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SOLITONS AND OTHER SOLUTIONS TO THE (3+1)-DIMENSIONAL EXTENDED KADOMTSEV-PETVIASHVILI EQUATION WITH POWER LAW NONLINEARITY

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Abstract. This paper studies the (3+1)-dimensional extended Kadomtsev-Petviashvili equation with power law nonlinearity that apperas in the study of multi-component plasmas. The solutions are obtained by several methods such as modified *F*-expansion method, exp-function method, G'/G expansion method, ansatz method, traveling wave hypothesis, the improved Jacobi's elliptic function method and Lie symmetry analysis. These method lead to several closed form exact solutions. Some of these solutions are topological, non-topological and singular solitons, cnoidal, snoidal waves. It is also shown that in the limiting case, these doubly periodic functions lead to singular periodic functions, complexitons and linear waves. The domain restrictions are also identfified in order for the soliton solutions to exist.

Key words: (3+1)-dimensional extended Kadomtsev-Petviashvili equation, power law nonlinearity, modified *F*-expansion method, G'/G expansion method.

1. INTRODUCTION

The theory of nonlinear evolution equations (NLEEs) is a very familiar area of research and study in the areas of Applied Mathematics and Theoretical Physics. Thses NLEEs appear in Fluid Dynamics, Plasma Physics, Nonlinear Optics, Mathematical Biology, Nuclear Physics and several other areas of Theoretical Physics [1-20]. While these equations are mostly studied in (1+1)-dimensions, it is important from the context of reality to address these NLEEs in multi-dimensions, in particular (3+1)-dimensions. This paper will address one such NLEE in (3+1)-D scenario.

In this paper, the Kadomtsev-Petviashvili (KP) equation in (3+1)-D will be studied. This is also known as the extended KP equation that appears in the study of multi-component plasmas [5, 14, 15]. In order to take a look at this equation from a generalized perspective, the power law nonlinearity will be considered so that the results of this paper will encompass the previously existing results. Thus the integrability aspects of the power law extended KP equation in (3+1)-D will be addressed.

There are several different tools of integrability of NLEEs that are available nowadays. These techniques reveal several different forms of solutions of these NLEEs. Some of the integrability tools will be applied to extract various kinds of solution to the (3+1)-D KP equation. They are the F-expansion method, G'/G-expansion method, exp-function method, improved Jacobi's elliptic function method, ansatz method, traveling wave solutions and finally the Lie symmetry analysis. Each of these methods are studied in separate sections where different form of nonlinear wave solutions will be extracted.

2. MODIFIED *F*-EXPANSION METHOD

We first describe the modified *F*-expansion method, then apply it to construct the traveling wave solutions for the generalized extended (3+1)-D KP equation.

2.1. DETAILS OF THE METHOD

Let us consider nonlinear partial differential equation (PDE) with independent variables x, y, z, t and dependent variable u as

$$\phi(u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, u_{zz}, u_{tt}, u_{xy}, u_{xt}, u_{yt}, \cdots) = 0, \tag{1}$$

where ϕ is in general a polynomial in u and its various partial derivatives. The main idea of the modified F-expansion method for solving eq. (1) consists of the following five steps.

***Step* 1.** Seek the traveling wave solution of eq. (1) in the form

$$u(x,t) = U(\xi = px + ky + lz - ct), \tag{2}$$

where c, p, k and l are constants. Inserting (2) into (1) yields an ordinary differential equation (ODE) of $U(\xi)$ as

$$\psi(U,U',U'',U''',\cdots)=0. \tag{3}$$

***Step* 2.** Suppose $U(\xi)$ can be expanded by a finite series of $\psi(\xi)$,

$$U(\xi)=\sum_{j=0}^{N}a_j\psi^j(\xi)+\sum_{j=1}^{N}a_{-j}\psi^{-j}(\xi) \qquad (a_N\neq 0), \tag{4}$$

where a_j and a_{-j} for $j=0,1,\cdots,N$ are constants to be determined, and an integer N is fixed by considering the balance between the highest order nonlinear terms and the highest order derivative of $U(\xi)$ in eq. (3), while $\psi(\xi)$ satisfies the nonlinear ODE (Riccati equation)

$$\psi'(\xi)=A+B\psi(\xi)+C\psi^2(\xi) \qquad (C\neq 0), \tag{5}$$

where A, B and C are parameters.

***Step* 3.** Substituting (4) into (3), and using (5) simultaneously, the left-hand side of (3) can be converted into a polynomial for $\psi^j(\xi)$ and $\psi^{-j}(\xi)$ for $j=0,1,\cdots,N$. Then setting the coefficients of the polynomial to zero yields a set of algebraic equations for a_j and $a_{-j}\,(j=0,1,\cdots,N)$ and c,k,l and L.

***Step* 4.** Solving the system of algebraic equations, probably with the aid of Mathematica, Maple or Matlab, then a_j, $a_{-j}\,(j=0,1,\cdots,N)$, c, k, l and L be expressed by A, B, C. Substituting these results into eq. (4), one can get a general form solution of eq. (1).

***Step* 5.** With the aid of Appendix (Table 1), from the general form of travelling wave solutions, we can give a series of soliton-like solutions, trigonometric function solutions and rational solutions of eq. (1).

2.2. APPLICATION TO THE GENERALIZED EXTENDED (3+1)-DIMENSIONAL KP EQUATION

We apply the F-expansion method to construct the traveling wave solutions for the generalized extended (3+1)-dimensional KP equation in the following form

$$(u_t+\alpha u^n u_x+\beta u_{xxx})_x+v\left(u_{xx}+u_{yy}+u_{zz}\right)+r\left(u_{xy}+u_{yz}+u_{xz}\right)=0, \tag{6}$$

where the α, β, v and r are parameters. We assume that

$$u(x,y,z,t)=U(\xi), \qquad \xi=px+ky+lz-ct, \tag{7}$$

where p, k, l and c are constants that to be determined later. We substitute eq. (7) into eq. (6). Integrating the obtained equation twice with respect to ξ and taking both constants of integration to be zero. Then we obtain the following ODE:

$$\left\{-cp+v\left(p^2+k^2+l^2\right)+r(pk+lk+pl)\right\}U+\frac{p^2\alpha}{n+1}U^{n+1}+p^4\beta U^{''}=0. \quad (8)$$

Multiplying eq. (8) by $U^{'}$ and integrating a third time with respect to ξ, one can write $\left\{-cp+v\left(p^2+k^2+l^2\right)+r(pk+lk+pl)\right\}U^2+\frac{2p^2\alpha}{(n+1)(n+2)}U^{n+2}+$

$$+p^4\beta\left(U^{'}\right)^2+L=0, \quad (9)$$

where L is a constant of integration to be determined later. We make transformation $U=W^{\frac{2}{n}}$. Thus, eq. (9) is transformed into the following ODE

$$\begin{aligned}&8p^4\beta\left(W^{'}\right)^2(\xi)+3n^3W^2(\xi)vp^2+3n^3W^2(\xi)vk^2+3n^3W^2(\xi)vl^2+\\&+3n^3W^2(\xi)rlk+n^4W^2(\xi)vl^2+4p^4\beta\left(W^{'}\right)^2(\xi)n^2+n^4W^2(\xi)vk^2-\\&-n^4W^2(\xi)cp+12p^4\beta\left(W^{'}\right)^2(\xi)n+2p^2\alpha W^4(\xi)n^2+n^4W^2(\xi)vp^2+\\&+n^4W^2(\xi)rpl+n^4W^2(\xi)rlk+n^4W^2(\xi)rkp-3n^3W^2(\xi)cp+\\&+2n^2W^2(\xi)vl^2-2n^2W^2(\xi)cp+2n^2W^2(\xi)vp^2+2n^2W^2(\xi)vk^2+\\&+2n^2W^2(\xi)rkp+2n^2W^2(\xi)rlk+2n^2W^2(\xi)rpl+3n^3W^2(\xi)rkp+\\&+3n^3W^2(\xi)rpl+L=0.\end{aligned} \quad (10)$$

Balancing W^4 with $\left(W^{'}\right)^2$ yields $N=1$. Therefore, according to the modified F-expansion method, we have

$$W(\xi)=\frac{a_{-1}}{\psi(\xi)}+a_0+a_1\psi(\xi), \quad (11)$$

where $\psi(\xi)$ satisfies eq. (5). Substituting (11) into eq. (10), and using eq. (5) simultaneously, the left-hand side of eq. (10) can be converted into a finite series in $\psi^j(\xi)(j=-4,\cdots,-1,0,1,\cdots,4)$, then setting each coefficient to zero, we get a set of overdetermined algebraic equations for a_{-1},a_0,a_1,c,l,k and L. Solving the obtained algebraic equations, gets the following solutions:

***Case* 1.** When $A=0$, yields

$$a_0=\mp\frac{p\beta B(3n+n^2+2)}{qn\alpha},\ a_1=\pm q\left(\frac{pC}{n}\right)$$

$$L=\frac{(n^2+3n+2)^2B^4\beta^2p^6}{2\alpha n^2},\ a_{-1}=0,\ k=k,\ l=l$$

$$c=\frac{b-2p^4\beta B^2}{n^2p}. \tag{12}$$

***Case* 2.** When $B=0$, yields

$$a_1=\pm q\left(\frac{pC}{n}\right),\ a_0=a_{-1}=0,\ l=l,$$

$$k=k,\ L=\frac{8\beta^2(3n+2+n^2)^2p^6C^2A^2}{\alpha n^2},$$

$$c=\frac{b+8p^4\beta AC}{n^2p}. \tag{13}$$

***Case* 3.** When $A=B=0$, yields

$$a_1=\pm q\left(\frac{pC}{n}\right),\ a_0=a_{-1}=0,\ l=l,\ k=k,$$

$$c=\frac{l^2v+k^2v+rpl+vp^2+klr+rkp}{p},\ L=0, \tag{14}$$

where $q=\frac{\sqrt{-2\alpha\beta(n^2+2+3n)}}{\alpha}$, and we assume $\alpha\beta(n^2+2+3n)<0$. Also in (12) and (13), we have

$$b=n^2vp^2+n^2rlk+n^2rkp+n^2vk^2+n^2vl^2+n^2rpl\ .$$

Substituting over solutions into eq. (11), and by using appendix we have:

***Solution* 1.** When $A=0$, $B=1$, $C=-1$ and $\psi(\xi)=\frac{1}{2}+\frac{1}{2}\tanh(\frac{1}{2}\xi)$ from appendix, by using *Case 1*, we have

$$W_1(\xi)=\frac{p\beta(2\cosh(\xi)-2+3n\cosh(\xi)-3n+n^2\cosh(\xi)-n^2)}{nq\alpha\sinh(\xi)},$$

so

$$u_1(x,y,z,t)=\left\{\frac{p\beta(2\cosh(\xi)-2+3n\cosh(\xi)-3n+n^2\cosh(\xi)-n^2)}{\sinh(\xi)nq\alpha}\right\}^{\frac{2}{n}},$$

where

$$\xi=px+ky+lz-\left(\frac{b-2p^4\beta B^2}{n^2p}\right)t.$$

***Solution* 2.** When $A=0$, $B=-1$, $C=1$ from Appendix (Table 1), then $\psi(\xi)=\frac{1}{2}-\frac{1}{2}\coth(\frac{1}{2}\xi)$, by using *Case 1*, we obtain

$$W_2(\xi)=\frac{p\sinh(\xi)\beta(3n+n^2+2)}{(\cosh(\xi)-1)nq\alpha},$$

so

$$u_2(x,y,z,t)=\left\{\frac{p\sinh(\xi)\beta(3n+n^2+2)}{(\cosh(\xi)-1)nq\alpha}\right\}^{\frac{2}{n}},$$

where

$$\xi=px+ky+lz-\left(\frac{b-2p^4\beta B^2}{n^2p}\right)t.$$

***Solution* 3.** When $B=0$, $A=\frac{1}{2}$, $C=\frac{-1}{2}$ and $\psi(\xi)=\coth(\xi)\pm csch(\xi)$ from appendix (Table 1), by using *Case 2*, we have

$$W_3(\xi)=\mp q\left\{\frac{p(\cosh(\xi)\pm 1)}{2n\sinh(\xi)}\right\},$$

and choosing $\psi(\xi)=\tanh(\xi)\pm i\,sech(\xi)$, gives

$$W_4(\xi)=\mp q\left\{\frac{p(\sinh(\xi)\pm I)}{2n\cosh(\xi)}\right\},$$

respectively, we have

$$u_3(x,y,z,t)=\left[\mp q\left\{\frac{p(\cosh(\xi)\pm 1)}{2n\sinh(\xi)}\right\}\right]^{\frac{2}{n}},$$

$$u_4(x,y,z,t)=\left[\mp q\left\{\frac{p(\sinh(\xi)\pm I)}{2n\cosh(\xi)}\right\}\right]^{\frac{2}{n}},$$

where

$$\xi = px+ky+lz-\left(\frac{b+8p^4\beta AC}{n^2 p}\right)t.$$

***Solution* 4.** When $A=1$, $B=0$, $C=-1$ and $\psi(\xi)=\tanh(\xi)$ from Appendix, by using *Case 2*, we obtain

$$W_5(\xi)=\mp q\left\{\frac{p\tanh(\xi)}{n}\right\},$$

and $\psi(\xi)=\coth(\xi)$, gets

$$W_6(\xi)=\mp q\left\{\frac{p\coth(\xi)}{n}\right\},$$

respectively, we obtain

$$u_5(x,y,z,t)=\left[\mp q\left\{\frac{p\tanh(\xi)}{n}\right\}\right]^{\frac{2}{n}},$$

$$u_6(x,y,z,t)=\left[\mp q\left\{\frac{p\coth(\xi)}{n}\right\}\right]^{\frac{2}{n}},$$

where

$$\xi = px+ky+lz-\left(\frac{b+8p^4\beta AC}{n^2 p}\right)t.$$

***Solution* 5.** By choosing $A=C=1/2$, $B=0$ and $\psi(\xi)=\sec(\xi)+\tan(\xi)$ from Appendix (Table 1), by using *Case 2*, yields

$$W_7(\xi)=\pm q\left\{\frac{p(1+\sin(\xi))}{2n\cos(\xi)}\right\},$$

and $\psi(\xi)=csc(\xi)-\cot(\xi)$, gives

$$W_8(\xi)=\mp q\left\{\frac{p(1-\cos(\xi))}{2n\sin(\xi)}\right\},$$

respectively, yields

$$u_7(x,y,z,t)=\left[\pm q\left\{\frac{p(1+\sin(\xi))}{2n\cos(\xi)}\right\}\right]^{\frac{2}{n}},$$

$$u_8(x,y,z,t)=\left[\mp q\left\{\frac{p(1-\cos(\xi))}{2n\sin(\xi)}\right\}\right]^{\frac{2}{n}},$$

where

$$\xi = px+ky+lz-\left(\frac{b+8p^4\beta AC}{n^2p}\right)t.$$

***Solution* 6.** We select $A=C=-1/2$, $B=0$ and $\psi(\xi)=\csc(\xi)+\cot(\xi)$ from Appendix (Table 1), by using *Case 2*, we have

$$W_9(\xi)=\mp q\left\{\frac{p(1+\cos(\xi))}{2n\sin(\xi)}\right\},$$

and $\psi(\xi)=\sec(\xi)-\tan(\xi)$, gets

$$W_{10}(\xi)=\pm q\left\{\frac{p(-1+\sin(\xi))}{2n\cos(\xi)}\right\},$$

so we obtain

$$u_9(x,y,z,t)=\left[\mp q\left\{\frac{p(1+\cos(\xi))}{2n\sin(\xi)}\right\}\right]^{\frac{2}{n}},$$

$$u_{10}(x,y,z,t)=\left[\pm q\left\{\frac{p(-1+\sin(\xi))}{2n\cos(\xi)}\right\}\right]^{\frac{2}{n}},$$

where

$$\xi = px+ky+lz-\left(\frac{b+8p^4\beta AC}{n^2p}\right)t.$$

***Solution* 7.** When $A=C=1(-1)$, $B=0$ and $\psi(\xi)=\tan(\xi)$ or $\cot(\xi)$ from appendix and *Case 2*, we have

$$W_{11}(\xi)=\pm q\left\{\frac{p\tan(\xi)}{n}\right\},$$

$$W_{12}(\xi) = \mp q\left\{\frac{p\cot(\xi)}{n}\right\},$$

then we obtain

$$u_{11}(x, y, z, t) = \left[\pm q\left\{\frac{p\tan(\xi)}{n}\right\}\right]^{\frac{2}{n}},$$

$$u_{12}(x, y, z, t) = \left[\mp q\left\{\frac{p\cot(\xi)}{n}\right\}\right]^{\frac{2}{n}},$$

where

$$\xi = px + ky + lz - \left(\frac{b + 8p^4\beta AC}{n^2 p}\right)t.$$

***Solution* 8.** When $A = B = 0$, $C \neq 0$ and $\psi(\xi) = \frac{-1}{C\xi + \lambda}$ (λ is an arbitrary constant) from Appendix (Table 1), by using *Case 3*, yields

$$W_{13}(\xi) = \mp q\left\{\frac{pC}{n(C\xi + \lambda)}\right\},$$

so

$$u_{13}(x, y, z, t) = \left[\mp q\left\{\frac{pC}{n(C\xi + \lambda)}\right\}\right]^{\frac{2}{n}},$$

where

$$\xi = px + ky + lz - \left(\frac{l^2 v + k^2 v + rpl + vp^2 + klr + rkp}{p}\right)t.$$

In solutions 1 through 8,

$$q = \frac{\sqrt{-2\alpha\beta(n^2 + 2 + 3n)}}{\alpha}$$

and it is assumed that

$$\alpha\beta(n^2 + 2 + 3n) < 0.$$

Also, in solutions 1 through 7,

$$b = n^2 vp^2 + n^2 rlk + n^2 rkp + n^2 vk^2 + n^2 vl^2 + n^2 rpl.$$

3. EXP-FUNCTION METHOD

The exp-function method was first proposed by He and Wu to solve differential equations. We now present briefly the main steps of the exp-function method that will be applied to generalized extended (3+1)-dimensional KP equation.

3.1. DETAILS OF THE METHOD

We consider the general nonlinear PDE of the type

$$P(u,u_x,u_y,u_z,u_t,u_{xx},u_{yy},u_{zz},u_{tt},u_{xy},u_{xt},u_{yt},\cdots)=0. \tag{15}$$

We introduce use the travelling wave transformation

$$u(x,y,z,t)=U(\xi),\quad \xi=px+ky+lz-ct, \tag{16}$$

therefore eq. (6) reduces to ODEs

$$F(U,U',U'',\cdots)=0, \tag{17}$$

where the prime denotes the derivation with respect to ξ. Also c, p, k, l are constants. The exp-function method is based on the assumption that the travelling wave solutions can be expressed in the following form

$$U(\xi)=\frac{\sum_{n=-w}^{d}a_n\exp(n\xi)}{\sum_{m=-s}^{q}b_m\exp(m\xi)}=\frac{a_{-w}\exp(-w\xi)+\cdots+a_d\exp(d\xi)}{b_{-s}\exp(-s\xi)+\cdots+b_q\exp(q\xi)}. \tag{18}$$

where w, d, s and q are positive integers which are known to be determined further, a_n and b_m are unknown constants. To determine the values of w and s, we balance the linear term of highest order in eq. (17) with the highest order nonlinear term. Similarly to determine the values of d and q.

3.2. APPLICATION TO THE GENERALIZED EXTENDED (3+1)-DIMENSIONAL KP EQUATION

We consider the generalized extended (3+1)-dimensional KP equation in the form of eq. (10) with the transformation

$$U(\xi=px+ky+lz-ct)=W^{\frac{2}{n}}(\xi) \tag{19}$$

We balance the linear term of highest order in eq. (10) with the highest order nonlinear term. By simple calculation, we have

$$\left(W'\right)^2 = \frac{c_1 \exp[(2w+4s)\xi] + \cdots}{c_2 \exp[6s\xi] + \cdots}, \tag{20}$$

and

$$W^4 = \frac{c_3 \exp[(4w+2s)\xi] + \cdots}{c_4 \exp[6s\xi] + \cdots}, \tag{21}$$

where c_i are determined coefficients only for simplicity. Balancing the highest order of exp-function in (20) and (21), we have

$$2w + 4s = 4w + 2s, \tag{22}$$

which leads to the result

$$w = s. \tag{23}$$

Similarly to determine values of d and q, we balance the linear term of lowest order in eq. (10)

$$\left(W'\right)^2 = \frac{\cdots + d_1 \exp[-(4q+2d)\xi]}{\cdots + d_2 \exp[-6q\xi]}, \tag{24}$$

and

$$W^4 = \frac{\cdots + d_3 \exp[-(2q+4d)\xi]}{\cdots + d_4 \exp[-6q\xi]}, \tag{25}$$

where d_i are determined coefficients. Therefore, we can obtain the following relation

$$-(4q+2d) = -(2q+4d), \tag{26}$$

which gives

$$q = d. \tag{27}$$

We can freely choose the values of w and d, but the final solution does not strongly depends upon the choice of values of w and d. For simplicity, we set $w = s = 1$ and $d = q = 1$, then from eq. (18), we have

$$W(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{28}$$

Substituting eq. (28) into eq. (25), equating the coefficients of all powers of $\exp(i\xi)$ $(i = 0, \pm 1, \pm 2, \cdots, \pm 6)$ to zero yields a set of algebraic equations for a_0, b_0,

a_1, a_{-1}, b_{-1}, b_1, c, l and k. Solving the system of algebraic equations with the aid of Maple, we obtain:

$$a_0 = a_0, b_0 = 0, b_{-1} = b_{-1}, a_1 = a_{-1} = 0$$

$$b_1 = \frac{\alpha a_0^2 n^2}{8p^2\beta b_{-1}(n^2+2+3n)}, \quad l = l,\ k = k,$$

$$c = \frac{b + 4p^4\beta}{n^2 p}. \tag{29}$$

Substituting eqs. (29) into eq. (28) yields

$$W(\xi) = \frac{a_0}{\dfrac{\alpha a_0^2 n^2 \exp(\xi)}{8p^2\beta b_{-1}(n^2+2+3n)} + b_{-1}\exp(-\xi)}. \tag{30}$$

From transformation (19), obtain the following general solution

$$u(x,y,z,t) = \left\{ \frac{a_0}{\dfrac{\alpha a_0^2 n^2 \exp(\xi)}{8p^2\beta b_{-1}(n^2+2+3n)} + b_{-1}\exp(-\xi)} \right\}^{\frac{2}{n}}, \tag{31}$$

where

$$b = n^2vl^2 + n^2rpl + n^2vk^2 + n^2rkp + n^2vp^2 + n^2rlk$$

and

$$\xi = px + ky + lz - \left(\frac{b + 4p^4\beta}{n^2 p} \right) t.$$

4. THE G'/G-EXPANSION METHOD

We will consider the traveling wave solutions of the generalized extended (3+1)-D KP equation by using the G'/G-expansion method.

4.1. DETAILS OF THE METHOD

We assume the given nonlinear PDE for $u(x,y,z,t)$ to be in the form

$$J(u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, u_{zz}, \cdots) = 0, \tag{32}$$

J is a polynomial with u and its derivatives as variable elements. To apply the method, we proceed as follows:

I. The transformation

$$\xi = px + ky + lz - ct, \tag{33}$$

where p, k, l and c are constants, converts eq. (32) to an ODE

$$F(U, U', U'', \cdots) = 0, \tag{34}$$

where prime denotes the derivative with respect to ξ.

II. Suppose that the solution of eq. (34) can be expressed by a polynomial in G'/G as follows:

$$U(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \qquad a_m \neq 0, \tag{35}$$

where $G = G(\xi)$ satisfies the second order linear ODE (LODE) in the form

$$G'' + \lambda G' + \mu G = 0, \tag{36}$$

where $a_m, \cdots, a_0, \lambda$ and μ are constants to be determined later. The positive integer m can be determined by balancing the highest order derivative with the highest order nonlinear terms in (34).

III. Substituting (35) into (34) and using the second order LODE, eq. (36), yields an algebraic equation involving powers of G'/G, collecting all terms with the same power of G'/G together, and then equating each coefficient of each powers of G'/G to zero.

IV. Solve the system of algebraic equations obtained from III, for a_m, $\cdots$, c, k, l, p, λ and μ by using of Maple. Depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, we get solutions of eq. (36). We can obtain exact solutions of the given system of equations for eq. (32).

4.2. APPLICATION TO THE GENERALIZED EXTENDED (3+1)-DIMENSIONAL KP EQUATION

Consider the generalized extended (3+1)-dimensional KP equation in the form of eq. (10) with the transformation

$$u(x, y, z, t) = U(\xi = px + ky + lz - ct) = W^{\frac{2}{n}}(\xi). \tag{37}$$

Balancing the highest order nonlinear term and a supreme derivative, we can get $m = 1$, so

$$W(\xi)=a_0+a_1\left(\frac{G'(\xi)}{G(\xi)}\right),\qquad a_1\neq 0, \tag{38}$$

where a_0 and a_1 are constants to be determined later.

Substituting eq. (38) into eq. (10) and collect all terms with the same power of $(\frac{G'}{G})$. Equating the coefficients of this polynomial in $(\frac{G'}{G})$ to zero, we can get the following equations

$$\left(\frac{G'}{G}\right)^0 :-n^4cpa_0^2+n^4vk^2a_0^2+n^4vl^2a_0^2+n^4vp^2a_0^2+2p^2\alpha n^2a_0^4$$
$$+3n^3vl^2a_0^2+3n^3vk^2a_0^2+3n^3vp^2a_0^2-3n^3cpa_0^2+2n^2vl^2a_0^2$$
$$+2n^2vp^2a_0^2+2n^2vk^2a_0^2+n^4rlka_0^2+8p^4\beta a_1^2\mu^2+n^4rpla_0^2$$
$$+n^4rkpa_0^2+2n^2rkpa_0^2+2n^2rlka_0^2+2n^2rpla_0^2+3n^3rkpa_0^2$$
$$+3n^3rpla_0^2+3n^3rlka_0^2+12p^4\beta na_1^2\mu^2+4p^4\beta n^2a_1^2\mu^2$$
$$-2n^2cpa_0^2=0,$$

$$\left(\frac{G'}{G}\right)^1 :-2a_1(n^4cpa_0-2n^2rpla_0-2n^2rkpa_0-3n^3rlka_0-n^4vp^2a_0$$
$$-n^4vk^2a_0-2n^2rlka_0-3n^3rkpa_0-8a_1p^4\beta\lambda\mu-12a_1p^4\beta n\lambda\mu$$
$$-n^4rkpa_0-n^4rpla_0-2n^2vk^2a_0+3n^3cpa_0-n^4rlka_0-n^4vl^2a_0$$
$$-4p^2\alpha n^2a_0^3-3n^3vl^2a_0-3n^3vk^2a_0-3n^3vp^2a_0-4a_1p^4\beta n^2\lambda\mu$$
$$-2n^2vl^2a_0+2n^2cpa_0-2n^2vp^2a_0-3n^3rpla_0)=0,$$

$$\left(\frac{G'}{G}\right)^2 :-a_1^2(2n^2cp-2n^2vp^2-2n^2vk^2-n^4vp^2-n^4vk^2-3n^3vp^2$$
$$-2n^2rpl-4p^4\beta n^2\lambda^2-n^4vl^2-16p^4\beta\mu-2n^2vl^2+3n^3cp$$
$$-3n^3vk^2-3n^3rlk-2n^2rlk-3n^3rkp-8p^4\beta\lambda^2-8p^4\beta n^2\mu$$
$$-12p^2\alpha n^2a_0^2-n^4rlk-n^4rpl-3n^3rpl-24p^4\beta n\mu-3n^3vl^2$$
$$-2n^2rkp+n^4cp-n^4rkp-12p^4\beta n\lambda^2)=0,$$

$$\left(\frac{G'}{G}\right)^3 :8p^2a_1^2(\alpha n^2a_0a_1+2\lambda\beta p^2+p^2\beta n^2\lambda+3p^2\beta n\lambda)=0,$$

$$\left(\frac{G'}{G}\right)^4 :2p^2a_1^2(\alpha n^2a_1^2+4p^2\beta+6p^2\beta n+2p^2\beta n^2)=0.$$

Solving this system by Maple gives

$$a_0 = a_0, a_1 = \pm q(\frac{p}{n}), l = l, k = k,$$

$$\mu = -\frac{a_0^2 \alpha n^2}{2\beta p^2 (n^2 + 3n + 2)}, c = \frac{s}{p},$$

$$\lambda = \mp q \frac{\alpha n a_0}{p\beta(n^2 + 2 + 3n)}. \quad (39)$$

Substituting eq. (39) into eq. (38) so we have

$$W = a_0 \pm q\left(\frac{p}{n}\right)\left(\frac{G'}{G}\right). \quad (40)$$

Since in (39), $\Delta = \lambda^2 - 4\mu = 0$ so we obtain only rational function solution

$$W(\xi) = a_0 + \frac{q^2 \alpha a_0}{2\beta(2 + 3n + n^2)} \pm \frac{qpc_2}{n(c_1 + c_2\xi)}, \quad (41)$$

so

$$u(x, y, z, t) = \left(a_0 + \frac{q^2 \alpha a_0}{2\beta(2 + 3n + n^2)} \pm \frac{qpc_2}{n(c_1 + c_2\xi)}\right)^{\frac{2}{n}}, \quad (42)$$

where

$$s = l^2 v + rpl + rkp + klr + vp^2 + k^2 v$$

and

$$\xi = px + ky + lz - \left(\frac{s}{p}\right)t.$$

Also,

$$q = \frac{\sqrt{-2\alpha\beta(n^2 + 2 + 3n)}}{\alpha}$$

and we assume that

$$\alpha\beta(n^2 + 2 + 3n) < 0.$$

5. ANSATZ METHOD

In this section the (3+1)-D extended KP equation that is going to be studied is going to be of the form

$$\left(q_t + aq^n q_x + bq_{xxx}\right)_x + c\left(q_{xx} + q_{yy} + q_{zz}\right) + k\left(q_{xy} + q_{yz} + q_{zx}\right) = 0 \quad (43)$$

The coefficients a, b, c and k are all constants. The method that is going to be adopted in this section is the ansatz method. Thus, the study will be split into the following three sections where the topological, non-topological and the singular solitary wave solutions will be obtained.

5.1. NON-TOPOLOGICAL SOLITONS

In this section, the starting hypothesis is going to be

$$q(x, y, z, t) = A\ sech^{p}\ \tau \quad (44)$$

where

$$\tau = B_1 x + B_2 y + B_3 z - vt \quad (45)$$

The unknown exponent p will be determined, in terms of the power law parameter n, from the balancing principle. Additionally, the parameter A is the amplitude of the solitary wave while the parameters B_j for $1 \le j \le 3$ are the widths in the x-, y- and z- directions respectively that are also related to the direction cosines of the solitary wave in $3D$. Finally, v is the velocity of the propagation of the solitary wave.

Substituting (44) into (43) yields

$$\begin{aligned}
&\Big[-p^2 vAB_1 + bAp^4 B_1^4 + cp^2 A\left(B_1^2 + B_2^2 + B_3^2\right) + \\
&+kp^2 A\left(B_1 B_2 + B_2 B_3 + B_1 B_3\right)\Big]\frac{1}{\cosh^{p}\tau} + \\
&+\Big[p(p+1)vAB_1 - bAB_1^4 p\left(p+1\right)\left\{p^2 + \left(p+2\right)^2\right\} - \\
&-cp(p+1)A\left(B_1^2 + B_2^2 + B_3^2\right) - -kp(p+1)A\left(B_1 B_2 + B_2 B_3 + B_1 B_3\right)\Big] \\
&\frac{1}{\cosh^{p+2}\tau} + \frac{ap^2\left(n+1\right)B_1^2 A^{n+1}}{\cosh^{p(n+1)}\tau} - \frac{ap\left[p\left(n+1\right)+1\right]B_1^2 A^{n+1}}{\cosh^{p(n+1)+2}\tau} + \\
&+\frac{bAB_1^4 p\left(p+1\right)\left(p+2\right)\left(p+3\right)}{\cosh^{p+4}\tau} = 0.
\end{aligned} \quad (46)$$

From (46), equating the exponents $p\left(n+1\right)+2$ and $p+4$ gives

$$p\left(n+1\right) + 2 = p + 4 \quad (47)$$

so that

$$p = \frac{2}{n}. \tag{48}$$

Again this same value of p is obtained on equating the exponents $p(n+1)$ and $p+2$.

Also from (48), the following restriction must hold

$$n > 0. \tag{49}$$

Now, from (46), the linearly independent functions are $1/\cosh^{p+j}\tau$ for $j = 0,2,4$. Therefore, setting their respective coefficients to zero yields

$$v = \frac{bp^2 B_1^4 + c\left(B_1^2 + B_2^2 + B_3^2\right) + k\left(B_1 B_2 + B_2 B_3 + B_1 B_3\right)}{B_1} \tag{50}$$

and

$$A = \left[\frac{2bB_1^2(n+2)(n+1)}{an^2}\right]^{\frac{1}{n}}, \tag{51}$$

which forces the constraint relation

$$ab > 0, \tag{52}$$

if n is an even integer. However, if n is an odd integer there is no such restriction but the soliton will be pointing downwards.

Thus, finally, the non-topological soliton solution to the (3+1)-D extended KP equation with power law nonlinearity (43) is given by

$$q(x,y,z,t) = \frac{A}{\cosh^{\frac{2}{n}}[B_1 x + B_2 y + B_3 z - vt]}, \tag{53}$$

where the amplitude A of the soliton is given by (61) while the velocity v is shown in (50). Note that this solution exist provided that $ab > 0$ and $n > 0$.

5.2. TOPOLOGICAL SOLITONS

In this case, the starting hypothesis is going to be

$$q(x,y,z,t) = A\tanh^p\tau, \tag{54}$$

where in this case the parameters A and B_j for $1 \le j \le 3$ are all free parameters, while v still represents the velocity of the solitary wave. The value of the unknown exponent p will be determined during the course of derivation of the soliton solution of (43). Now, substituting (54) into (43) gives

$$\begin{aligned}
&-pvAB_1\left\{(p-1)\tanh^{p-2}\tau-2p\tanh^{p}\tau+(p+1)\tanh^{p+2}\tau\right\}+\\
&+apA^{n+1}B_1^2\left\{(np+p-1)\tanh^{np+p-2}\tau-2p(n+1)\tanh^{np+p}\tau+\right.\\
&\left.+(np+p+1)\tanh^{np+p+2}\tau\right\}+bpAB_1^4\left[(p-1)(p-2)(p-3)\tanh^{p-4}\tau+\right.\\
&+(p+1)(p+2)(p+3)\tanh^{p+4}\tau-2(p-1)\left\{p^2+(p-2)^2\right\}\tanh^{p-2}\tau-\\
&-2(p+1)\left\{p^2+(p+2)^2\right\}\tanh^{p+2}\tau+\\
&\left.+\left[4p^3+(p-1)^2(p-2)+(p+1)^2(p+2)\right]\tanh^{p}\tau\right]+\\
&+cpA\left(B_1^2+B_2^2+B_3^2\right)\left\{(p-1)\tanh^{p-2}\tau-2p\tanh^{p}\tau+(p+1)\tanh^{p+2}\tau\right\}+\\
&+pAk\left(B_1B_2+B_2B_3+B_1B_3\right)\left\{(p-1)\tanh^{p-2}\tau-2p\tanh^{p}\tau+\right.\\
&\left.+(p+1)\tanh^{p+2}\tau\right\}=0.
\end{aligned} \tag{55}$$

Then, equating the exponents $np+p+2$ and $p+4$ in (55)

$$np+p+2=p+4 \tag{56}$$

that gives

$$p=\frac{2}{n}, \tag{57}$$

which is also obtained by equating the exponents pairs $np+p$ and $p+2$, $np+p-2$ and p.

Now from (55) the linearly independent functions are $\tanh^{p+j}\tau$ for $j=0,\pm 2,\pm 4$. Hence setting their respective coefficients to zero yields a set of algebraic equations:

$$\begin{aligned}
&-pvAB_1(p-1)-2bpAB_1^4(p-1)\left\{p^2+(p-2)^2\right\}+\\
&+cpA\left(B_1^2+B_2^2+B_3^2\right)(p-1)+pAk\left(B_1B_2+B_2B_3+B_1B_3\right)(p-1)=0,
\end{aligned} \tag{58}$$

$$\begin{aligned}
&2p^2vAB_1+bpAB_1^4\left[4p^3+(p-1)^2(p-2)+(p+1)^2(p+2)\right]-\\
&-2cp^2A\left(B_1^2+B_2^2+B_3^2\right)-2p^2Ak\left(B_1B_2+B_2B_3+B_1B_3\right)+\\
&+apA^{n+1}B_1^2(np+p-1)=0,
\end{aligned} \tag{59}$$

$$\begin{aligned}
&-pvAB_1(p+1)-2bpAB_1^4(p+1)\left\{p^2+(p+2)^2\right\}+\\
&+cpA\left(B_1^2+B_2^2+B_3^2\right)(p+1)+pAk\left(B_1B_2+B_2B_3+B_1B_3\right)(p+1)-\\
&-2ap^2A^{n+1}B_1^2(n+1)=0,
\end{aligned} \tag{60}$$

$$apA^{n+1}B_1^2(np+p+1)+bpAB_1^4(p+1)(p+2)(p+3)=0 \tag{61}$$

$$bpAB_1^4(p-1)(p-2)(p-3)=0 \tag{62}$$

To solve (62), we have considered the the following two cases:

5.2.1. Case 1: $p=1$. This yields

$$n=2, \tag{63}$$

Further substitution of $p=1$ into (58)-(61) gives

$$v=\frac{c\left(B_1^2+B_2^2+B_3^2\right)+k\left(B_1B_2+B_2B_3+B_1B_3\right)-2bB_1^4}{B_1} \tag{64}$$

and

$$A=B_1\sqrt{-\frac{6b}{a}}, \tag{65}$$

which shows that solitons will exist for

$$ab<0 \tag{66}$$

5.2.2. Case 2: $p=2$. This yields

$$n=1, \tag{67}$$

By substituting $p=2$ into (58)-(61), we obtain

$$v=\frac{c\left(B_1^2+B_2^2+B_3^2\right)+k\left(B_1B_2+B_2B_3+B_1B_3\right)-8bB_1^4}{B_1} \tag{68}$$

and

$$A=-\frac{12bB_1^2}{a}. \tag{69}$$

The third case when $p=3$ is not considered here as it does not yield an unique value of the free parameter A.

Lastly, we can determine the topological soliton solutions for the (3+1)-D extended KP equation with power law nonlinearity given by (43) when we substitute (64) and (65) in (54) with the respective constraint (66) for the first case of solution or we substitute (68) and (69) in (54) for the second case of solution as

$$q(x,y,z,t)=A\tanh^{\frac{2}{n}}\left(B_1x+B_2y+B_3z-vt\right). \tag{70}$$

5.3. SINGULAR SOLITONS

In this case, the starting hypothesis is given by

$$q(x,y,z,t)=A\,csch^{\,p}\,\tau \tag{71}$$

Substituting (71) into (43) implies

$$\begin{aligned}&-p^2vAB_1\,csch^{\,p}\,\tau-p(p+1)vAB_1\,csch^{\,p+2}\,\tau+\\&+ap^2\left(n+1\right)B_1^2A^{n+1}csch^{\,p(n+1)}\,\tau+ap\left[p\left(n+1\right)+1\right]B_1^2A^{n+1}csch^{\,p(n+1)+2}\,\tau+\\&+bAp^4B_1^4csch^{\,p}\,\tau+bAB_1^4p\left(p+1\right)\left\{p^2+\left(p+2\right)^2\right\}csch^{\,p+2}\,\tau+\\&+bAB_1^4p\left(p+1\right)\left(p+2\right)\left(p+3\right)csch^{\,p+4}\,\tau+cp^2A\left(B_1^2+B_2^2+B_3^2\right)csch^{\,p}\,\tau+\\&+cp(p+1)A\left(B_1^2+B_2^2+B_3^2\right)csch^{\,p+2}\,\tau+kp^2A\left(B_1B_2+B_2B_3+\right.\\&\left.+B_1B_3\right)csch^{\,p}\,\tau+kp(p+1)A\left(B_1B_2+B_2B_3+B_1B_3\right)csch^{\,p+2}\,\tau=0\end{aligned} \tag{72}$$

Now, from (72), equating the exponents $p\left(n+1\right)+2$ and $p+4$ leads to

$$p\left(n+1\right)+2=p+4, \tag{73}$$

which gives

$$p=\frac{2}{n}. \tag{74}$$

It needs to be noted that the same value of p is yielded when the exponents $p\left(n+1\right)$ and $p+2$ are equated with each other.

From (72) setting the coefficients of $csch^{\,p+j}\,\tau$ to zero where $j=0,2,4$, since these are linearly independent functions, gives

$$v=\frac{bAp^2B_1^4+c\left(B_1^2+B_2^2+B_3^2\right)+k\left(B_1B_2+B_2B_3+B_1B_3\right)}{B_1} \tag{75}$$

and

$$A=\left[-\frac{2bB_1^2\left(n+2\right)\left(n+1\right)}{an^2}\right]^{\frac{1}{n}}, \tag{76}$$

which shows that solitons will exist for

$$ab<0 \tag{77}$$

if n is an even integer. However, if n is an odd integer there is no such restriction but the soliton will be pointing downwards.

Thus, finally, the singular soliton solution to the (3+1)-D extended KP equation with power law nonlinearity (43) is given by

$$q(x,y,z,t) = A\, csch^{\frac{2}{n}}\left(B_1 x + B_2 y + B_3 z - vt\right), \tag{78}$$

where the velocity v and the amplitude A are given by (75) and (76) respectively while the conditions of existence of this solution are $n > 0$ and $ab < 0$.

6. TRAVELING WAVE SOLUTION

In this section, the traveling wave solution to (43) will be obtained. The starting hypothesis here is

$$q(x,y,z,t) = g\left(B_1 x + B_2 y + B_3 z - vt\right), \tag{79}$$

where g is the functional form of the nonlinear wave that solves (43). Substituting (79) into (43) leads to the relation

$$\begin{aligned}&\left(-vg' + aB_1 g^n g' + bg'''\right)' + \left\{c\left(B_1^2 + B_2^2 + B_3^2\right) + \right.\\ &\left.+k\left(B_1 B_2 + B_2 B_3 + B_3 B_1\right)\right\} g'' = 0,\end{aligned} \tag{80}$$

where g' represents dg/ds, while g'' represents $d^2 g/ds^2$ and so on, with s being

$$s = B_1 x + B_2 y + B_3 z - vt. \tag{81}$$

Integrating (80) twice and chosing the integration constant to be zero, since the search is for a soliton solution, yields

$$bg'' = (v - \alpha)g - \frac{aB_1}{n+1} g^{n+1}, \tag{82}$$

where

$$\alpha = c\left(B_1^2 + B_2^2 + B_3^2\right) + k\left(B_1 B_2 + B_2 B_3 + B_3 B_1\right). \tag{83}$$

Multiplying both sides of (82) by g' and integrating, while keeping the integration constant to be zero, yields

$$\left(g'\right)^2 = g^2 \left\{\frac{v-\alpha}{b} - \frac{2aB_1}{(n+1)(n+2)b}\right\}. \tag{84}$$

From (84), separating variables and integrating once again leads to

$$g(s) = g\left(B_1x + B_2y + B_3z - vt\right) = A\ sech^{\frac{2}{n}}\left[B\left(B_1x + B_2y + B_3z - vt\right)\right], \quad (85)$$

where

$$A = \left[\frac{(v-\alpha)(n+1)(n+2)}{2aB_1}\right]^{\frac{1}{n}} \quad (86)$$

and

$$B = \frac{n}{2}\sqrt{\frac{v-\alpha}{b}}\,. \quad (87)$$

Equations (86) and (87) impose the constraints given by

$$b(v-\alpha) > 0 \quad (88)$$

and

$$a(v-\alpha) > 0\,, \quad (89)$$

when n is an even integer. However, if n is an odd integer, the soliton will be simply pointing downwards.

Hence, finally, the 1-soliton solution of the extended KP equation with power law nonlinearity, that is obtained by traveling wave hypothesis, is given by

$$q(x,y,z,t) = A\ \mathrm{sech}^{\frac{2}{n}}\left[B\left(B_1x + B_2y + B_3z - vt\right)\right], \quad (90)$$

where the amplitude of the soliton is given by (86) with the constraints that are seen in (24) and (25).

7. IMPROVED JACOBI'S ELLIPTIC FUNCTION METHOD

7.1. DETAILS OF THE METHOD

In this section, we introduce a simple description of the Improved Jacobi elliptic function method (IJEF) method, for a given partial differential equation

$$G(u, u_x, u_y, u_z, u_{xy}, \ldots) = 0. \quad (91)$$

We like to know whether travelling waves (or stationary waves) are solutions of Eq. (43). The first step is to unite the independent variables x, y, z and t into one particular variable through the new variable

$$\zeta = px + ky + lz - ct, \quad u(x,y,t) = U(\zeta)\,,$$

where c is wave speed, and reduce Eq. (119) to an ODE

$$G(U, U', U'', U''', \ldots) = 0. \quad (92)$$

Our main goal is to derive exact or at least approximate solutions, if possible, for this ODE. For this purpose, let us simply U as the expansion in the form,

$$u(x, y, t) = U(\zeta) = \sum_{i=0}^{N} a_i \psi^i + \sum_{i=1}^{N} a_{-i} \psi^{-i}, \quad (93)$$

where

$$\psi' = \sqrt{q_0 + q_2\psi^2 + q_4\psi^4}, \quad (94)$$

the highest degree of $\frac{\mathrm{d}^p U}{\mathrm{d}\zeta^p}$ is taken as

$$O\left(\frac{\mathrm{d}^p U}{\mathrm{d}\zeta^p}\right) = N + p, \qquad p = 1, 2, 3, \cdots, \quad (95)$$

$$O\left(U^q \frac{\mathrm{d}^p U}{\mathrm{d}\zeta^p}\right) = (q+1)N + p, \quad q = 0, 1, 2, \cdots, p = 1, 2, 3, \cdots, \quad (96)$$

where q_0, q_2 and q_4 are constants, and N in Eq. (93) is a positive integer that can be determined by balancing the nonlinear term(s) and the highest order derivatives. Normally, N is a positive integer, so that an analytic solution in closed form may be obtained. Substituting Eqs. (93–96) into Eq. (92) and comparing the coefficients of each power of $\psi(\zeta)$ in both sides, to get an over-determined system of nonlinear algebraic equations with respect to k, l, c, a_0, a_i, $\cdots$. Solving the over-determined system of nonlinear algebraic equations by use of Mathematica. The relations between values of q_0, q_2 and q_4 and corresponding JEF solution $\psi(\zeta)$ of Eq. (93) are given in Table 2 (Appendix). Substitute the values of A, B, C and the corresponding JEF solution $\psi(\zeta)$ chosen from Table 2 in the Appendix, into the general form of solution, then an ideal periodic wave solution expressed by JEF can be obtained.

In Table 2, of the Appendix, cn (ζ) and dn (ζ) are the Jacobi elliptic cosine function and the JEF of the third kind, respectively and also

$$\mathrm{cn}^2(\zeta) = 1 - \mathrm{sn}^2(\zeta), \qquad \mathrm{dn}^2(\zeta) = 1 - m^2\mathrm{sn}^2(\zeta), \quad (97)$$

with the modulus m $(0 < m < 1)$. When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, *i.e.*,

$$\mathrm{sn}\zeta \to \tanh\zeta, \quad \mathrm{cn}\zeta \to \mathrm{sech}\zeta, \quad \mathrm{dn}\zeta \to \mathrm{sech}\zeta,$$

when $m \to 0$, the Jacobi functions degenerate to the triangular functions, *i.e.*,

$$sn\zeta \to \sin\zeta, \quad cn\zeta \to \cos\zeta \quad \text{and} \quad dn \to 1.$$

***Solution* 2.** When $q_0 = 0,\ q_2 = -1,\ q_4 = 1$ and $\psi(\xi) = \csc(\xi)$ or $\psi(\xi) = \sec(\xi)$ from Table 2, by using *Case 1*, we have

$$W_2(\xi) = \pm\left\{\frac{p\beta(3n + n^2 + 2)}{qn\alpha} + q\left(\frac{p}{n}\right)\csc(\xi)\right\}, \tag{98}$$

$$W_3(\xi) = \pm\left\{\frac{p\beta(3n + n^2 + 2)}{qn\alpha} + q\left(\frac{p}{n}\right)\sec(\xi)\right\}, \tag{99}$$

so

$$u_2(x, y, z, t) = \left\{\frac{p\beta(3n + n^2 + 2)}{qn\alpha} + q\left(\frac{p}{n}\right)\csc(\xi)\right\}^{\frac{2}{n}}, \tag{100}$$

$$u_3(x, y, z, t) = \left\{\frac{p\beta(3n + n^2 + 2)}{qn\alpha} + q\left(\frac{p}{n}\right)\sec(\xi)\right\}^{\frac{2}{n}}, \tag{101}$$

where

$$\xi = px + ky + lz - \left(\frac{b - 2p^4\beta}{n^2 p}\right)t.$$

When $q_2 = 0$, $q_0 = 1/2$, $q_4 = -1/2$ and $\psi(\xi) = \mathrm{cn}[\xi, \frac{1}{\sqrt{2}}]$ from Table 2, by using *Case* 2, we have

$$W_4(\xi) = \mp q\left\{\frac{p\,\mathrm{cn}[\xi, \frac{1}{\sqrt{2}}]}{2n}\right\}, \tag{102}$$

$$u_4(x, y, z, t) = \left\{\mp q\frac{p\,\mathrm{cn}[\xi, \frac{1}{\sqrt{2}}]}{2n}\right\}^{\frac{2}{n}}, \tag{103}$$

where

$$\xi = px + ky + lz + \left(\frac{b + 8p^4\beta}{4n^2 p}\right)t.$$

When $q_2 = 0$, $q_0 = 1$, $q_4 = -1$ and $\psi(\xi) = \mathrm{dn}[\xi, \sqrt{2}]$ or $\psi(\xi) = \mathrm{sc}[\xi, \sqrt{2}]$ from Table 2, by using *Case* 2, we have

$$W_5(\xi) = \mp q \left\{ \frac{p\,\mathrm{dn}[\xi, \sqrt{2}]}{n} \right\}, \tag{104}$$

$$W_6(\xi) = \mp q \left\{ \frac{p\,\mathrm{sc}[\xi, \sqrt{2}]}{n} \right\}, \tag{105}$$

$$u_5(x, y, z, t) = \left\{ \mp q \frac{p\,\mathrm{dn}[\xi, \sqrt{2}]}{n} \right\}^{\frac{2}{n}}, \tag{106}$$

$$u_6(x, y, z, t) = \left\{ \mp q \frac{p\,\mathrm{sc}[\xi, \sqrt{2}]}{n} \right\}^{\frac{2}{n}}, \tag{107}$$

where

$$\xi = px + ky + lz + \left(\frac{b + 8p^4\beta}{n^2 p} \right) t.$$

When $q_2 = 0$, $q_0 = -1/2$, $q_4 = 1/2$ and $\psi(\xi) = \mathrm{nc}[\xi, \frac{1}{\frac{1}{\sqrt{2}}}]$ from Table 2 (Appendix), by using *Case* 2, we have

$$W_7(\xi) = \pm q \left\{ \frac{p\,\mathrm{nc}[\xi, \frac{1}{\frac{1}{\sqrt{2}}}]}{2n} \right\}, \tag{108}$$

$$u_7(x, y, z, t) = \left\{ \mp q \frac{p\,\mathrm{nc}[\xi, \frac{1}{\frac{1}{\sqrt{2}}}]}{2n} \right\}^{\frac{2}{n}}, \tag{109}$$

where

$$\xi = px + ky + lz + \left(\frac{b + 8p^4\beta}{4n^2 p}\right)t.$$

When $q_2 = 0$, $q_0 = -1$, $q_4 = 1$ and $\psi(\xi) = \mathrm{nd}[\xi, \sqrt{2}]$ or $\psi(\xi) = \mathrm{cs}[\xi, \sqrt{2}]$ from Table 2 (Appendix), by using *Case* 2, we have

$$W_8(\xi) = \pm q\left\{\frac{p\,\mathrm{nd}[\xi, \sqrt{2}]}{n}\right\}, \tag{110}$$

$$W_9(\xi) = \pm q\left\{\frac{p\,\mathrm{cs}[\xi, \sqrt{2}]}{n}\right\}, \tag{111}$$

$$u_8(x, y, z, t) = \left\{\mp q\frac{p\,\mathrm{nd}[\xi, \sqrt{2}]}{n}\right\}^{\frac{2}{n}}, \tag{112}$$

$$u_8(x, y, z, t) = \left\{\mp q\frac{p\,\mathrm{cs}[\xi, \sqrt{2}]}{n}\right\}^{\frac{2}{n}}, \tag{113}$$

where

$$\xi = px + ky + lz + \left(\frac{b + 8p^4\beta}{n^2 p}\right)t.$$

When $q_2 = 0$, $q_0 = 1$, $q_4 = 3/4$ and $\psi(\xi) = \mathrm{sd}[\xi, \frac{1}{\sqrt{2}}]$ from Appendix (Table 2), by using *Case* 2, we have

$$W_{10}(\xi) = \pm q\left\{\frac{3p\,\mathrm{sd}[\xi, \frac{1}{\sqrt{2}}]}{4n}\right\}, \tag{114}$$

$$u_{10}(x, y, z, t) = \left\{\mp q\frac{3p\,sd[\xi, \frac{1}{\sqrt{2}}]}{4n}\right\}^{\frac{2}{n}}, \tag{115}$$

where

$$\xi = px + ky + lz - \left(\frac{b + 8p^4\beta}{4n^2 p}\right)t.$$

When $q_2 = 0$, $q_0 = -1/4$, $q_4 = 1$ and $\psi(\xi) = \mathrm{ds}[\xi, \frac{1}{\sqrt{2}}]$ from Table 2 (Appendix), by using *Case* 2, we have

$$W_{11}(\xi) = \pm q \left\{ \frac{p\,\mathrm{ds}[\xi, \frac{1}{\sqrt{2}}]}{n} \right\}, \tag{116}$$

$$u_{11}(x, y, z, t) = \left\{ \mp q \frac{p\,\mathrm{ds}[\xi, \frac{1}{\sqrt{2}}]}{n} \right\}^{\frac{2}{n}}, \tag{117}$$

where

$$\xi = px + ky + lz + \left(\frac{b + 8p^4\beta}{4n^2 p}\right)t.$$

When $q_2 = 0$, $q_0 = 1/4$, $q_4 = 1/4$ and $\psi(\xi) = \mathrm{ns}[\xi, \frac{1}{\sqrt{2}}] + \mathrm{cs}[\xi, \frac{1}{\sqrt{2}}]$ from Table 2 of the Appendix, by using *Case* 2, we have

$$W_{12}(\xi) = \pm q \left\{ \frac{p\left[\mathrm{ns}\left(\xi, \frac{1}{\sqrt{2}}\right) + \mathrm{cs}\left(\xi, \frac{1}{\sqrt{2}}\right)\right]}{4n} \right\}, \tag{118}$$

$$u_{13}(x, y, z, t) = \left\{ \mp q \frac{p\left[\mathrm{ns}\left(\xi, \frac{1}{\sqrt{2}}\right) + \mathrm{cs}\left(\xi, \frac{1}{\sqrt{2}}\right)\right]}{4n} \right\}^{\frac{2}{n}}, \tag{119}$$

where

$$\xi = px + ky + lz - \left(\frac{b + 8p^4\beta}{16n^2 p}\right)t.$$

When $q_2 = 0$, $q_0 = 1/4$, $q_4 = 1/2$ and $\psi(\xi) = \text{ns}[\xi, \sqrt{2}] + \text{ds}[\xi, \sqrt{2}]$ from Table 2 (Appendix), by using *Case* 2, we have

$$W_{14}(\xi) = \pm q\left\{\frac{p\left[n\text{s}\left(\xi, \sqrt{2}\right) + \text{ds}\left(\xi, \sqrt{2}\right)\right]}{2n}\right\}, \tag{120}$$

$$u_{14}(x, y, z, t) = \left\{\mp q\frac{p\left[\text{ns}\left(\xi, \sqrt{2}\right) + \text{ds}\left(\xi, \sqrt{2}\right)\right]}{2n}\right\}^{\frac{2}{n}}, \tag{121}$$

where

$$\xi = px + ky + lz - \left(\frac{b + 8p^4\beta}{8n^2 p}\right)t.$$

When $q_2 = 0$, $q_0 = 1/2$, $q_4 = 1/2$ and $\psi(\xi) = \text{sn}[\xi, \sqrt{2}] + \text{ics}[\xi, \sqrt{2}]$ from Table 2 (Appendix), by using *Case* 2, we have

$$W_{15}(\xi) = \pm q\left\{\frac{p\left[\text{sn}\left(\xi, \sqrt{2}\right) + \text{ics}\left(\xi, \sqrt{2}\right)\right]}{2n}\right\}, \tag{122}$$

$$u_{15}(x, y, z, t) = \left\{\mp q\frac{p\left[\text{sn}\left(\xi, \sqrt{2}\right) + \text{ics}\left(\xi, \sqrt{2}\right)\right]}{2n}\right\}^{\frac{2}{n}}, \tag{123}$$

where

$$\xi = px + ky + lz - \left(\frac{b + 8p^4\beta}{4n^2 p}\right)t,$$

which in the limiting case gives rise to complexiton solutions.

8. LIE SYMMETRY ANALYSIS

The Lie group method [3, 9, 10] is sometimes also called symmetry analysis. Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones. Some of the recent contributions in this field are available in 2011 and 2012 [8, 11].

In this section, we will investigate the symmetries of equation (1.2) by using Lie classical method. Firstly, let us consider a one-parameter Lie group of infinitesimal transformation

$$\begin{aligned} x^* &\to x + \varepsilon\xi(x, y, z, t, q) \\ y^* &\to y + \varepsilon\phi(x, y, z, t, q) \\ z^* &\to z + \varepsilon\zeta(x, y, z, t, q) \\ t^* &\to t + \varepsilon\tau(x, y, z, t, q) \\ q^* &\to q + \varepsilon\eta(x, y, z, t, q), \end{aligned} \tag{124}$$

with a small parameter $\varepsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$\begin{aligned} V = {} & \xi(x, y, z, t, q)\frac{\partial}{\partial x} + \phi(x, y, z, t, q)\frac{\partial}{\partial y} + \\ & +\zeta(x, y, z, t, q)\frac{\partial}{\partial z} + \tau(x, y, z, t, q)\frac{\partial}{\partial t} + \eta(x, y, z, t, q)\frac{\partial}{\partial q}. \end{aligned} \tag{125}$$

The symmetry group of equation (43) will be generated by the vector field of the form (125). Applying the fourth prolongation $\mathrm{pr}^{(4)}V$ of V to equation (43), we find that the coefficient functions ξ, ϕ, ζ, τ and η must satisfy the symmetry condition

$$\begin{aligned} & \eta^{tx} + 2anq^{n-1}q_x\eta^x + an(n-1)q^{n-2}q_x^2\eta + aq^n\eta^{xx} + anq^{n-1}q_{xx}\eta + b\eta^{xxxx} + \\ & +c(\eta^{xx} + \eta^{yy} + \eta^{zz}) + k(\eta^{xy} + \eta^{yz} + \eta^{zx}) = 0, \end{aligned} \tag{126}$$

where $\eta^{tx}, \eta^{xx}, \eta^{yy}, \eta^{zz}, \eta^{xy}, \eta^{yz}, \eta^{zx}$ and η^{xxxx} are coefficients of $\mathrm{pr}^{(4)}V$. Substituting the values of $\eta^{tx}, \eta^{xx}, \eta^{yy}, \eta^{zz}, \eta^{xy}, \eta^{yz}, \eta^{zx}$ and η^{xxxx} in (126) and replace q_{xt} by (43). On substituting the coefficients of different differentials equal to zero lead to the system of determining equations.

Solving this system of determining equations provides following forms for the infinitesimal elements ξ, ϕ, ζ, τ and η

$$\eta = 0,$$

$$\xi = \frac{(C_2 + C_3)k^2 t + (C_5 t - (C_2 + C_3)y + C_7)k - C_5 y}{k},$$

$$\zeta = C_2 y + C_5 t + C_3 z + C_4,$$

$$\tau = C_1$$

$$\phi = 2\left(-\frac{k^4 C_3 t}{8} + (-\frac{3cC_2}{8}t + \frac{C_3}{8}y - \frac{C_6}{8})k^3 + \right.$$
$$+(cC_2 t + \frac{C_2}{2}y - C_5 t - C_3 z)\frac{ck^2}{4} + \tag{127}$$
$$\left. + \frac{c^2((C_2 + C_3)ct - \frac{C_2}{4} + \frac{C_6}{2})k + c^3 C_5 t}{(c - \frac{k}{2})(k)(c + \frac{k}{2})} \right),$$

for $k \neq 2c$ and $k \neq -2c$.

Solving determining equations or $k = 2c$, we have following form of the infinitesimals

$$\eta = 0, \quad \xi = (-kC_5 + C_3)t + C_5 y - \frac{C_3}{k}y + C_6,$$
$$\phi = (-kC_5 + C_3)t + f(z - y), \quad \zeta = C_3 t + (z - 2y)C_5 + C_4, \tag{128}$$
$$\tau = (z - y)C_2 + C_1,$$

where $f(z - y)$ is any arbitrary function.

Corresponding vector fields are given as

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = (z - y)\frac{\partial}{\partial t}, \quad V_3 = t\frac{\partial}{\partial x} - \frac{y}{k}\frac{\partial}{\partial x} + t\frac{\partial}{\partial z} + t\frac{\partial}{\partial y},$$
$$V_4 = \frac{\partial}{\partial z}, \quad V_5 = -kt\frac{\partial}{\partial y} + (y - kt)\frac{\partial}{\partial x} + (z - 2y)\frac{\partial}{\partial z}, \tag{129}$$
$$V_6 = \frac{\partial}{\partial x}, \quad V_f = f(z - y)\frac{\partial}{\partial y}.$$

Now solving determining equations for $k = -2c$, we have following form of the infinitesimals

$$\eta = 0, \quad \xi = -(cC_5 - \frac{C_3}{6}) - 6t - 3C_5 y + \frac{C_3}{2c}y + C_6,$$
$$\phi = (6cC_5 - C_3)t + g(y + z + 4ct), \quad \zeta = C_3 t + (z + 2y)C_5 + C_4, \tag{130}$$
$$\tau = (y + z + 4ct)C_1 + C_2,$$

where $g(y + z + 4ct)$ is any arbitrary function.

Corresponding vector fields are

$$V_1=(z+y+4ct)\frac{\partial}{\partial t},\quad V_2=\frac{\partial}{\partial t},\quad V_3=-t\frac{\partial}{\partial y}+(t+\frac{y}{2c})\frac{\partial}{\partial x}+t\frac{\partial}{\partial z},$$
$$V_4=\frac{\partial}{\partial z},\quad V_5=6ct\frac{\partial}{\partial y}-3(2ct+y)\frac{\partial}{\partial x}+(z+2y)\frac{\partial}{\partial z},\tag{131}$$
$$V_6=\frac{\partial}{\partial x},\quad V_g=g(y+z+4ct)\frac{\partial}{\partial y}.$$

8.1. SYMMETRY REDUCTIONS AND EXACT SOLUTIONS

In this section we will consider reductions of equation (43) and will derived some interesting exact solutions. We will consider following two cases.

Case 1. When $k=2c$. In this case for the generator $e_1V_1+e_2V_4+$ $+e_3V_6+e_4V_f$, from (129) we have following similarity variables

$$\rho=e_4^{-1}\int\frac{1}{f(z-y)}\mathrm{d}y+e_1^{-1}t+e_2^{-1}z+e_3^{-1}x,\; q=F(\rho),\tag{132}$$

where ρ is new independent variable and $F(\rho)$ is new dependent variable.

Substituting new variables from (132) in (43), we obtain ordinary differential equation (ODE). Now integrating the ODE twice with respect to ρ and taking constants of integration equal to zero, we have

$$F''''+\alpha\frac{F^{n+1}}{n+1}+\beta F=0,\tag{133}$$

where $(')$ denotes derivative with respect to ρ and

$$\alpha=\frac{a}{b}e_3^2,\beta=\frac{e_3^2e_2^2+ce_1e_2^2e_3^2+ce_1e_3^4+2ce_1e_2e_3^3}{be_1e_2^2}.\tag{134}$$

Subcase i: $n=2$. Corresponding to (133), with condition $e_3=1$, we have following solutions of main equation (43)

$$\text{(i) } q=\frac{\sqrt{-3\alpha\beta}\tanh\left(C_1+1/2\sqrt{2}\sqrt{\beta}(e_4^{-1}\int\frac{1}{f(z-y)}\mathrm{d}y+e_1^{-1}t+e_2^{-1}z+e_3^{-1}x)\right),}{\alpha}$$
$$\text{(ii) } q=\frac{\sqrt{-6\alpha\left(-C_3{}^2+\beta\right)}}{\alpha}\,\mathrm{sn}\left(C_2+C_3(e_4^{-1}\int\frac{1}{f(z-y)}\mathrm{d}y+e_1^{-1}t+e_2^{-1}z+e_3^{-1}x),\frac{\sqrt{-C_3{}^2+\beta}}{C_3}\right)\tag{135}$$
$$\text{(iii) } q=\sqrt{6}C_3\,\mathrm{dn}\left(C_2+C_3(e_4^{-1}\int\frac{1}{f(z-y)}\mathrm{d}y+e_1^{-1}t+e_2^{-1}z+e_3^{-1}x),\frac{\sqrt{2C_3{}^2+\beta}}{C_3}\right)\frac{1}{\sqrt{\alpha}},$$

where C_1, C_2, C_3 are arbitrary constants and α, β are given by (134).

Subcase ii: $n=1$. Corresponding to (133), with condition $e_3 = 1$, we have following solutions of main equation (43)

$$\text{(i) } q = \frac{\beta}{\alpha} - 3\frac{\beta\left(\tanh\left(C_1 + 1/2\sqrt{\beta}(e_4^{-1}\int \frac{1}{f(z-y)}\mathrm{d}y + e_1^{-1}t + e_2^{-1}z + e_3^{-1}x)\right)\right)^2}{\alpha}$$

$$\text{(ii) } q = \left(-\beta + \frac{\beta C_1^{\,2}}{\sqrt{C_1^{\,4} - C_1^{\,2} + 1}} + \frac{\beta}{\sqrt{C_1^{\,4} - C_1^{\,2} + 1}}\right)\alpha^{-1} \tag{136}$$

$$-\frac{3\beta C_1^{\,2}}{\alpha\sqrt{C_1^{\,4} - C_1^{\,2} + 1}}\left(\operatorname{sn}\left(C_2 - 1/2\sqrt{\frac{\beta}{\sqrt{C_1^{\,4} - C_1^{\,2} + 1}}}(e_4^{-1}\int \frac{1}{f(z-y)}\mathrm{d}y + e_1^{-1}t + e_2^{-1}z + e_3^{-1}x), C_1\right)\right)^2$$

$$\text{(iii) } q = -2\frac{\beta}{\alpha} + 3\frac{\beta\left(\csc\left(C_1 - 1/2\sqrt{-\beta}(e_4^{-1}\int \frac{1}{f(z-y)}\mathrm{d}y + e_1^{-1}t + e_2^{-1}z + e_3^{-1}x)\right)\right)^2}{\alpha},$$

where C_1, C_2 are arbitrary constants and α, β are given by (134).

Case 2. $k = -2c$. From (131), consider the generator $e_1V_2 + e_2V_4 + {}$ $+ e_3V_6 + e_4V_g$, we have following similarity variables

$$\rho = e_1^{-1}t + e_2^{-1}z + e_3^{-1}x + e_4^{-1}\int \frac{1}{g(y+z+4ct)}\mathrm{d}y, q = F(\rho), \tag{137}$$

where ρ is new independent variable and $F(\rho)$ is new dependent variable.

Substituting new variables from (137) in (43), we obtain an ODE. Now integrating the ODE twice with respect to ρ and taking constants of integration equal to zero, we have

$$F'''' + \alpha\frac{F^{n+1}}{n+1} + \beta F = 0, \tag{138}$$

where $(\,')$ denotes derivative with respect to ρ and

$$\alpha = \frac{a}{b}e_3^2, \beta = \frac{e_3^2 e_2^2 + ce_1e_2^2e_3^2 + ce_1e_3^4 - 2ce_1e_2e_3^3}{be_1e_2^2}. \tag{139}$$

Subcase i: $n=2$. Corresponding to (138), with condition $e_3 = 1$, we have following solutions of main equation (43)

$$\text{(i) } q=\frac{\sqrt{-3\alpha\beta}\tanh\left(C_1+1/2\sqrt{2}\sqrt{\beta}(e_1^{-1}t+e_2^{-1}z+e_3^{-1}x+e_4^{-1}\int\frac{1}{g(y+z+4ct)}\mathrm{d}y)\right)}{\alpha}$$

$$\text{(ii) } q=\frac{\sqrt{-6\alpha\left(-C_3{}^2+\beta\right)}}{\alpha}\operatorname{sn}\left(C_2+C_3(e_1^{-1}t+e_2^{-1}z+e_3^{-1}x+e_4^{-1}\int\frac{1}{g(y+z+4ct)}\mathrm{d}y),\frac{\sqrt{-C_3{}^2+\beta}}{C_3}\right) \tag{140}$$

$$\text{(iii) } q=\sqrt{6}C_3\operatorname{dn}\left(C_2+C_3(e_1^{-1}t+e_2^{-1}z+e_3^{-1}x+e_4^{-1}\int\frac{1}{g(y+z+4ct)}\mathrm{d}y),\frac{\sqrt{2C_3{}^2+\beta}}{C_3}\right)\frac{1}{\sqrt{\alpha}},$$

where C_1, C_2, C_3 are arbitrary constants and α, β are given by (139).

Subcase ii: $n=1$. Corresponding to (138), with condition $e_3=1$, we have following solutions of main equation (43)

$$\text{(i) } q=\frac{\beta}{\alpha}-3\frac{\beta\left(\tanh\left(C_1+1/2\sqrt{\beta}(e_1^{-1}t+e_2^{-1}z+e_3^{-1}x+e_4^{-1}\int\frac{1}{g(y+z+4ct)}\mathrm{d}y)\right)\right)^2}{\alpha}$$

$$\text{(ii) } q=\left(-\beta+\frac{\beta C_1{}^2}{\sqrt{C_1{}^4-C_1{}^2+1}}+\frac{\beta}{\sqrt{C_1{}^4-C_1{}^2+1}}\right)\alpha^{-1} \tag{141}$$

$$-\frac{3\beta C_1{}^2}{\alpha\sqrt{C_1{}^4-C_1{}^2+1}}\left(\operatorname{sn}\left(C_2-1/2\sqrt{\frac{\beta}{\sqrt{C_1{}^4-C_1{}^2+1}}}(e_1^{-1}t+e_2^{-1}z+e_3^{-1}x+e_4^{-1}\int\frac{1}{g(y+z+4ct)}\mathrm{d}y),C_1\right)\right)^2$$

$$\text{(iii) } q=-2\frac{\beta}{\alpha}+3\frac{\beta\left(\csc\left(C_1-1/2\sqrt{-\beta}(e_1^{-1}t+e_2^{-1}z+e_3^{-1}x+e_4^{-1}\int\frac{1}{g(y+z+4ct)}\mathrm{d}y)\right)\right)^2}{\alpha},$$

where C_1, C_2 are arbitrary constants and α, β are given by (139).

9. CONCLUSIONS

This paper gives a very comprehensive account of the extended KP equation in (3+1)-D with power law nonlinearity. Several kinds of solutions are revealed by the aid of multiple integration techniques. There are three kinds of solitons that are obtained here. They are topological solitons that are also known as shock waves, non-topological solitons and singular solitons. Their respective domain restrictions are identified for these respective solitons to exist.

The improved JEF method displayed several forms of double periodic functions including the cnoidal and snoidal waves. In the limiting cases, these degenerate into the linear waves or solitons as well as complexitons, depending on the solution structure. The modified F-expansion method also gave a wide range of solutions that are in terms of the hyperbolic functions as well as rational solutions. Many special cases of these solutions are going to be in terms of topological or non-topological or singular solitons. One of the most powerful integration tol that was used in this paper is the Lie symmetry analysis that retrieved a few more solutions to the equation under study.

These wide variety of results are going to be truly useful in the area of multi-component plasmas where this equation appears. In fact the generalization to power law nonlinearity will be of added advantage as the special cases where $n = 1$ or $n = 2$, the latter being refered to as the modified KP equation, fall out easily from this generalized setting. Additionally, considering the power law nonlinearity displays the results on a generalized tone that easily allows the mathematical analysis of the results.

In future, this equation will be further studied. One of the many aspects that needs to be addressed is the soliton perturbation theory. In order to study this, the conservation laws needs to be obtained. Based on these conservation laws, the adiabatic parameter dynamics of the solitons will be obtained in presence of concrete perturbations terms. In addition to these deterministic perturbation terms, stochastic perturbation terms will also be considered. In this case, the corresponding Langevin equation will be derived and the mean free dynamics of the soliton parameters will be obtained. These results will be reported in the future.

APPENDIX

Table 1

Relations between values of (A, B, C) and corresponding $\psi(\xi)$ in Riccati equation

$$\psi'(\xi) = A + B\psi(\xi) + C\psi^2(\xi) \qquad (C \neq 0)$$

A	B	C	ψ
0	1	–1	$\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{1}{2}\xi\right)$
0	–1	1	$\frac{1}{2}-\frac{1}{2}\coth\left(\frac{1}{2}\xi\right)$
$\frac{1}{2}$	0	$\frac{-1}{2}$	$\coth(\xi) \pm \operatorname{csch}(\xi)$, $\tanh(\xi) \pm i\operatorname{sech}(\xi)$
1	0		$\tanh(\xi)$, $\coth(\xi)$
$\frac{1}{2}$	0	–1	$\sec(\xi)+\tan(\xi)$, $\csc(\xi)-\cot(\xi)$
$\frac{-1}{2}$	0	$\frac{1}{2}$	$\csc(\xi)+\cot(\xi)$, $\sec(\xi)-\tan(\xi)$
1(–1)	0	1(–1)	$\tan(\xi)(\cot(\xi))$
0	0	$\neq 0$	$\frac{-1}{C\xi+\lambda}$

Table 1 (continued)

arbitrary constant	0	0	$A\xi$
arbitrary constant	$\neq 0$	0	$\frac{\exp(B\xi)-A}{B}$

Table 2

Relation between values of (q_0, q_2, q_4) and corresponding ψ

q_0	q_2	q_4	$\psi(\zeta)$
1	$-1-m^2$	m^2	$\mathrm{sn}(\zeta)$ or $\mathrm{cd}(\zeta)=\frac{\mathrm{cn}(\zeta)}{\mathrm{dn}(\zeta)}$
$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{cn}(\zeta)$
m^2-1	$2-m^2$	-1	$\mathrm{dn}(\zeta)$
m^2	$-1-m^2$	1	$\mathrm{ns}(\zeta)\ \frac{1}{\mathrm{sn}(\zeta)}$ or $\mathrm{dc}(\zeta))=\frac{\mathrm{dn}(\zeta)}{\mathrm{cn}(\zeta)}$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{nc}(\zeta)=\frac{1}{\mathrm{cn}(\zeta)}$
-1	$2-m^2$	m^2-1	$\mathrm{nd}(\zeta)=\frac{1}{\mathrm{dn}(\zeta)}$
1	$2-m^2$	$1-m^2$	$\mathrm{sc}(\zeta)=\frac{\mathrm{sn}(\zeta)}{\mathrm{cn}(\zeta)}$
1	$2m^2-1$	$-m^2(-1-m^2)$	$\mathrm{sd}(\zeta)=\frac{\mathrm{sn}(\zeta)}{\mathrm{dn}(\zeta)}$
$1-m^2$	$2-m^2$	1	$\mathrm{cs}(\zeta)=\frac{\mathrm{cn}(\zeta)}{\mathrm{sn}(\zeta)}$
$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{ds}(\zeta)=\frac{\mathrm{dn}(\zeta)}{\mathrm{sn}(\zeta)}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}(\zeta)+\mathrm{cs}(\zeta)$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{2}$	$\mathrm{nc}(\zeta)+\mathrm{sc}(\zeta)$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{ns}(\zeta)+\mathrm{ds}(\zeta)$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{sn}(\zeta)+i\mathrm{cs}(\zeta)$

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