Romanian Reports in Physics, Vol. 64, Supplement, P. 1399–1412, 2012

Dedicated to Professor Ioan-Iovitz Popescu's 80th Anniversary

VARIETIES OF EXACT SOLUTIONS FOR THE (2+1)-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION WITH THE TRAPPING POTENTIAL

WEI-PING ZHONG[*1,2], MILIVOJ R. BELIĆ[2] , DUMITRU MIHALACHE[3], BORIS A. MALOMED[4], TINGWEN HUANG[2]

[1] Shunde Polytechnic, Department of Electronic and Information Engineering, Guangdong Province, Shunde 528300, China
[2] Texas A&M Univsersity at Qatar, P.O. Box 23874 Doha, Qatar
[3] "Horia Hulubei" National Institute for Physics and Nuclear Engineering, P.O. Box MG-6, 077125 Bucharest-Magurele, Romania
[4]Tel Aviv University, Faculty of Engineering, School of Electrical Engineering, Department of Physical Electronics, IL-69978 Tel Aviv, Israel and ICFO Institut de Ciencies Fotoniques, E-08860 Castelldefels, Barcelona, Spain
*E-mail: zhongwp6@126.com

Received October 12, 2012

Abstract. The separation of variables and Hirota's bilinearization technique are applied to solve, in Cartesian coordinates, the (2+1)-dimensional nonlinear Schrödinger (NLS) equation with specially chosen external trapping potentials. The inverse problem is introduced and solved, aimed at finding the trapping potential that supports the prescribed form of the solution. It consists in appropriately selecting two arbitrary functions, which describe the exact solutions of the NLS equation, and defining the trapping potential through Hirota's bilinear forms, which involve the two functions. A variety of localized wave structures, such as single and multiple dromions ("dromion lattices"), ring, parabolic, and breather modes, and two-dimensional square-shaped peakons, are found. The stability of these solutions is checked by propagating them numerically for long distances.

Key words: Hirota's bilinearization technique, trapping potentials, dromions, two-dimensional peakons.

1. INTRODUCTION

The search for physically meaningful nonlinear integrable systems has produced a number of classical results of the modern nonlinear science [1]. Diverse approximate methods have been elaborated for realistic settings where exact solutions are not available, see Ref. [2] for relevant reviews and books. It is commonly known that the Fourier transformation and the separation of variables (SoV) methods are most often used for solving linear equations with constant coefficients. The celebrated inverse-scattering transform, based on the Lax-pair representation, can be viewed as a natural extension of the Fourier transform for the study of integrable [1] and nearly integrable [2] nonlinear systems. While usually it is difficult to exactly solve nonlinear physical models by means of the SoV approach, a procedure of this type was put forward, in the form of *symmetry constraints* for Lax pairs of integrable systems [3]. Further, a special type of the SoV approach for obtaining new solutions of certain (2+1)-dimensional [(2+1)D] integrable models was developed in Refs. [4, 5]. Some physically relevant (2+1)D integrable systems, such as the Nizhnik-Novikov-Veselov [6] and Davey-Stewartson (DS) equations [7], and the asymmetric version of the former one [8], have also been investigated by means of this method.

The nonlinear Schrödinger (NLS) equation, along with its various extensions, is a ubiquitous model of the modern nonlinear science [1, 2]. With the advent of symbolic computation algorithms, sophisticated methods of algebraic manipulation have become feasible for equations of the NLS type. Recently, exact solutions of NLS equations with variable coefficients, depending on the evolutional variable (z), have been obtained in the self-similar form [9–11], and the means of the homogeneous-balance and F-expansion techniques [12–15].

In cases when general exact solutions to (2+1)D equations are not available, a relevant problem is to find particular solutions in an exact form. In this work, we use a special type of the SoV approach to construct special forms of the (2+1)D NLS equation, which admit particular exact solutions. Two such equations are derived by means of an *a priori* ansatz related to the Hirota's bilinear operator. After solving these equations, we obtain a rich variety of solutions for the (2+1)D NLS equation with two arbitrary functions, $p(z,x)$ and $p(z,y)$, where x and y are the transverse coordinates. Some particularly interesting localized structures are produced by appropriately selecting these two arbitrary functions.

This paper is organized as follows. Section 2 introduces the (2+1)D NLS equation and presents the solution method. Section 3 deals with different forms of 2D spatial solitons and the associated trapping potentials in the NLS equation which support those solitons. In Section 4, we test the stability of the solutions by numerical simulations. A summary of the results is given in Section 5.

2. VARIABLE SEPARATION APPROACH FOR (2+1)D NLS EQUATION

We describe the propagation of a light beam along axis z in the Kerr medium with transverse plane (x,y), using the scaled (2+1)D NLS equation for the complex field amplitude u:

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+|u|^2 u+V(z,x,y)u=0, \tag{1}$$

where $V(z,x,y)$ is an external potential function. To establish an appropriate SoV procedure for Eq. (1), we aim to convert the model into a multi-linear form, and, accordingly, extend the Hirota's bilinear operator for a general SoV ansatz. Then, by substituting the SoV ansatz into Eq. (1), we will find some nontrivial SoV solutions.

The starting point for obtaining exact solutions by means of the Hirota's method is substitution $u=\frac{g}{f}$, where f and g are real and complex functions, respectively. Inserting this into Eq. (1), we arrive at the following bilinear-form equations:

$$\left[\mathrm{i}D_z+\frac{1}{2}\left(D_x^2+D_y^2\right)\right](gf)=0, \tag{2a}$$

$$(D_x^2+D_y^2-2V)ff=2|g|^2, \tag{2b}$$

where D_x, D_y and D_z are the usual Hirota's bilinear operators, which are widely used in the study of various nonlinear systems [16,17], and are defined as follows: $D_x^n[gf]=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n g(x)f(x')\Big|_{x=x'}$. To find some physically relevant solutions of Eq. (2), we introduce the following SoV ansatz [6-8]:

$$f=a_1+a_2 p(z,x)q(z,y), \tag{3a}$$

$$g=T(z,x)R(z,y)\mathrm{e}^{\mathrm{i}[t(z,x)+r(z,y)]}, \tag{3b}$$

where $a_1(\neq 0)$ and $a_2(\neq 0)$ are arbitrary constants, and $p(z,x)$, $q(z,y)$, $T(z,x)$, $R(z,y)$, $t(z,x)$ and $r(z,y)$ are arbitrary functions of the indicated variables. Then we substitute Eqs. (3) into Eqs. (2), and separately equate the real and imaginary parts to zero. We are thus left with a system of differential equations for $p(z,x)$, $q(z,y)$, $T(z,x)$, $R(z,y)$, $t(z,x)$, and $r(z,y)$:

$$p_x q_y=T^2R^2, \tag{4a}$$

$$\begin{aligned}&a_1a_2\left(qp_{xx}+pq_{yy}\right)+a_2^2q^2\left(pp_{xx}-p_x^2\right)+a_2^2p^2\left(qq_{yy}-q_y^2\right)-q_yp_x=\\&=2V\left(a_1+a_2pq\right)^2\end{aligned} \tag{4b}$$

$$-\left(t_z + r_z\right)RT + \frac{1}{2}\left(T_{xx}R + TR_{yy} - TRt_x^2 - TRr_y^2\right) = 0\,, \tag{5a}$$

$$TR\left(p_{xx}q + pq_{yy}\right) - 2T_x R p_x q - 2TR_y p q_y = 0\,, \tag{5b}$$

$$RT_z + TR_z + \frac{1}{2}\left(2T_x R t_x + 2TR_y r_y + TRt_{xx} + TRr_{yy}\right) = 0\,, \tag{5c}$$

$$TR\left(p_z q + pq_z\right) + 2TR\left(t_x p_x q + r_y p q_y\right) = 0\,. \tag{5d}$$

From Eq. (4a), we see that p and T are the functions of coordinates (x,z), while q and R are the functions of (y,z). Therefore, Eq. (4a) enforces the separation of the variables and provides solutions for R and T in terms of q and p:

$$R = \delta_1\sqrt{\left|\lambda q_y\right|}\,, \quad T = \delta_2\sqrt{\left|\lambda^{-1} p_x\right|}\,, \tag{6}$$

where $\delta_1^2 = \delta_2^2 = 1$, and $\lambda\,(\neq 0)$ is an arbitrary real constant. Likewise, Eq. (4b) can be used to express the potential V in terms of q and p. The arbitrariness of functions $p(x,z)$ and $q(z,y)$, which appear in Eq. (4b), implies that the associated class of potential functions $V(z,x,y)$ may be a rich one. The freedom in the choice of the solution of Eq. (4b) makes it feasible to consider the following *inverse problem*, as it was defined, in similar 1D and 2D contexts [18]: First, select appropriate expressions for $p(x,z)$ and $q(z,y)$ to describe desired nonlinear localized modes, and then find the corresponding trapping potential V which produces such solutions. Following this route, we construct some physically relevant exact solutions, along with the associated potential $V(z,x,y)$, determined by Eq. (4b). The purpose of Eqs. (5) is to determine the phases $t(z,x)$ and $r(z,y)$; obviously the system of these equations is over-defined, but, upon the inclusion of q and p, some of them turn into identities.

We treat the inverse problem in the following way: for a given set of arbitrary functions $p(x,z)$ and $q(y,z)$, functions t, r and T, R can be expressed in terms of p and q by means of Eqs. (5) and (6). Then, substituting Eq. (6) into Eq. (3), we obtain quite a generic solution to Eq. (1) in the following form:

$$u(x,y,z) = \frac{\delta_1\delta_2\sqrt{p_x q_y}}{a_1 + a_2 pq}\exp\left\{\mathrm{i}\left[t(z,x) + r(z,y)\right]\right\}. \tag{7}$$

In this paper, we aim to find localized solutions, i.e., those subject to a constraint that $|u(x,y)|$ should approach zero at least in one direction, $|x| \to \infty$ or $|y| \to \infty$ (or in both directions simultaneously). From Eq. (7), we obtain the expression for the local intensity:

$$I(x,y,z) \equiv |u|^2 = \frac{|p_x q_y|}{(a_1 + a_2 pq)^2} . \tag{8}$$

Because of the arbitrariness of functions $p(x,z)$ and $q(y,z)$ appearing in Eq. (8), the optical-intensity field may acquire quite a rich structure, with the corresponding trapping potential $V(z,x,y)$ calculated from Eq. (4b). Typical examples of several types of localized solutions are presented below, fixing, for simplicity's sake, $a_2 = 1$ in Eq. (8). Also, we confine the attention to the intensity distributions, without displaying phases t and r as they are produced by Eqs. (5).

3. EXACT SOLUTIONS OF THE (2+1)D NLS EQUATION

From Eq. (8), one sees that the arbitrary selection of functions p and q gives rise to singularities in the distribution of the intensity at $a_1 + a_2 pq = 0$. Therefore, functions p and q must be carefully selected, so as to avoid the singularities. In particular, we will demonstrate that, selecting the arbitrary functions as multiple straight-line nonlinear waves parallel to the x and y axes, respectively, then Eq. (8) gives rise to *multiple dromion solutions*.

3.1. LOCALIZED DROMION SOLUTIONS

A *dromion* is located at a cross point of underlying ("ghost") straight or curved line solitons subject to suitable dispersion relations [19, 20]. It can feature diverse structures, such as multiple peaks – in particular, dipole and twin dromions, with peaks of opposite and equal signs, respectively.

Usually, multi-dromion solutions are built by multiple straight-line solitons subject to appropriate conditions. The dromions are localized around the cross points of the corresponding straight lines. Here we restrict ourselves to the simplest choice of the arbitrary functions appropriate for the construction of the dromions, *viz.*,

$$p = \exp\left(x^2 + \sin z\right), \quad q = \exp y^2 . \tag{9}$$

Figure 1a–c show the structure of the single-dromion solution at different propagation distances, $z = 0, 3, 6$, for $a_1 = 1$. The corresponding potential $V(z = 0, x, y)$, which is shown in Fig. 1d, has the shape of a protruding square with a very deep well in the center. It is seen from Eq. (9) that the structure varies periodically along z. Accordingly, the (quite small) change of the amplitude of the solution is a result of the periodic modulation of function $\sin z$.

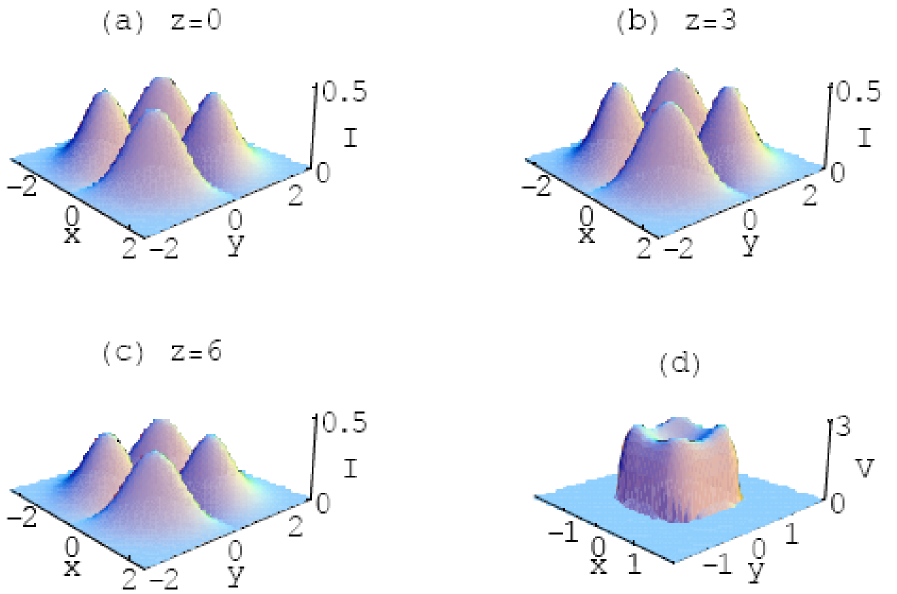

Fig. 1 – a), b), c) The dromion-type localized solution at different values of the propagation distance, $z = 0,3,6$, generated by the pair of functions p and q chosen as per Eq. (9); d) the 2D potential supporting this solution at $z = 0$.

If some a periodicity in x and y is included into functions $p(x,z)$ and $q(y,z)$, one can obtain dromion solutions with oscillating tails in some spatial directions. For instance, taking

$$p = \exp\left[1.001x + \frac{\sin(3x)}{3} + z^2\right] + 1, \; q = \mathrm{e}^y, \tag{10}$$

we obtain a dromion featuring oscillations along x and the localization along y, as shown in Figs. 2a, b and c (the coefficient 1.001 is taken slightly different from 1 to prevent the vanishing of p_x at points $x = \pi(1 + 2n)$, see Eq. (7)).

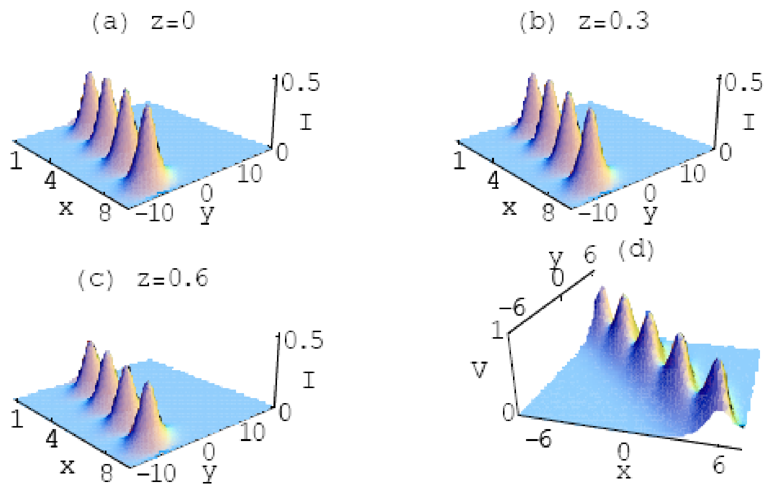

Fig. 2 – a), b), c) An oscillating dromion at distances $z = 0, 0.3, 0.6$ for $a_1 = 1$, generated by the pair of functions p and q chosen as per Eq. (10); d) the trapping potential supporting this solution at $z = 0$.

Another class of exact solutions, which oscillate in one direction and are localized in the other (cf. the solution shown in Fig. 2), can also be constructed by means of the methods outlined above. Figure 3 displays such a solution to Eq. (1) oscillating in the y direction. This solution corresponds to $a_1 = 1$ and functions

$$p = \exp\left(x + z^2\right),\ q = \exp\left(\sin y\right). \tag{11}$$

The shape of the solution resembles periodically modulated states known in other physical systems, such as vortex streets [21].

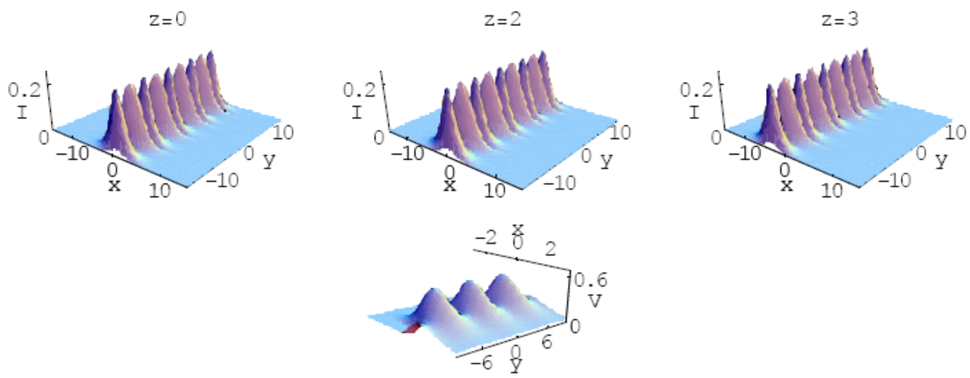

Fig. 3 – A breather-like solution generated by the pair of functions p and q chosen as per Eq. (11), is shown at $z = 0, 2, 3$ (the top panel). The corresponding trapping potential at $z = 0$ is shown in the bottom panel.

Further, Fig. 4a displays a "dromion lattice" solution for $a_1 = 4$, and for p, q being given by

$$p = \sin x, \; q = \sin y. \tag{12}$$

In Fig. 4b we depict the corresponding potential function V. Note that the individual dromions building the lattice are located at the cross points of the "ghost" (underlying) straight-line solitons.

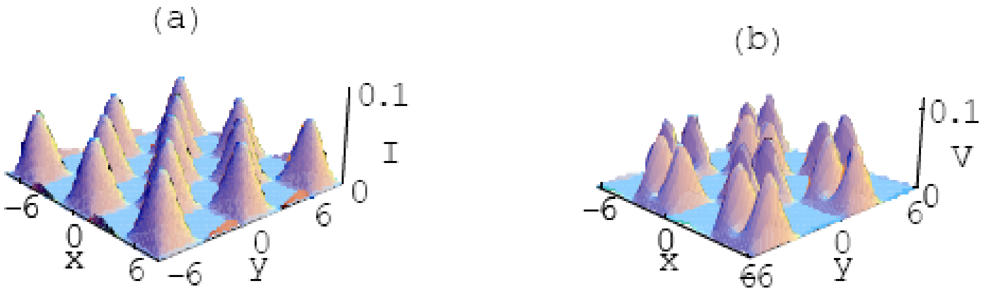

Fig. 4 – a) A dromion lattice for $a_1 = 4$, generated by the pair of functions p and q chosen as per Eq. (12); b) the periodic potential supporting the lattice.

Multiple-dromion solutions have been found in many other models [22, 23] – for instance, the dromion solutions of the DS equation [22], which were constructed by the summations of exponential functions. The dromion solutions reported in the present work have a potentially richer structure, as we may use the summations of arbitrary localized functions, see Eq. (8).

3.2. RING-LIKE SOLUTIONS

A ring-like nonlinear wave can be viewed as a curved-line nonlinear wave closing on itself. Furthermore, it can feature different wave structures, such as plateau, basin, bowl, saddle, etc. [5]. Here we choose

$$p = \exp\left(-x^2 + z^2\right), \; q = \exp\left(-y^2\right), \tag{13}$$

with $a_1 = 1$ and $z = 3$. The shape of the respective ring-like nonlinear waves and supporting potential are displayed in Fig. 5. Note that the shape of the potential function also has a ring-like structure, with values varying from positive to negative with the decrease of the ring's radius.

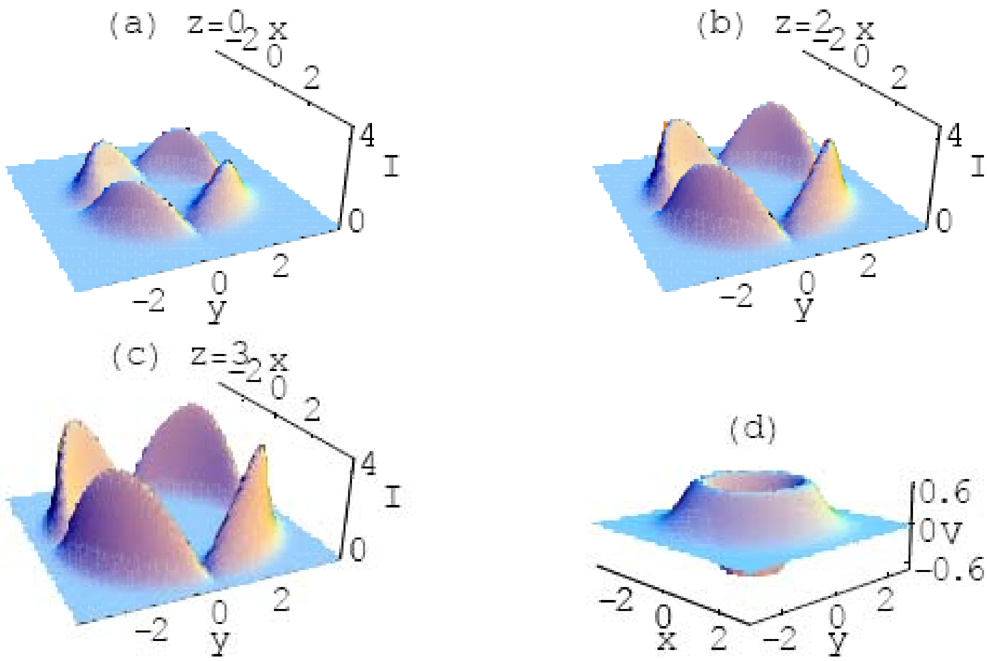

Fig. 5 – a), b), c) A ring-like nonlinear localized wave shown at $z = 0,2,3$, which is generated by the pair of functions p and q chosen as per Eq. (13); d) the corresponding supporting potential at $z = 3$.

3.3. PARABOLA-SHAPED LOCALIZED SOLUTIONS

Parabola-like localized solutions can be considered as curved nonlinear waves. A curved-line nonlinear wave of an integrable model is defined as a solution which is finite on the curved line and decays exponentially away from it [24]. Figure 6 (a) shows an example of parabola-shaped solutions for this type, based on functions

$$p(x,z) = \exp\left(x + z^2\right),\ q(y,z) = \exp\left(-y^2\right), \tag{14}$$

with $a_1 = 1$. Another example, with

$$p = \exp\left(|x| + z^2\right),\ q = e^y, \tag{15}$$

is shown in Fig. 6b for $a_1 = 3$.

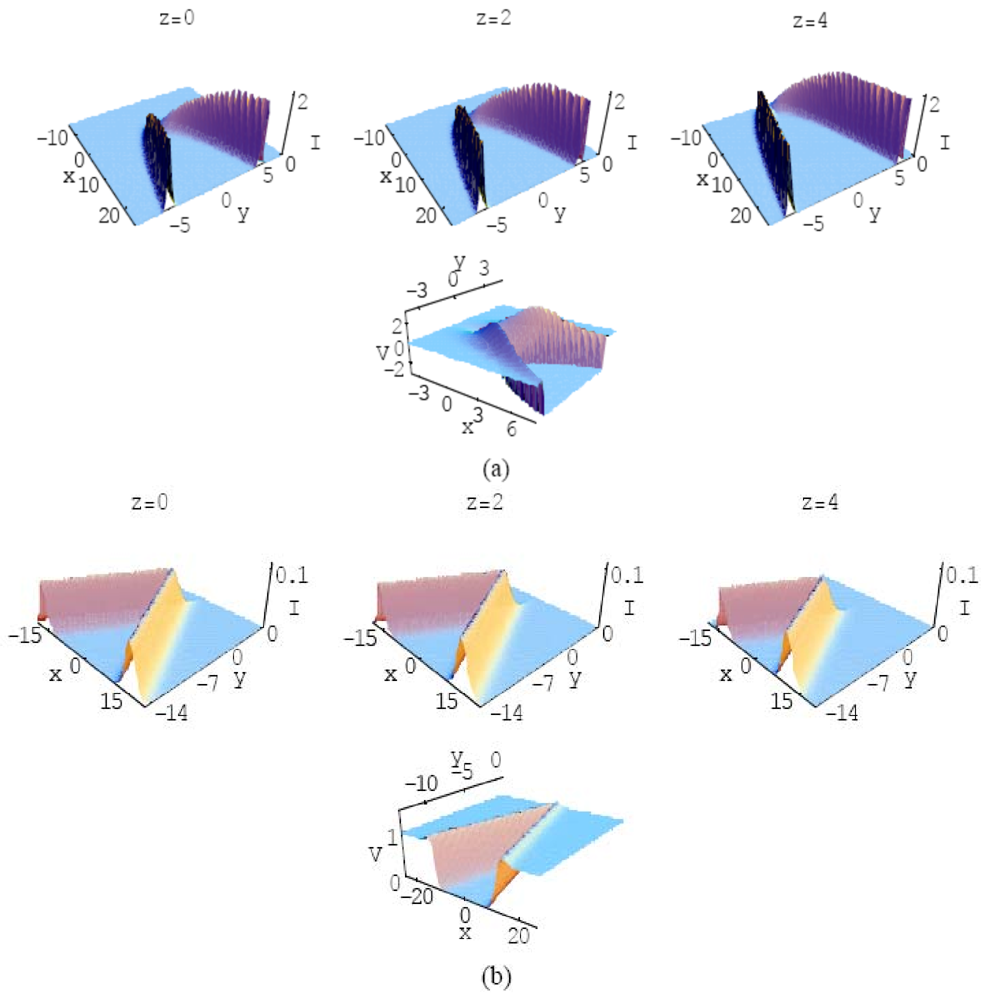

Fig. 6 – The top column: examples of parabolic-shaped waves corresponding to pairs of functions (14) and (15) – parts (a) and (b), respectively, shown at $z = 0, 1, 3$. The corresponding trapping potentials (bottom panels) at $z = 3$.

3.4. SQUARE-SHAPED SOLUTIONS

A (2+1)D square-shaped soliton may be defined as a *peakon*, i.e., a solution with a singularity of its derivatives at the center [25]. In particular, choosing the arbitrary functions, introduced above, as

$$p = \exp\left(|x| + z^2\right),\ q = \mathrm{e}^{|y|}, \tag{16}$$

one can obtain a square-like peakons which is shown in Fig. 7a for $a_1 = 3$ and $z = 0$. Another noteworthy example of an exact solution is a hybrid of square-shaped and curved-line solitons, shown in Fig. 7b, where the arbitrary functions are selected as

$$p = \exp\left(|x| + z^2\right) \quad \text{and} \quad q = \exp\left(-|y+6|\right) + \exp\left(-|y-6|\right), \tag{17}$$

and the solution is shown for $a_1 = 1$ and $z = 0$.

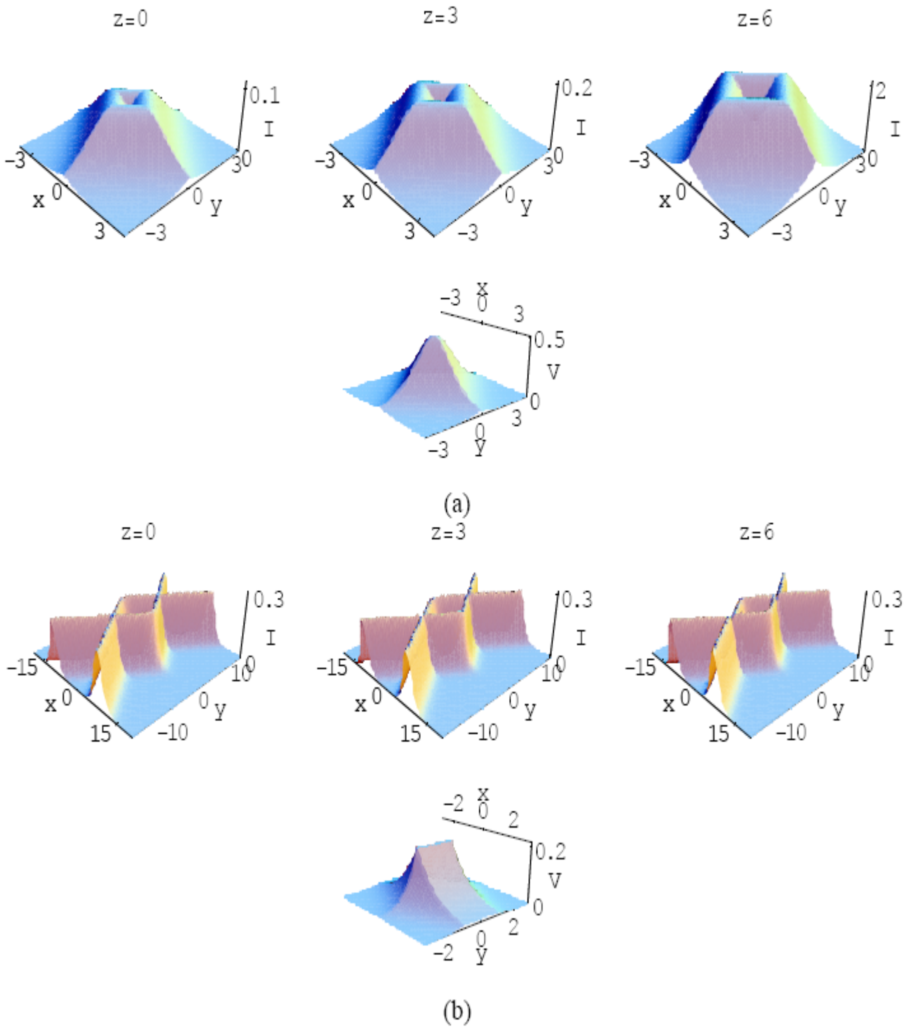

Fig. 7 – Examples of square-shaped solutions generated by the pair of functions p and q chosen as per Eqs. (16) and (17) (top panels in parts (a) and (b), respectively). The associated trapping potentials are shown in the bottom panels for $z = 0$.

4. STABILITY ANALYSIS

The stability of the above solutions is a crucial issue because, obviously, only stable (or weakly unstable) self-trapped beams can be observed in optical media. To study the stability, we consider the evolution of perturbed localized states represented by solutions to equation (1) generated by initial conditions in the form of

$$u\left(z=0,x,y\right)=w_r\left(x,y\right)\left[1+\rho\left(x,y\right)\right], \quad (18)$$

where $\rho\left(x,y\right)$ is a random-noise function with variance σ^2_{noise}. For example, by means of direct simulations of Eq. (1) we have tested the stability of solutions (7) corresponding to functions p and q chosen as per Eqs. (9) and (10) [the respective unperturbed solutions are displayed in Figs. 1 and 2]. The simulations were performed using the split-step beam-propagation method, similar to the way it was implemented in Ref. [26]. Figure 8 demonstrates that the weakly perturbed solutions are stable, keeping the initial shape in the course of the evolution. Similar results were produced by direct simulations of the perturbed evolution of other solutions.

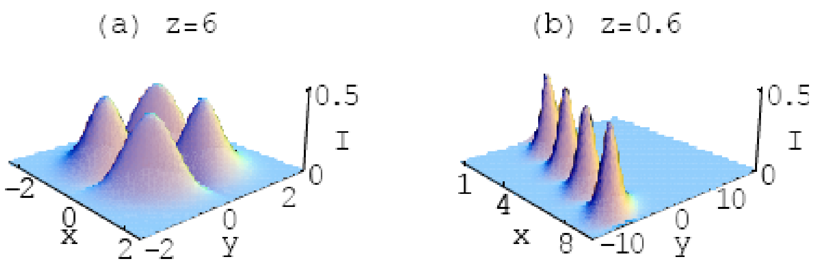

Fig. 8 – The results of numerical simulations of Eq. (1) with perturbed initial conditions (18). Panels (a) and (b) display the intensity distributions corresponding to the perturbed versions of solutions generated by Eqs. (9) and (10), respectively, with variance $\sigma^2_{\text{noise}} = 0.01$ of the random initial perturbations.

5. CONCLUSIONS

We have developed an approach to solve the multidimensional NLS equation based on the solution of the inverse problem, *i.e.,* looking for a potential which supports a nonlinear localized or delocalized mode of a desired form [18]. This approach has been applied to the (2+1)D NLS equation written in the Cartesian coordinates, with the 2D trapping potential. The separation of variables and Hirota's bilinear method have been combined to construct a wide class of solutions, in the framework of our inverse-problem technique. The solutions depend on the set of two arbitrary functions, see Eq. (8). Exact solutions have thus been produced for single and multiple dromions, ring- and parabola-shaped wave forms, peakons (square-shaped modes), and spatial counterparts of breathers. The stability of the so-obtained solutions was tested by dint of direct simulations.

Acknowledgments. This work was supported by the National Natural Science Foundation of China under Grant No. 61275001 and by the Natural Science Foundation of Guangdong Province, China, under Grant No. 1015283001000000. Work at the Texas A&M University at Qatar was supported by the NPRP 09-462-1-074 project with the Qatar National Research Foundation.

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