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PLANE WAVE SOLUTIONS OF THE FIELD EQUATIONS OF GENERAL RELATIVITY - II

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Abstract. In this paper, we have deduced the metric of space-time with a plane symmetry have been introduced and studied by Taub (1951) for $Z = (t/z)$-type plane gravitational waves and obtained some new solutions of the field equations of general relativity than previously discovered. Furthermore we have discussed these solutions with solutions of space-time (Takeno 1961) has (t/z)-property, but no plane symmetry (in the sense of Taub) in general.

Key words: plane symmetry, plane gravitational waves, curvature tensor, Ricci tensor etc.

1. INTRODUCTION

Having known some of the beauties of plane symmetry A.H. Taub (1951); Bondi (1957); Bondi, Pirani-Robinson (1959) has defined and studied plane wave solutions by considering the concept of group of motions of space-time which plays a fundamental role in plane gravitational waves. According to Taub (1951) the space-time that admits the three parameter group of motions of the Euclidean plane is said to possess plane symmetry and is called as plane symmetric space-time. Such space-time has many properties equivalent to those of spherical symmetry. The plane symmetric space-time has been extensively studied by many authors from various standpoints. Some of them are Takeno (1957), Peres (1959), Nordtvedt and Pegele (1962), Karade and Gaikwad (1990), Karade and Adhav (1994), Karade and Katore (2000) and Karade and Deshmukh (2000) etc. By assuming the space-time has plane symmetry, Takeno (1957) has reduced the line element of space-time with a plane symmetry have been introduced and studied by Taub (1951) as

$$ds^2 = -A\left(dx^2 + dy^2\right) - Cdz^2 + Cdt^2, \tag{1.1}$$

where A and C are functions of Z and $Z = z - t$ and deduced that the space-time (1.1) be the plane wave solutions of field equations (1.3) of general relativity. In this paper, we intend to transform the space-time (1.1) by using suitable transformation. In brief the paper is organized as follows: field equations are presented in section 2. In section 3, we have transformed the metric (1.1). Section 4 related to the computations of curvature tensor and Ricci tensor. Components of electromagnetic energy momentum tensor obtained in section 5. Section 6 leads to solutions of field equation while we have considered electromagnetic field except gauge transformation in section 7. In last section we have compared the solutions with solutions given by Takeno (1961) space-time having no plane symmetry.

2. FIELD EQUATIONS

The field equations with electromagnetic field without matter in general relativity is given by

$$R_{ij} = -8\pi E_{ij}, \tag{2.1}$$

where R_{ij} is the Ricci tensor and E_{ij} is the electromagnetic energy tensor which is defined by

$$E_{ij} = \frac{1}{4} g_{ij} F_{kl} F^{kl} - F_{ik} F_{jl} g^{kl}, \tag{2.2}$$

where F_{ij} is the antisymmetric field tensor and g_{ij} is the fundamental tensor of space-time.

The antisymmetric field tensor F_{ij} satisfies the following generalized Maxwell equations

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0, \tag{2.3}$$

and

$$F^{ij}_{;j} = J^{\mu}, \tag{2.4}$$

where J^{μ} is the charge current four vectors and $\left(i, j, k, l, \mu = 1, 2, 3, 4\right)$. Here a semicolon (;) and a comma (,) followed by indices denote covariant and partial derivatives respectively. Here we consider pure radiation field only, hence we assume that the charge current four vectors is identically zero. Hence equation (2.4) becomes

$$F^{ij}_{;j} = 0. \tag{2.5}$$

3. TRANSFORMATION OF (1.1)

The metric (1.1) can be put in the form

$$\mathrm{d}s^2 = -A_1\left(\mathrm{d}x_1^{\ 2} + \mathrm{d}y_1^{\ 2}\right) - C_1\mathrm{d}z_1^{\ 2} + C_1\mathrm{d}t_1^{\ 2}. \tag{3.1}$$

Employing the transformation

$$x_1 = x\ , \qquad y_1 = y\ , \quad z_1 = h_1 z^2 + k_1\ , \qquad t_1 = h_1 z^2 - k_1\ , \tag{3.2}$$

where h_1 and k_1 are arbitrary functions of $Z\left(= t/z\right)$ subject to the following two conditions (3.3) and (3.5).

The first condition is

$$\det\left(\partial x_1^i / \partial x^j\right) = -4\ h_1 \overline{k_1} \neq 0. \tag{3.3}$$

The metric (3.1) becomes

$$\begin{aligned}\mathrm{d}s^2 = -A_1\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) - 4C_1\Big[&\left(2\overline{k_1}/z\right)\left(zh_1 - \overline{h_1}t\right)\mathrm{d}z\mathrm{d}t - \\ &-\left(t\overline{k_1}/z^2\right)\left(2zh_1 - \overline{h_1}t\right)\mathrm{d}z^2 + \left(\overline{h_1}\,\overline{k_1}\right)\mathrm{d}t^2\Big]\ ,\end{aligned} \tag{3.4}$$

where a bar over a letter denotes the derivative with respect to Z. In the metric (3.4) the term $\mathrm{d}z\mathrm{d}t$ will vanish if

$$\overline{k_1}\left(zh_1 - \overline{h_1}t\right) = 0 \quad \text{i.e} \quad zh_1 - \overline{h_1}t = 0\ . \tag{3.5}$$

Since $\overline{k_1} \neq 0$ from (3.3) or $h_1 = \xi_1 Z$, where $\xi_1 = \overline{h_1}$ being arbitrary function of Z. Putting the value of h_1 in (3.4) we have

$$\mathrm{d}s^2 = -A_1\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + \left(4C_1\xi_1\overline{k_1}\right)Z^2\mathrm{d}z^2 - \left(4C_1\xi_1\overline{k_1}\right)\mathrm{d}t^2. \tag{3.6}$$

Further writing $-\left(4B_1\xi_1\overline{k_1}\right) = C$, the metric (3.6) becomes

$$\mathrm{d}s^2 = -A\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) - CZ^2\mathrm{d}z^2 + C\mathrm{d}t^2, \tag{3.7}$$

where $A = A_1$, $C = -4C_1\xi_1\overline{k_1}$ and $k_1 = \left(z_1 - t_1\right)/2 = u/2$, $u \equiv \left(z_1 - t_1\right)$. As A, C and u are given above are all functions of $Z = \left(t/z\right)$, the metric (3.7) is of $Z = \left(t/z\right)-$ type. The signature condition require that $C_1\xi_1\overline{k_1} < 0$ or $C_1\xi_1\overline{k_1} > 0$. In the latter case, the roles of z and t must be interchanged and when this interchange is performed, the metric takes the same form as in the former case. So we have only to deal with the former case, which can be shown to hold by the choice of ξ_1 in a suitable domain of *Z*.

4. SPACE-TIME HAVING BOTH PLANE SYMMETRY AND THE (t/z) -PROPERTY

[A] *We assumed that our space-time has plane symmetry and the components of fundamental tensor* g_{ij} *and electromagnetic field tensor* F_{ij} *have the functions of single variable* $Z \equiv (t/z)$ *– type in a co-ordinate system suitably chosen in which the plane gravitational waves* g_{ij} *and electromagnetic waves* F_{ij} *are propagating in the direction of z-axis have* (t/z) *- property.*

We have

$$g_{ij} = g_{ij}(Z) \ ; \quad Z = t/z , \tag{4.1}$$

where $A = A(Z)$ and $C = C(Z)$ and $A = A(Z)$ must satisfy

$$Z_{,3} = \varphi Z_{,4} . \tag{4.2}$$

Using $Z = t/z$ the equation (4.2) implies $\phi = -Z$ and the non-vanishing components of g_{ij} and g^{ij} are given by the metric (3.7). Here we use the notations (x^1, x^2, x^3, x^4) corresponding to (x, y, z, t).

The non-vanishing components of the Christoffel symbol, the curvature tensor R_{ijkl} and Ricci tensor R_{ij} made from (3.7) are respectively

$$Z\begin{Bmatrix}3\\11\end{Bmatrix} = Z\begin{Bmatrix}3\\22\end{Bmatrix} = \begin{Bmatrix}4\\11\end{Bmatrix} = \begin{Bmatrix}4\\22\end{Bmatrix} = \frac{\bar{A}}{2Cz},$$

$$\frac{1}{Z}\begin{Bmatrix}1\\13\end{Bmatrix} = \frac{1}{Z}\begin{Bmatrix}2\\23\end{Bmatrix} = -\begin{Bmatrix}1\\14\end{Bmatrix} = -\begin{Bmatrix}2\\24\end{Bmatrix} = -\frac{\bar{A}}{2Az},$$

$$\begin{Bmatrix}3\\33\end{Bmatrix} = -Z\begin{Bmatrix}3\\34\end{Bmatrix} = -\frac{1}{Z}\begin{Bmatrix}4\\33\end{Bmatrix} = -\frac{\left(2C + \bar{C}Z\right)}{2Cz},$$

$$\frac{1}{Z}\begin{Bmatrix}4\\34\end{Bmatrix} = -\begin{Bmatrix}4\\44\end{Bmatrix} = Z\begin{Bmatrix}3\\44\end{Bmatrix} = -\frac{\overline{C}}{2Cz}, \tag{4.3}$$

$$\frac{R_{1313}}{Z^2} = \frac{R_{2323}}{Z^2} = -\frac{R_{2324}}{Z} = -\frac{R_{1314}}{Z} = R_{1414} = R_{2424} = \Phi, \tag{4.4}$$

and

$$\frac{R_{33}}{Z^2} = -\frac{R_{34}}{Z} = R_{44} = \frac{2\Phi}{A}, \tag{4.5}$$

where

$$\Phi = \Phi\ (Z) = \frac{\overline{\overline{A}}}{2z^2} - \frac{\overline{A}^2}{4Az^2} - \frac{\overline{A}\,\overline{C}}{2Cz^2}. \tag{4.6}$$

And a bar over a kernel letter means the derivative with respect to Z.

From equations (4.3) and (4.5), we have

$$\begin{Bmatrix}a\\1a\end{Bmatrix} = \begin{Bmatrix}2\\2a\end{Bmatrix} = 0, \qquad \begin{Bmatrix}a\\3a\end{Bmatrix} = -Z\begin{Bmatrix}a\\4a\end{Bmatrix}, \qquad Z\begin{Bmatrix}a\\4a\end{Bmatrix} = \frac{\overline{m}}{2m} + \frac{1}{Z} + \frac{\overline{C}}{C}, \tag{4.7}$$

where

$$m = g_{11}g_{22} - g_{12}^2, \tag{4.8}$$

$$R = g^{ij}R_{ij} = 0. \tag{4.9}$$

From (4.9) we see that the scalar curvature of space-time vanishes. This is the necessary condition satisfying the field equation (2.1).

5. ELECTROMAGNETIC FIELD

To calculate the components of energy tensor E_{ij} we shall assume that the electromagnetic field F_{ij} is transverse electromagnetic and its components are functions of Z and $Z = (t/z)$ in the co-ordinate system under consideration. That is

$$zF_{ij} = f_{ij}(Z), \quad Z = t/z. \tag{5.1}$$

Here the transverse electromagnetic means that both electric and magnetic fields have no surviving components in the direction of the wave propogation. Since we are considering the waves propagating in the positive direction of the z-axis, they are expressed as

$$F_{12} = F_{34} = 0\,. \tag{5.2}$$

If we substitute (5.2) into (2.3), the non-vanishing components of F_{ij} becomes

$$zF_{23} = \rho \qquad tF_{24} = -\rho + c_1\ , \quad zF_{31} = \sigma + c_2\ , \quad tF_{14} = \sigma, \tag{5.3}$$

where c_1 and c_2 are arbitrary constants and ρ and σ are functions of $Z = (t/z)$.

From (3.7) and (5.3), we have the non-vanishing components of F^{ij}

$$tZF^{23} = \frac{\rho}{\sqrt{mC}}, \quad tF^{24} = -\frac{(-\rho + c_1)}{\sqrt{mC}},$$

$$tZF^{31} = \frac{(\sigma + c_2)}{\sqrt{mC}}, \quad tF^{14} = \frac{-\sigma}{\sqrt{mC}}, \tag{5.4}$$

and

$$t^2 F_{ij} F^{ij} = \frac{4}{\sqrt{mC}}\left(\rho c_1 + \sigma c_2 - \frac{1}{2}\left(c_1^2 - c_2^2\right)\right). \tag{5.5}$$

From (2.2), (3.7) (5.3), (5.4) and (5.5), we have obtained the non-vanishing components of energy tensor E_{ij} as follows:

$$t^2 E_{11} = \frac{\mu}{C}, \quad t^2 E_{12} = \frac{-(\rho c_1 + \rho c_2)}{C},$$

$$z^2 E_{33} = t^2 E_{44} = \frac{a + \mu}{\sqrt{m}}, \ ztE_{34} = \frac{-a}{\sqrt{m}} + \frac{(\rho c_1 - \rho c_2)}{\sqrt{m}} \quad , \tag{5.6}$$

where

$$a = \sigma^2 + \rho^2\,, \quad \mu = -\rho c_1 + \sigma c_2 + \frac{1}{2}\left(c_1{}^2 + c_2{}^2\right). \tag{5.7}$$

Using (5.3), (4.3) and (5.4), we find that (5.8) satisfied identically.

$$F^{ij}_{;j} = 0\,. \tag{5.8}$$

6. SOLUTION OF THE FIELD EQUATIONS

If we substitute (4.5) and (5.6) in (2.1), we have

$$\rho c_1 - \sigma c_2 = 0, \tag{6.1}$$

$$\rho c_1 + \sigma c_2 = 0, \tag{6.2}$$

$$\mu = 0 \tag{6.3}$$

$$2\varphi t^2 = -8\pi a, \tag{6.4}$$

where $a = \sigma^2 + \rho^2$ and $\varphi = \dfrac{\bar{\bar{A}}}{2z^2} - \dfrac{\bar{A}^2}{4Az^2} - \dfrac{\bar{A}\bar{C}}{2Cz^2}$.

Solving (6.1) and (6.2), we obtain

$$c_1 = c_2 = 0. \tag{6.5}$$

And accordingly the non-vanishing independent components of F_{ij}, F^{ij} and E_{ij} becomes

$$tF_{24} = -zF_{23} = -\rho\,, \quad tF_{14} = zF_{31} = \sigma, \tag{6.6}$$

$$tZF^{23} = tF^{24} = \frac{\rho}{\sqrt{mC}}\,, \quad tZF^{31} = tF^{14} = \frac{\sigma}{\sqrt{mC}}\,,$$

and

$$z^2 E_{33} = -ztE_{34} = t^2 E_{44} = \frac{\sigma^2 + \rho^2}{\sqrt{m}}, \tag{6.7}$$

where ρ and σ are any functions of $Z = t/z$.

Again
$$F_{ij}F^{ij} = 0, \tag{6.8}$$

satisfied identically by any F_{ij} of the form (6.6). Equation (6.8) shows that electromagnetic field F_{ij} is null. The equation (6.4) is equivalent to

$$Z^2 P^* = -8\pi\left(\sigma^2 + \rho^2\right)\,, \tag{6.9}$$

where

$$P^* = \bar{\bar{A}} - \frac{\bar{A}^2}{2A} - \frac{\bar{A}\bar{C}}{C} = \frac{1}{2\sqrt{m}}\left[\frac{\bar{m}^2}{4m} - \bar{\bar{m}} - \frac{\bar{m}\bar{C}}{C}\right]. \tag{6.10}$$

Conversely, it is evident that g_{ij} given by (3.7) and F_{ij} given by (6.6) and (5.2) satisfy all field equations (2.1) (2.3) and (2.5) only if the relation (6.9) is satisfied. Hence (6.9) is only one relation among the three functions A, B and a and there exist infinitely many sets of $\left(A, B, a\right)$ satisfying this relation. Thus we have the result:

THEOREM 1. *The most general solutions of the field equations (2.1) (2.3) and (2.5) is composed of the* g_{ij} *given by (3.7) and the* F_{ij} *given by (5.2) and (6.6) both satisfy the one condition (6.9) where* P^* *is defined by (6.10).*

7. ELECTROMAGNETIC FIELD EXCEPT GAUGE TRANSFORMATIONS

In this section we deduce F_{ij} given by (6.6) from (2.1) and (2.3).

THEOREM. *Let* g_{ij} *given by (3.7) and let it satisfy (2.1) and (2.3) together with some* F_{ij}. *If we assume the following (7.1) then* F_{ij} *becomes of the form (6.6).*

Proof. We assume the components of four potentials (k_i) whose existence is assured by the generalized Maxwell equations (2.3) are functions of *Z* in the co-ordinate system under consideration except for any gauge transformation. This corresponds to the condition (4.1) which asserts that the components of the gravitational potential are functions of *Z*.

We define

$$F_{ij} \quad \text{as} \quad F_{ij} = k_{j,i} - k_{i,j}, \tag{7.1}$$

where $k_i = k_i(Z)$ the arbitrary functions of are Z.

Let g_{ij} be given by (3.7) and let it satisfy (2.1) and (2.3) together with F_{ij}. Then from (7.1) with (5.1), we have

$$Zf_{14} = f_{31} = \sigma, \quad Zf_{24} = -f_{23} = -\rho, \quad f_{34} = \nu, \quad f_{12} = 0. \tag{7.2}$$

We have put, $\sigma = -Z\overline{k_1}$, $\rho = -Z\overline{k_2}$, $\nu = -\ \overline{k_3} + Z\overline{k_4}$, where ρ and σ are any functions of Z. The non-vanishing independent components of F^{ij} and E_{ij} becomes

$$zF^{24} = zZF^{23} = \frac{\rho}{\sqrt{m}CZ}, \quad zF^{14} = zZF^{13} = -\frac{\sigma}{\sqrt{m}CZ}$$

$$zF^{34} = -\frac{\nu}{Z^2C^2}, \qquad F^{12} = 0 \quad \text{and} \quad z^2F_{ij}F^{ij} = -\frac{2\nu^2}{Z^2C^2}. \tag{7.3}$$

The $(a\,\alpha)-$ components $(a = 1, 2;\ \ \alpha = 3, 4)$ of E_{ij} becomes from (2.2),

$$z^2E_{13} = -z^2ZE_{14} = -\frac{\sigma\nu}{ZC}, \qquad z^2E_{23} = -z^2ZE_{24} = \frac{\rho\nu}{ZC}.$$

From $E_{aa} = 0$ which is obtained from (3.7) and (4.3), we have the following two cases to consider.

Case i: *when* $\nu = 0$*,* F_{ij} *becomes (5.2) and (6.6) and the relation (6.9), where* P^* *defined by (6.10) is again satisfied.*

Case ii: *when* $\nu \neq 0$*, we must have* $\sigma = \rho = 0$*, which contradicts to* $E_{ab} = 0$*. Hence*

$$t^2 E_{ab} = -g_{ab} \frac{\nu^2}{2C^2} \neq 0. \tag{7.4}$$

Next the tensor ${}^*F_{ij}$, the dual of F_{ij} is defined by [see Takeno, 1957]

$${}^*F_{ij} = \frac{1}{2} \omega_{ijkl} F^{kl}, \tag{7.5}$$

where ω_{ijkl} being the tensor which is antisymmetric with respect to each pair of indices and $\omega_{1234} = \sqrt{-g}$.

From (3.7), (4.2), (4.3) and (7.5), we have the following theorem.

THEOREM 2. $R_{ijkl} F^{kl} = R_{ijkl}\, {}^*F^{kl} = R_{ijkl} F^{jk} = R_{ijkl}\, {}^*F^{jk} = 0.$

8. ASSUMPTIONS

The assumptions on which the present research is based are [A*] and [B]. In this section we intend to generalize the results obtained in section 4 and 5 by obtaining some new solutions of the field equations. First we give up the symmetry between *x* and *y*, the orthogonal between hyper surfaces *x* = constant and *y* = constant and that between *z* = constant and *t* = constant. In the line element of space-time and put the following assumption in place of [A]:

A*. *The line element of the space-time is given by*

$$\mathrm{d}s^2 = -A\mathrm{d}x^2 - 2D\mathrm{d}x\mathrm{d}y - B\mathrm{d}y^2 - Z^2\left(C - E\right)\mathrm{d}z^2 - 2ZE\mathrm{d}z\mathrm{d}t + \left(C + E\right)\mathrm{d}t^2, \tag{8.1}$$

where A, B, C, D and E are functions of Z, the quadratic form ($\mathrm{d}l^2 = A\mathrm{d}x^2 + 2D\mathrm{d}x\mathrm{d}y + B\mathrm{d}y^2$) *is positive definite, and C > 0.*

B. *The space-time having (t/z)-property.*

The Curvature and Ricci Tensor for (8.1). The non-vanishing components of the contravarient tensor g^{ij} from the line element (2.1.3) are

$$(g^{ij}) = \begin{bmatrix} -\frac{B}{m} & \frac{D}{m} & 0 & 0 \\ \frac{D}{m} & -\frac{A}{m} & 0 & 0 \\ 0 & 0 & -\frac{C+E}{C^2Z^2} & -\frac{E}{C^2Z} \\ 0 & 0 & -\frac{E}{C^2Z} & \frac{C-E}{C^2} \end{bmatrix}, \tag{8.2}$$

where $m = (AB - D^2) > 0$, $A,\ B > 0,\ C > |E|$.

The non-vanishing components of the Christoffel symbol are

$$\frac{1}{Z}\Gamma^1_{13} = -\Gamma^1_{14} = \frac{1}{2mz}(D\overline{D} - B\overline{A}), \quad \frac{1}{Z}\Gamma^1_{23} = -\Gamma^1_{24} = \frac{1}{2mz}(D\overline{B} - B\overline{D}),$$

$$\frac{1}{Z}\Gamma^2_{13} = -\Gamma^2_{14} = \frac{1}{2mz}(D\overline{A} - A\overline{D}), \quad \Gamma^3_{44} = \frac{(E/C - Z\overline{C}/2C + L)}{Z^2 z},$$

$$\Gamma^4_{44} = \frac{(Z\overline{C}/2C + L)}{Zz}, \quad Z\Gamma^3_{11} = \Gamma^4_{11} = \frac{\overline{A}}{2Cz}, \quad Z\Gamma^3_{22} = \Gamma^4_{22} = \frac{\overline{B}}{2Cz},$$

$$\frac{1}{Z}\Gamma^2_{23} = -\Gamma^2_{24} = \frac{1}{2mz}(D\overline{D} - A\overline{B}), \quad Z\ \ \Gamma^3_{12} = \Gamma^4_{12} = \frac{\overline{D}}{2Cz},$$

$$\Gamma^3_{33} = \frac{(-1 - E/C - Z\overline{C}/2C + L)}{z}, \quad \Gamma^4_{34} = \frac{(E/C - Z\overline{C}/2C - L)}{z},$$

$$\Gamma^3_{34} = \frac{(1 + Z\overline{C}/2C - L)}{Zz}, \quad \Gamma^4_{33} = \frac{Z(1 - 2E/C + Z\overline{C}/2C + L)}{z}, \tag{8.3}$$

where $L = \frac{Z\overline{E}}{2C} + \frac{E^2}{C^2} - \frac{ZE\overline{C}}{C^2}$, etc. Here bar $(-)$ over a letter means derivative with respect to Z.

The components of the curvature tensor R_{ijkl} are easily calculated, and its non-vanishing components are

$$\frac{1}{Z^2}R_{1313} = -\frac{1}{Z}R_{1314} = R_{1414} = \frac{U'}{z^2},$$

$$\frac{1}{Z^2}R_{1323} = -\frac{1}{Z}R_{1324} = R_{1424} = \frac{W'}{z^2},$$

$$\frac{1}{Z^2}R_{2323} = -\frac{1}{Z}R_{2324} = R_{2424} = \frac{V'}{z^2}. \tag{8.4}$$

The non-vanishing components of Ricci tensor are obtained from (8.4) are as follows

$$\frac{R_{33}}{Z^2} = -\frac{R_{34}}{Z} = R_{44} = \frac{Q'}{z^2}, \tag{8.5}$$

where $Q' = \dfrac{AV' + BU' - 2DW'}{m}$,

$$2U' = \overline{\overline{A}} - \frac{(A\overline{D}^2 + B\overline{A}^2 - 2D\overline{A}\overline{D})}{2m} - \frac{\overline{A}\left(\overline{C} - E/Z\right)}{C},$$

$$2V' = \overline{\overline{B}} - \frac{(A\overline{B}^2 + B\overline{D}^2 - 2D\overline{B}\overline{D})}{2m} - \frac{\overline{B}\left(\overline{C} - E/Z\right)}{C},$$

$$2W' = \overline{\overline{D}} - \frac{(B\overline{A}\overline{D} + A\overline{B}\overline{D} - D\overline{A}\overline{B} - D\overline{D}^2)}{2m} - \frac{\overline{D}\left(\overline{C} - E/Z\right)}{C}, \text{ etc.}$$

Thus we have:

1) From (8.4) and (8.5) it is interesting that $R_{ij} = 0$ not equivalent to $R_{ijkl} = 0$. But it is not necessarily the case with (3.7) [see (4.4) and (4.5)]. Furthermore if we put $A=B$ and $D=E=0$ in the results of this section, we have the corresponding formulas obtained in section 4, 5 and 6 concerning the space-time (3.7) having plane symmetry.

2) Also from (8.4), the space-time is Minkowskian when and only when $U'=V'=W'=0$. Hence the space-time mentioned above (8.1), therefore not Minkowskian in general. However when the space-time (3.7) having plane symmetry, we have $U'= V'=\Phi$ and $W' = 0$.

3) It should be clear that whatever the solutions obtained in this paper which are quite of new type those are completely different from solutions dealt in section 8. Further they can be easily reduced to the results previously obtained under the different conditions.

4) The most general solution of the field equations (2.1), (2.3) and (2.5) is composed of the fundamental tensor g_{ij} given by (3.7) and electromagnetic field F_{ij} given by (6.6) both satisfying only one relation (6.9). The solution (3.7), (5.3) and (5.4) with (6.9) represents a system in which g_{ij} and F_{ij} are propagating in the positive direction of z-axis of the generalized cartesian co-ordinates.

5) The equation (6.9) shows that the space-time and electromagnetic field cannot be independent of each other. This means that (6.9) indicates that the electromagnetic field takes part in the structure of space-time through its energy tensor F_{ij}.

6) Furthermore when the electromagnetic field is absent *i.e.* when $\rho = \sigma = 0$, space-time becomes Minkowskian and hence field equation (2.1) reduces to $R_{ij} = 0$ and g_{ij} become a solution of the purely gravitational equations for empty space-time.

7) It is interesting to note that the Takeno's (1957, 1961) formats are retained in our results.

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