

## APPLICATION OF HOMOTOPY ANALYSIS TRANSFORM METHOD TO FRACTIONAL BIOLOGICAL POPULATION MODEL

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*Abstract.* In this paper, we present an algorithm of the homotopy analysis transform method (HATM) which is a combination of Laplace transform method and the homotopy analysis method (HAM) to solve generalized biological population models. The fractional derivatives are described by Caputo sense. The proposed method presents a procedure of constructing the set of base functions and gives the high-order deformation equations in a simple form. The proposed scheme provides the solution in the form of a rapidly convergent series. Three examples are used to illustrate the preciseness and effectiveness of the proposed method. The results show that the HATM is very efficient, simple and can be applied to other nonlinear problems.

*Key words:* Laplace transform, homotopy analysis transform method, biological population model, Mittag-Leffler function.

### 1. INTRODUCTION

Fractional differential equations have gained importance and popularity during the past three decades or so, mainly due to its demonstrated applications in numerous seemingly diverse fields of science and engineering. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. The fractional differential equations are also used in modeling of many chemical processes, mathematical biology and many other problems in physics and engineering [1–16].

Nonlinear problems are important for engineers, physicists and mathematicians namely because most physical system are nonlinear in nature. However, the nonlinear equations are difficult to solve and lead to interesting phenomena, *e.g.* chaos. The investigation of the exact solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. There are many approaches for seeking exact solutions, such as, Hirota's method, Bäcklund and Darboux transformations, Painlevé expansions. Recently, many alternative methods used for solving both nonlinear and linear differential equations of physical interest. The Adomian decomposition method (ADM) [17–18], the homotopy perturbation method (HPM) [19–23], the homotopy analysis method (HAM) [24–30], the variational iteration method (VIM) [31–38] and other methods have been used to solve linear and nonlinear problems. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Laplace decomposition method (LDM) [39–43] and the homotopy perturbation transform method (HPTM) [44]. Very recently, the homotopy analysis method (HAM) is combined with the well-known Laplace transform to produce a highly effective technique called the homotopy analysis transform method (HATM) [45, 46] for handling many nonlinear problems.

The fractional optimal control problems have been solved by Baleanu *et al.* [48]. Golmankhaneh *et al.* have employed the homotopy perturbation method (HPM) for solving a system of Schrödinger-Korteweg-de Vries equations [49].

In this paper, we consider the nonlinear fractional-order biological population model in the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) + f(u), \quad (1)$$

with the given initial condition

$$u(x, y, 0) = f_0(x, y), \quad (2)$$

where  $u$  denotes the population density and  $f$  represents the population supply due to birth and deaths. This nonlinear fractional biological population model is obtained by replacing the first time derivative term in the corresponding biological population model by a fractional derivative of order  $\alpha$  with  $0 < \alpha \leq 1$ . The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of  $\alpha = 1$  the fractional biological population model reduces to the standard biological population model. Some aspects of such a model have been studied previously by other researchers [51, 52]. In this paper, further we apply the homotopy analysis transform method (HATM)

to solve the fractional biological population models. The objective of the present paper is to modify the homotopy analysis method (HAM) to solve nonlinear fractional biological population models. The homotopy analysis transform method (HATM) is a combination of the homotopy analysis method (HAM) and Laplace transform method. The advantage of this method is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. The fact that the HATM solves nonlinear problems without using Adomian's polynomials and He's polynomials is a clear advantage of this technique over the Adomian's decomposition method (ADM) and the homotopy perturbation transform method (HPTM). The plan of our paper is as follows: Brief definitions of the fractional calculus are given in Section 2. The HATM is presented in Section 3. In Section 4, three numerical examples are solved to illustrate the applicability of the considered method. Conclusions are presented in Section 5.

## 2. BASIC DEFINITIONS

In this section, we mention the following basic definitions of fractional calculus.

*Definition 1.* The Riemann-Liouville fractional integral operator of order  $\alpha > 0$ , of a function  $f(t) \in C_{\mu}$ ,  $\mu \geq -1$  is defined as [5]:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0), \quad (3)$$

$$J^0 f(t) = f(t). \quad (4)$$

For the Riemann-Liouville fractional integral we have:

$$J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} t^{\alpha + \gamma}. \quad (5)$$

*Definition 2.* The fractional derivative of  $f(t)$  in the Caputo sense is defined as [10]:

$$\begin{aligned} D^{\alpha} f(t) &= J^{n-\alpha} D^n f(t) = \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \end{aligned} \quad (6)$$

for  $n-1 < \alpha \leq n$ ,  $n \in N$ ,  $x > 0$ .

*Definition 3.* The Laplace transform of the Caputo derivative is given by Caputo [10]; see also Kilbas *et al.* [13] in the form

$$L[D^\alpha f(t)] = s^\alpha L[f(t)] - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(0+), \quad n-1 < \alpha \leq n. \quad (7)$$

*Definition 4.* The Mittag-Leffler is defined as [47]:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (\alpha \in C, \operatorname{Re}(\alpha) > 0). \quad (8)$$

### 3. HATM FOR GENERALIZED BIOLOGICAL POPULATION MODEL

We consider the generalized biological population model of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + ku^a(1 - ru^b), \quad (9)$$

$t > 0$ ,  $x, y \in \mathbb{R}$ ,  $0 < \alpha \leq 1$ , with the initial condition

$$u(x, y, 0) = f_0(x, y). \quad (10)$$

Taking the Laplace transform on both sides of equation (9) subject to the initial condition (10), we have

$$L[u(x, y, t)] - \frac{1}{s} f_0(x, y) - \frac{1}{s^\alpha} L \left[ \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + ku^a(1 - ru^b) \right] = 0. \quad (11)$$

We define the nonlinear operator

$$N[\phi(x, y, t; q)] = L[\phi(x, y, t; q)] - \frac{1}{s} f_0(x, y) - \frac{1}{s^\alpha} L \left[ \frac{\partial^2}{\partial x^2} (\phi^2(x, y, t; q)) + \frac{\partial^2}{\partial y^2} (\phi^2(x, y, t; q)) + k\phi^a(x, y, t; q)(1 - r\phi^b(x, y, t; q)) \right], \quad (12)$$

where  $q \in [0, 1]$  and  $\phi(x, y, t; q)$  is a real function of  $x, y, t$  and  $q$ . We construct a homotopy as follows

$$(1 - q)L[\phi(x, y, t; q) - u_0(x, y, t)] = \hbar q H(t) N[\phi(x, y, t)], \quad (13)$$

where “L” denotes the Laplace transform,  $q \in [0, 1]$  is the embedding parameter,  $H(t)$  denotes a nonzero auxiliary function,  $\hbar \neq 0$  is an auxiliary parameter,  $u_0(x, y, t)$  is an initial guess of  $u(x, y, t)$  and  $\phi(x, y, t; q)$  is a unknown function. Obviously, when the embedding parameter  $q = 0$  and  $q = 1$ , it holds

$$\phi(x, y, t; 0) = u_0(x, y, t), \quad \phi(x, y, t; 1) = u(x, y, t), \quad (14)$$

respectively. Thus, as  $q$  increases from 0 to 1, the solution  $\phi(x, y, t; q)$  varies from the initial guess  $u_0(x, y, t)$  to the solution  $u(x, y, t)$ . Expanding  $\phi(x, y, t; q)$  in Taylor series with respect to  $q$ , we have

$$\phi(x, y, t; q) = u_0(x, y, t) + \sum_{m=1}^{\infty} u_m(x, y, t)q^m, \quad (15)$$

where

$$u_m(x, y, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, y, t; q)}{\partial q^m} \right|_{q=0}. \quad (16)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter  $\hbar$ , and the auxiliary function are properly chosen, the series (15) converges at  $q = 1$ , then we have

$$u(x, y, t) = u_0(x, y, t) + \sum_{m=1}^{\infty} u_m(x, y, t), \quad (17)$$

which must be one of the solutions of the original nonlinear equations. According to the definition (17), the governing equation can be deduced from the zero-order deformation (13). Define the vectors

$$\vec{u}_m = \{u_0(x, y, t), u_1(x, y, t), \dots, u_m(x, y, t)\}. \quad (18)$$

Differentiating the zero<sup>th</sup>-order deformation equation (13)  $m$ -times with respect to  $q$  and then dividing them by  $m!$  and finally setting  $q=0$ , we get the following  $m^{\text{th}}$ -order deformation equation:

$$\mathcal{L}[u_m(x, y, t) - \chi_m u_{m-1}(x, y, t)] = \hbar q H(t) \mathfrak{R}_m(\vec{u}_{m-1}). \quad (19)$$

Applying the inverse Laplace transform, we have

$$u_m(x, y, t) = \chi_m u_{m-1}(x, y, t) + \hbar \mathcal{L}^{-1}[q H(t) \mathfrak{R}_m(\vec{u}_{m-1})], \quad (20)$$

where

$$\mathfrak{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(x, y, t; q)]}{\partial q^{m-1}} \right|_{q=0}, \quad (21)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (22)$$

#### 4. APPLICATIONS

In this section, we use the HATM to solve the generalized biological population models.

*Example 4.1.* Consider the following generalized biological population model:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + u(1 - ru), \quad (23)$$

with the initial condition

$$u(x, y, 0) = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x + y)\right]. \quad (24)$$

Applying the Laplace transform subject to the initial condition, we have

$$\begin{aligned} & \mathcal{L}[u(x, y, t)] - \frac{1}{s} \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x + y)\right] - \\ & - \frac{1}{s^\alpha} \mathcal{L}\left[\frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + u(1 - ru)\right] = 0. \end{aligned} \quad (25)$$

The nonlinear operator is

$$\begin{aligned} & N[\phi(x, y, t; q)] = \mathcal{L}[\phi(x, y, t; q)] - \frac{1}{s} \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x + y)\right] - \\ & - \frac{1}{s^\alpha} \mathcal{L}\left[\frac{\partial^2}{\partial x^2}(\phi^2(x, y, t; q)) + \frac{\partial^2}{\partial y^2}(\phi^2(x, y, t; q)) + \right. \\ & \left. + \phi(x, y, t; q)(1 - r\phi(x, y, t; q))\right], \end{aligned} \quad (26)$$

and thus

$$\begin{aligned} \mathfrak{R}_m(\bar{u}_{m-1}) = & \mathcal{L}(u_{m-1}) - (1 - \chi_m) \frac{1}{s} \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x + y)\right] - \frac{1}{s^\alpha} \mathcal{L}\left[\frac{\partial^2}{\partial x^2}\left(\sum_{r=0}^{m-1} u_r u_{m-1-r}\right)\right] \\ & + \frac{\partial^2}{\partial y^2}\left(\sum_{r=0}^{m-1} u_r u_{m-1-r}\right) + u_{m-1} - r\left(\sum_{r=0}^{m-1} u_r u_{m-1-r}\right). \end{aligned} \quad (27)$$

The  $m^{\text{th}}$ -order deformation equation is given by

$$\mathbb{L}[u_m(x, y, t) - \chi_m u_{m-1}(x, y, t)] = \hbar \mathfrak{R}_m(\bar{u}_{m-1}). \quad (28)$$

Applying the inverse Laplace transform, we have

$$u_m(x, y, t) = \chi_m u_{m-1}(x, y, t) + \hbar \mathbb{L}^{-1}[\mathfrak{R}_m(\bar{u}_{m-1})]. \quad (29)$$

Solving the above equation (29), for  $m = 1, 2, 3, \dots$ , we get

$$\begin{aligned} u_1(x, y, t) &= -\hbar \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right] \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2(x, y, t) &= -\hbar(1+\hbar) \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right] \frac{t^\alpha}{\Gamma(\alpha+1)} + \\ &\quad + \hbar^2 \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3(x, y, t) &= -\hbar(1+\hbar)^2 \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right] \frac{t^\alpha}{\Gamma(\alpha+1)} + \\ &\quad + 2\hbar^2(1+\hbar) \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \\ &\quad - \hbar^3 \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right] \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned} \quad (30)$$

$\vdots$

and so on.

Taking  $\hbar = -1$ , the solution is given by

$$\begin{aligned} u(x, y, t) &= \sum_{m=0}^{\infty} u_m(x, y, t) = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right] \sum_{m=0}^{\infty} \frac{(t^\alpha)^m}{\Gamma(m\alpha+1)} = \\ &= \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right] E_\alpha(t^\alpha). \end{aligned} \quad (31)$$

If we put  $\alpha = 1$ , we obtain the exact solution:

$$u(x, y, t) = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right] e^t = \exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y) + t\right], \quad (32)$$

which is in full agreement with the results obtained by El-Sayed *et al.* [50] and Arafa *et al.* [51].

*Example 4.2.* Consider the following generalized biological population model:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + ku, \quad (33)$$

with the initial condition

$$u(x, y, 0) = \sqrt{xy}. \quad (34)$$

Applying the Laplace transform subject to the initial condition, we have

$$\mathbb{L}[u(x, y, t)] - \frac{1}{s}\sqrt{xy} - \frac{1}{s^\alpha} \mathbb{L} \left[ \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + ku \right] = 0. \quad (35)$$

The nonlinear operator is

$$\begin{aligned} N[\phi(x, y, t; q)] = & \mathbb{L}[\phi(x, y, t; q)] - \frac{1}{s}\sqrt{xy} - \frac{1}{s^\alpha} \mathbb{L} \left[ \frac{\partial^2}{\partial x^2}(\phi^2(x, y, t; q)) + \right. \\ & \left. + \frac{\partial^2}{\partial y^2}(\phi^2(x, y, t; q)) + k\phi(x, y, t; q) \right], \end{aligned} \quad (36)$$

and thus

$$\begin{aligned} \mathfrak{R}_m(\bar{u}_{m-1}) = & \mathbb{L}[u_{m-1}] - (1 - \chi_m) \frac{1}{s}\sqrt{xy} - \frac{1}{s^\alpha} \mathbb{L} \left[ \frac{\partial^2}{\partial x^2} \left( \sum_{r=0}^{m-1} u_r u_{m-1-r} \right) + \right. \\ & \left. + \frac{\partial^2}{\partial y^2} \left( \sum_{r=0}^{m-1} u_r u_{m-1-r} \right) + ku_{m-1} \right]. \end{aligned} \quad (37)$$

The  $m^{\text{th}}$ -order deformation equation is given by

$$\mathbb{L}[u_m(x, y, t) - \chi_m u_{m-1}(x, y, t)] = \hbar \mathfrak{R}_m(\bar{u}_{m-1}). \quad (38)$$

Applying the inverse Laplace transform, we have

$$u_m(x, y, t) = \chi_m u_{m-1}(x, y, t) + \hbar \mathbb{L}^{-1}[\mathfrak{R}_m(\bar{u}_{m-1})]. \quad (39)$$

Solving the above equation (39), for  $m = 1, 2, 3, \dots$ , we get



$$\begin{aligned}
u_1(x, y, t) &= -\hbar k \sqrt{xy} \frac{t^\alpha}{\Gamma(\alpha + 1)}, \\
u_2(x, y, t) &= -\hbar(1 + \hbar)k \sqrt{xy} \frac{t^\alpha}{\Gamma(\alpha + 1)} + \hbar^2 k^2 \sqrt{xy} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \\
u_3(x, y, t) &= -\hbar(1 + \hbar)^2 k \sqrt{xy} \frac{t^\alpha}{\Gamma(\alpha + 1)} + 2\hbar^2(1 + \hbar)k^2 \sqrt{xy} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \\
&\quad - \hbar^3 k^3 \sqrt{xy} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)},
\end{aligned} \tag{40}$$

and so on.

Taking  $\hbar = -1$ , the solution is given by

$$u(x, y, t) = \sum_{m=0}^{\infty} u_m(x, y, t) = \sqrt{xy} \sum_{m=0}^{\infty} \frac{(kt^\alpha)^m}{\Gamma(m\alpha + 1)} = \sqrt{xy} E_\alpha(kt^\alpha). \tag{41}$$

If we put  $\alpha = 1$ , we obtain the exact solution:

$$u(x, y, t) = \sqrt{xy} e^{kt}, \tag{42}$$

which is in full agreement with the results given by El-Sayed *et al.* [50] and Arafa *et al.* [51].

*Example 4.3.* Consider the following generalized biological population model:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + u, \tag{43}$$

with the initial condition

$$u(x, y, 0) = \sqrt{\sin x \cdot \sinh y}. \tag{44}$$

Applying the Laplace transform subject to the initial condition, we have

$$L[u(x, y, t)] - \frac{1}{s} \sqrt{\sin x \cdot \sinh y} - \frac{1}{s^\alpha} L \left[ \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + u \right] = 0. \tag{45}$$

The nonlinear operator is

$$\begin{aligned}
N[\phi(x, y, t; q)] &= L[\phi(x, y, t; q)] - \frac{1}{s} \sqrt{\sin x \cdot \sinh y} - \frac{1}{s^\alpha} L \left[ \frac{\partial^2}{\partial x^2} (\phi^2(x, y, t; q)) + \right. \\
&\quad \left. + \frac{\partial^2}{\partial y^2} (\phi^2(x, y, t; q)) + \phi(x, y, t; q) \right],
\end{aligned} \tag{46}$$

and thus

$$\begin{aligned} \mathfrak{R}_m(\bar{u}_{m-1}) = & L[u_{m-1}] - (1 - \chi_m) \frac{1}{s} \sqrt{\sin x \cdot \sinh y} - \\ & - \frac{1}{s^\alpha} L \left[ \frac{\partial^2}{\partial x^2} \left( \sum_{r=0}^{m-1} u_r u_{m-1-r} \right) + \frac{\partial^2}{\partial y^2} \left( \sum_{r=0}^{m-1} u_r u_{m-1-r} \right) + u_{m-1} \right]. \end{aligned} \quad (47)$$

The  $m^{\text{th}}$ -order deformation equation is given by

$$L[u_m(x, y, t) - \chi_m u_{m-1}(x, y, t)] = \hbar \mathfrak{R}_m(\bar{u}_{m-1}). \quad (48)$$

Applying the inverse Laplace transform, we have

$$u_m(x, y, t) = \chi_m u_{m-1}(x, y, t) + \hbar L^{-1}[\mathfrak{R}_m(\bar{u}_{m-1})]. \quad (49)$$

Solving the above equation (49), for  $m = 1, 2, 3, \dots$ , we get

$$\begin{aligned} u_1(x, y, t) &= -\hbar \sqrt{\sin x \cdot \sinh y} \frac{t^\alpha}{\Gamma(\alpha + 1)}, \\ u_2(x, y, t) &= -\hbar(1 + \hbar) \sqrt{\sin x \cdot \sinh y} \frac{t^\alpha}{\Gamma(\alpha + 1)} + \\ &+ \hbar^2 \sqrt{\sin x \cdot \sinh y} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \\ u_3(x, y, t) &= -\hbar(1 + \hbar)^2 \sqrt{\sin x \cdot \sinh y} \frac{t^\alpha}{\Gamma(\alpha + 1)} + \\ &+ 2\hbar^2(1 + \hbar) \sqrt{\sin x \cdot \sinh y} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \hbar^3 \sqrt{\sin x \cdot \sinh y} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}, \end{aligned} \quad (50)$$

and so on.

Taking  $\hbar = -1$ , the solution is given by

$$\begin{aligned} u(x, y, t) &= \sum_{m=0}^{\infty} u_m(x, y, t) = \sqrt{\sin x \cdot \sinh y} \sum_{m=0}^{\infty} \frac{(t^\alpha)^m}{\Gamma(m\alpha + 1)} = \\ &= \sqrt{\sin x \cdot \sinh y} E_\alpha(t^\alpha). \end{aligned} \quad (51)$$

If we put  $\alpha = 1$ , we obtain the exact solution:

$$u(x, y, t) = \sqrt{\sin x \cdot \sinh y} e^t, \quad (52)$$

which is in full agreement with the results given by El-Sayed *et al.* [50] and Arafa *et al.* [51].

## 5. CONCLUSIONS

In this paper, the homotopy analysis transform method (HATM) has been successfully applied to obtain the exact solutions of the generalized biological population equations subject to some initial conditions. The results obtained using the scheme presented here agree well with the analytical solutions and the numerical results obtained by Adomian's decomposition method (ADM) [50] and homotopy analysis method (HAM) [51]. However, El-Sayed *et al.* [50] have shown that ADM does not converge in general, in particular, when the method is applied to linear operator equations. It was also shown that ADM is equivalent to Picard iteration method, and therefore it might diverge. The homotopy analysis transform method (HATM) is another technique used to derive an analytic solution for nonlinear operators. It provides us with a simple way to adjust and control the convergence region of solution series by choosing proper values for auxiliary parameter  $\hbar$  and auxiliary function  $H(t)$ . The results reveal that HATM a very powerful and efficient technique in finding analytical solutions for wide classes of nonlinear differential equations.

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