beta\gamma}+\frac{2mkc}{\beta}\sqrt{\frac{\beta\delta}{\alpha\gamma}}\tanh\left[k(x-ct)\right]-\\&-\frac{\alpha}{2\beta}\left\{\frac{(\alpha+2\gamma)c}{3\alpha\gamma}+2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\tanh\left[k(x-ct)\right]\right\}^2+K,\end{aligned} \tag{41}$$

which are shock wave solutions of the family of Boussinesq-Burgers equations (3) and (4).

2.2. ANSATZ 2

To seek the travelling wave solutions of Eq. (19), we assume another ansatz solution in the form [19, 20]

$$U(\xi)=a_0+\sum_{j=1}^{l}\mathrm{sn}^{j-1}(\xi)\left[a_j\mathrm{sn}(\xi)+b_j\mathrm{cn}(\xi)\right], \tag{42}$$

where $\mathrm{sn}(\xi)=\mathrm{sn}(\xi,m)$ and $\mathrm{cn}(\xi)=\mathrm{cn}(\xi,m)$. m is a is the modulus of the elliptic function and l is a parameter which can be found by balancing the highest-order linear term with the nonlinear terms. By balancing the highest-order derivative U'' term with the nonlinear U^3 term in Eq. (36), we get $l=1$. Therefore, the solution (36) reduces to

$$U(\xi)=a_0+a_1\mathrm{sn}(\xi)+b_1\mathrm{cn}(\xi), \tag{43}$$

where a_0, a_1 and b_1 are coefficients to be calculated from Eq. (19). From (43), we get

$$\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2} = -\left(1+m^2\right)a_1\mathrm{sn}(\xi) + 2m^2a_1\mathrm{sn}^3(\xi) - b_1\mathrm{cn}(\xi) + 2m^2b_1\mathrm{sn}^2(\xi)\mathrm{cn}(\xi), \quad (44)$$

$$U^2 = a_0^2 + a_1^2\mathrm{sn}^2(\xi) + b_1^2\mathrm{cn}^2(\xi) + 2a_0a_1\mathrm{sn}(\xi) + 2a_0b_1\mathrm{cn}(\xi) + 2a_1b_1\mathrm{sn}(\xi)\mathrm{cn}(\xi) \quad (45)$$

$$U^3 = a_0^3 + 3a_0b_1^2 + \left(3a_0^2a_1 + 3a_1b_1^2\right)\mathrm{sn}(\xi) + \left(3a_0^2b_1 + b_1^3\right)\mathrm{cn}(\xi) +$$
$$+\left(a_1^3 - 3a_1b_1^2\right)\mathrm{sn}^3(\xi) + \left(3a_1^2b_1 - b_1^3\right)\mathrm{sn}^2(\xi)\mathrm{cn}(\xi) +$$
$$+\left(3a_0a_1^2 - 3a_0b_1^2\right)\mathrm{sn}^2(\xi) + 6a_0a_1b_1\mathrm{sn}(\xi)\mathrm{cn}(\xi). \quad (46)$$

Substituting Eqs. (43–46) into Eq. (19), we obtain the following 7 independent parametric equations:

$$\eta_1a_0 + \eta_2a_0^2 + \eta_2b_1^2 - \eta_3a_0\left(a_0^2 + 3b_1^2\right) = 0, \quad (47)$$

$$-\left(1+m^2\right)a_1 + \eta_1a_1 + 2\eta_2a_0a_1 - 3a_1\eta_3\left(a_0^2 + b_1^2\right) = 0, \quad (48)$$

$$\left(a_1^2 - b_1^2\right)\left(\eta_2 - 3a_0\eta_3\right) = 0, \quad (49)$$

$$2m^2a_1 - \eta_3a_1\left(a_1^2 - 3b_1^2\right) = 0, \quad (50)$$

$$-b_1 + 2\eta_2a_0b_1 + \eta_1b_1 - \eta_3b_1\left(3a_0^2 + b_1^2\right) = 0, \quad (51)$$

$$2m^2b_1 - \eta_3b_1\left(3a_1^2 - b_1^2\right) = 0, \quad (52)$$

$$2a_1b_1\left(\eta_2 - 3\eta_3a_0\right) = 0. \quad (53)$$

From Eqs. (49) and (53), it follows that

$$a_0 = \frac{\eta_2}{3\eta_3}. \quad (54)$$

In view of Eqs. (50) and (52), one obtains

$$a_1^2 = -b_1^2. \quad (55)$$

Also, if we take Eqs. (48) and (51) it follows that

$$b_1^2 = -\frac{2m^2}{\eta_3}. \quad (56)$$

Further, we get from inserting Eq. (56) into Eq. (47) the constraint condition:

$$9\eta_1\eta_3 + 2\eta_2^2 = 0. \tag{57}$$

Substituting Eq. (18) into Eqs. (54–56) yields the coefficients a_0, a_1 and b_1 as functions of the model coefficients α, β, γ and δ as

$$a_0 = \frac{(\alpha + 2\gamma)c}{3\alpha\gamma}, \; a_1 = 2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}, \; b_1 = \pm i2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}. \tag{58}$$

According to Eqs. (5), (8), (37) and (58), we obtain the following periodic solutions of Boussinesq-Burgers equations (3) and (4):

$$u(x,t) = \frac{(\alpha + 2\gamma)c}{3\alpha\gamma} + 2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{sn}\left[k(x-ct)\right] \pm i2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{cn}\left[k(x-ct)\right], \tag{59}$$

$$\begin{aligned} v(x,t) = {} & \frac{(\alpha + 2\gamma)c^2}{3\alpha\beta\gamma} + \frac{2mkc}{\beta}\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{sn}\left[k(x-ct)\right] \pm i\frac{2mkc}{\beta}\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{cn}\left[k(x-ct)\right] - \\ & -\frac{\alpha}{2\beta}\left\{\frac{(\alpha + 2\gamma)c}{3\alpha\gamma} + 2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{sn}\left[k(x-ct)\right] \pm i2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{cn}\left[k(x-ct)\right]\right\}^2 + K. \end{aligned} \tag{60}$$

As $m \to 1$, then $\mathrm{sn}(\xi) = \tanh(\xi)$ and $\mathrm{cn}(\xi) = \mathrm{sech}(\xi)$. Thus, the solutions (59) and (60) reduces to

$$\begin{aligned} u(x,t) = {} & \frac{(\alpha + 2\gamma)c}{3\alpha\gamma} + 2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\tanh(\xi)\left[k(x-ct)\right] \pm \\ & \pm i2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{sech}(\xi)\left[k(x-ct)\right], \end{aligned} \tag{61}$$

$$\begin{aligned} v(x,t) = {} & \frac{(\alpha + 2\gamma)c^2}{3\alpha\beta\gamma} + \frac{2mkc}{\beta}\sqrt{\frac{\beta\delta}{\alpha\gamma}}\tanh(\xi)\left[k(x-ct)\right] \pm \\ & \pm i\frac{2mkc}{\beta}\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{sech}(\xi)\left[k(x-ct)\right] \\ & -\frac{\alpha}{2\beta}\left\{\frac{(\alpha + 2\gamma)c}{3\alpha\gamma} + 2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\tanh(\xi)\left[k(x-ct)\right] \pm \right. \\ & \left. \pm i2mk\sqrt{\frac{\beta\delta}{\alpha\gamma}}\mathrm{sech}(\xi)\left[k(x-ct)\right]\right\}^2 + K. \end{aligned} \tag{62}$$

These results show the complexiton solutions to the Boussinesq-Burgers equation.

3. EXPONENTIAL FUNCTION METHOD

3.1. DETAİLS OF THE METHOD

We consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_{xx}, u_{tx}, \cdots) = 0. \tag{63}$$

Using the transformation

$$u(x,t) = u(\xi), \xi = kx + ct, \tag{64}$$

where k, c are constants, we can rewrite eq. (59) in the following nonlinear ODE:

$$Q(u, cu', u', u'', cu'' \cdots) = 0, \tag{65}$$

where the prime denotes the derivation with respect to ξ. According to exp-function method, we assume that the solution of eq. (61) can be expressed in the form

$$u(\xi) = \frac{\sum_{n=-w}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \tag{66}$$

where w, d, p and q are positive integers which could be freely chosen, a_n and b_m are unknown constants to be determined. To determine the values of d and q, we balance the linear term of highest order in eq. (61) with the highest order nonlinear term. Similarly, to determine the values of w and p, we balance the linear term of lowest order in eq. (61) with the lowest order nonlinear term.

3.2. APPLICATION TO BOUSSINESQ-BURGERS EQUATION

We consider the Boussinesq-Burgers equations [1],

$$u_t = -2uu_x + \frac{1}{2}v_x, \tag{67}$$

$$v_t = \frac{1}{2}u_{xxx} - 2(uv)_x. \tag{68}$$

Introducing the transformation

$$\xi = kx + ct, \tag{69}$$

we can convert eqs. (63) and (64) into ordinary differential equations

$$cu' + 2kuu' - \frac{1}{2}kv' = 0, \tag{70}$$

$$cv' + 2k(uv' + vu') - \frac{1}{2}k^3u''' = 0, \tag{71}$$

where the prime denotes the derivative with respect to ξ. Integrating eqs. (63) and (64) once and setting the constant of integration to zero gives

$$cu + ku^2 - \frac{1}{2}kv = 0, \tag{72}$$

$$cv - \frac{1}{2}k^3u'' + 2kuv = 0. \tag{73}$$

Solving for v from eq. (68), gives

$$v = \frac{2(cu + ku^2)}{k} \tag{74}$$

Substituting v from (70) into (69) gives

$$\frac{2c(cu + ku^2)}{k} - \frac{1}{2}k^3u'' + 4u(cu + ku^2) = 0. \tag{75}$$

In order to determine the values of p, w, d and q, we balance u'' with u^3 that leads to

$$u'' = \frac{c_1 \exp[(3p + w)\xi] + \cdots}{c_2 \exp[4p\xi] + \cdots}, \tag{76}$$

and

$$u^3 = \frac{c_3 \exp[(3w + p)\xi] + \cdots}{c_4 \exp[4p\xi] + \cdots}, \tag{77}$$

where c_i are determined coefficients. Balancing highest order of exp-functions in (72) and (73), we have

$$3p + w = 3w + p, \tag{78}$$

which gives

$$p = w. \tag{79}$$

Similarly, balancing the lowest order of exp-functions in the following equations

$$u'' = \frac{\cdots + d_1 \exp[-(3q + d)\xi]}{\cdots + d_2 \exp[-4q\xi]}, \tag{80}$$

and

$$u^3 = \frac{\cdots + d_3 \exp[-(2d + q)\xi]}{\cdots + d_4 \exp[-4q\xi]}, \tag{81}$$

where d_i are determined coefficients, yields

$$-(3q + d) = -(3d + q), \tag{82}$$

which gives

$$q = d. \tag{83}$$

We can freely choose the values of w and d, but the final solution does not strongly depend upon the choice of values of w and d. For simplicity, we set $p = w = 1$ and $d = q = 1$, then eq. (62) becomes

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{84}$$

Substituting eq. (80) into eq. (71), and by the help of Maple, equating to zero the coefficients of all powers of $e^{n\xi}$ yields a set of algebraic equations for $a_0, b_0, a_1, a_{-1}, b_{-1}, b_1, c, k$. Solving this system of algebraic equations, we obtain the following results:

3.2.1. Case 1

We have: $a_1 = 0,\ a_{-1} = 0,\ b_1 = 0,\ b_0 = \pm\frac{2a_0}{k},\ c = \mp\frac{1}{2}k^2.$ (85)

So

$$u(x,t) = \frac{a_0 k}{\pm 2a_0 + kb_{-1}\mathrm{e}^{-(kx \mp \frac{1}{2}k^2 t)}}, \tag{86}$$

where a_0, b_{-1}, k are arbitrary constants.

3.2.2. Case 2

We have: $a_1 = 0,\ a_{-1} = 0,\ b_{-1} = 0,\ c = \pm\dfrac{2a_0^2}{b_0^2},\ k = \mp\dfrac{2a_0}{b_0}.$ (87)

So

$$u(x,t) = \frac{a_0}{b_1 \mathrm{e}^{kx \pm \frac{2a_0^2}{b_0^2}t} + b_0}, \tag{88}$$

where a_0, b_0, b_1 are arbitrary constants.

3.2.3. Case 3

We have: $a_0 = 0,\ a_1 = 0,\ b_1 = 0,\ b_{-1} = \pm\dfrac{2a_{-1}}{k},\ c = \mp\dfrac{1}{2}k^2.$ (89)

So the following exact solution

$$u(x,t) = \frac{ka_{-1}e^{-(kx \mp \frac{1}{2}k^2 t)}}{kb_0 \pm 2a_{-1}\mathrm{e}^{-(kx \mp \frac{1}{2}k^2 t)}}, \tag{90}$$

where b_0, a_{-1}, k are arbitrary constants.

4. G′/G METHOD

In this section, we first describe the G'/G-expansion method, then apply it to construct the traveling wave solutions for the Boussinesq-Burgers equations.

4.1. DETAILS OF THE METHOD

We suppose that the given nonlinear partial differential equation for $u(x,t)$ to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \tag{91}$$

where P is a polynomial in its arguments. The essence of the G'/G-expansion method can be presented in the following steps:

Step 1: Seek traveling wave solutions of eq. (87) by taking

$$u(x,t) = u(\xi),\ \ \xi = kx + ct, \tag{92}$$

and transform Eq. (87) to the ordinary differential equation

$$Q(u,u',u'',\cdots)=0, \tag{93}$$

where prime denotes the derivative with respect to ξ.

Step 2: If possible, integrate eq. term by term one or more times. This yields constants of integration. For simplicity, the integration constants can be set to zero.

Step 3: Introduce the solution $u(\xi)$ of Eq. (89) in the finite series form

$$u(\xi)=\sum_{i=0}^{m} a_i\left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{94}$$

where a_i are real constants with $a_m \neq 0$ to be determined, m is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''+\lambda G'+\mu G=0, \tag{95}$$

where λ and μ are real constants to be determined later.

Step 4: The positive integer m can be determined by considering the homogeneous balance the highest order derivatives and highest order nonlinearity appearing in the ODE.

Step 5: Substituting into (89) and using (91), collecting all terms with the same power of G'/G together, and then equating each coefficient of each powers of G'/G to zero, yields a set of algebraic equations for a_i, c, k, λ and μ.

Step 6: We solve the system with the aid of a computer algebra system (CAS), such as Maple. Depending on the sign of the discriminant $\Delta=\lambda^2-4\mu$, the solutions of Eq. (91) are well known to us. Finally, therefore, we can obtain exact solutions of the given eq.

4.2. APPLICATION TO BOUSSINESQ-BURGERS EQUATION

In this section, we will demonstrate the G'/G-expansion method on one of the well-known nonlinear evolution equations, namely, the Boussinesq-Burgers equations. According to step 4, with balancing the u^3 and u'' in Eq. (71), we get $3m=m+2$, hence $m=1$. We assume that Eq. (71) has the following formal solution:

$$u = a_0 + a_1\left(\frac{G'}{G}\right),\ a_1 \neq 0, \tag{96}$$

where a_0 and a_1 are constants to be determined later.

Substituting Eq. (92) along with eq. (91) into Eq. (71) and collecting all terms with the same power of G'/G together, the left-hand side of Eq. (71) are converted into polynomial in G'/G. Setting each coefficient of each powers of G'/G to zero, we derive a set of algebraic equations for a_0, a_1, k, c, λ, μ as follows:

$$\left(\frac{G'}{G}\right)^0 : \frac{2c^2 a_0}{k} + 6ca_0^2 - \frac{1}{2}k^3 a_1\lambda\mu + 4ka_0^3 = 0, \tag{97}$$

$$\left(\frac{G'}{G}\right)^1 : \frac{2a_1c^2}{k} + 12a_1ca_0 - \frac{1}{2}a_1k^3\lambda^2 - a_1k^3\mu + 12a_1ka_0^2 = 0,$$

$$\left(\frac{G'}{G}\right)^2 : 6a_1^2c + 12a_1^2a_0k - \frac{3}{2}k^3a_1\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^3 : -k^3a_1 + 4a_1^3k = 0.$$

Solving the above system, we have the following two sets of solutions:

$$\text{Set } 1: a_1 = \frac{k}{2},\ c = -2a_0k + \frac{1}{2}\lambda k^2,\ \mu = \frac{2a_0(-2a_0 + k\lambda)}{k^2}, \tag{98}$$

$$\text{Set } 2: a_1 = -\frac{k}{2},\ c = -2a_0k - \frac{1}{2}\lambda k^2,\ \mu = -\frac{2a_0(2a_0 + k\lambda)}{k^2}, \tag{99}$$

where a_0 and λ are arbitrary constants.

Substituting the solutions (94), (95) into (92), we have two types of traveling wave solutions of Eq. (71).

(i) Since $\Delta = \left(\frac{k\lambda - 4a_0}{k}\right)^2 > 0$, from (94), we have

$$u_1(x,t) = a_0 + \frac{1}{2}k\left\{-\frac{1}{2}\lambda + \frac{\beta_1(c_1\cosh(\beta_1\xi) + c_2\sinh(\beta_1\xi))}{c_2\cosh(\beta_1\xi) + c_1\sinh(\beta_1\xi)}\right\}, \tag{100}$$

where $\beta_1 = \pm\left(\dfrac{k\lambda - 4a_0}{2k}\right)$ and $\xi = kx - 2a_0kt + \dfrac{1}{2}\lambda k^2 t$.

(ii) Since $\Delta = \left(\dfrac{k\lambda + 4a_0}{k}\right)^2 > 0$, from (95), we obtain the following solution

$$u_2(x,t) = a_0 - \frac{1}{2}k\left\{-\frac{1}{2}\lambda + \frac{\beta_2(c_1\cosh(\beta_2\xi) + c_2\sinh(\beta_2\xi))}{c_2\cosh(\beta_2\xi) + c_1\sinh(\beta_2\xi)}\right\}, \tag{101}$$

where $\beta_2 = \pm\left(\dfrac{k\lambda + 4a_0}{2k}\right)$ and $\xi = kx - 2a_0kt - \dfrac{1}{2}\lambda k^2 t$.

(iii) When $\Delta = \lambda^2 - 4\mu = 0$ from (94) and (95), respectively we have

$$u(x,t) = \pm\frac{kc_2}{2(c_1 + c_2(kx + ct))}. \tag{102}$$

We observe that the case of $\Delta < 0$ does not arise in the BBE.

5. TRAVELING WAVE SOLUTIONS

In this section, the traveling wave solutions to the Boussinesq-Burgers equation will be obtained by direct integration of the ODE (10). Now (10) can be rewritten as

$$U'' = aU - bU^2 + c_1U^3, \tag{103}$$

where

$$a = \frac{1}{\delta k^2}\left(\frac{c^2}{\beta} - \gamma K\right), \tag{104}$$

$$b = \frac{(\alpha + 2\gamma)c}{2\beta\delta k^2}, \tag{105}$$

$$c_1 = -\frac{\alpha\gamma}{2\beta\delta k^2}. \tag{106}$$

Now, multiplying both sides of (103) by U' and integrating, while choosing the integration constant to be zero as the search is for soliton solutions, and then separating variables leads to

$$B(x-ct)=\sqrt{6}\int\frac{\mathrm{d}s}{U\sqrt{3c_1U^2-4bU+6a}}, \tag{107}$$

which leads to the 1-soliton solution as

$$U(x,t)=\frac{A}{D+\cosh[B(x-ct)]}, \tag{108}$$

where

$$A=\frac{6a}{\sqrt{4b^2-18ac_1}} \tag{109}$$

and

$$D=\frac{18abc}{\left(2b^2+9ac_1\right)\sqrt{4b^2-18ac_1}}, \tag{110}$$

which shows that the soliton solution will exist provided

$$9ac_1<2b^2. \tag{111}$$

Finally, $V(x,t)$ can be obtained from (8). Thus, the 1-soliton solution $U(x,t)$, that is obtained by the traveling wave hypothesis is given by (108) which will exist provided (111) holds true.

6. CONCLUSIONS

To summarise, we have derived the exact periodic solutions of Boussinesq-Burgers equations by using the Jacobi elliptic function expansion method. As a consequence, some types of travelling wave solutions are obtained in the limit cases $m\to 1$, which contain solitary wave solution, shock wave or envelope solitary wave solutions. Importantly, these exact solutions can be useful to understand the dynamical process obtained from the considered model.

Additionally, the exp-function method and the G'/G method are applied to obtain addition solution to the equation of study. Finally, a 1-soliton solution to this equation is also obtained. In future, the conserved quantities of this equation will be obtained and they will be reported in future publications.

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